CHAPTER 2

Nonautonomous dynamical systems

The formulation of an autonomous dynamical system as a group or semi-group of mappings depends on the fact that such systems depend only on the elapsed time $t - t_0$ since starting and not directly on the current time t or starting time t_0 themselves. For a nonautonomous system both the current time t and starting time t_0 are important rather than just their difference.

The most natural generalization of a semi-group formalism to nonautonomous dynamical systems is the *two-parameter semi-group* or *process* formalism of a nonautonomous dynamical system, where both t and t_0 are the parameters. The process formulation will be treated in the first section of this chapter.

An alternative method includes an autonomous dynamical system as a driving mechanism which is responsible for, e.g., the temporal change of the vector field of a nonautonomous dynamical system. This leads to the *skew product flow* formalism of a nonautonomous dynamical system, which is discussed in the second part of this chapter.

1. Processes formulation

Consider an initial value problem for a nonautonomous ordinary differential equation in $\mathbb{R}^d$,

$$\dot{x} = f(t, x)\,, \qquad x(t_0) = x_0\,. \tag{2.1}$$

In contrast to the autonomous case, the solutions may now depend separately on the actual time t and the starting time t_0 rather than only on the elapsed time $t - t_0$ since starting. For example, the scalar initial value problem

$$\dot{x} = -2tx\,, \qquad x(t_0) = x_0\,,$$

has the explicit solution

$$x(t) = x(t, t_0, x_0) = x_0\, e^{-(t^2 - t_0^2)} \quad \text{for all } \; t, t_0, x_0 \in \mathbb{R}\,,$$

and $t^2 - t_0^2 = (t - t_0)^2 + 2(t - t_0)t_0$ cannot be expressed in terms of $t - t_0$ alone. Assuming global existence and uniqueness of solutions in forward time, the solutions form a continuous mapping $(t, t_0, x_0) \mapsto x(t, t_0, x_0) \in \mathbb{R}^d$ for $t \geq t_0$ with $t, t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$ fulfilling the initial value and evolution properties

(i) $x(t_0, t_0, x_0) = x_0$ for all $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^d$,
(ii) $x(t_2, t_0, x_0) = x\big(t_2, t_1, x(t_1, t_0, x_0)\big)$ for all $t_0 \leq t_1 \leq t_2$ and $x_0 \in \mathbb{R}^d$.

The evolution property (ii) is a consequence of the causality principle that the solutions are determined uniquely by their initial values (for the given differential equation).

1.1. Definition. Solution mappings of nonautonomous ordinary differential equations are one of the main motivations for the process formulation of an abstract nonautonomous dynamical system on a metric state space (X, d) and time set $\mathbb{T}$, where $\mathbb{T} = \mathbb{R}$ for a continuous time process and $\mathbb{T} = \mathbb{Z}$ for a discrete time process.

The following definition originates from Dafermos [**62**] and Hale [**90**].

Definition 2.1 (Process formulation). A *process* is a continuous mapping $(t, t_0, x_0) \mapsto \phi(t, t_0, x_0) \in X$ for $t, t_0 \in \mathbb{T}$ and $x_0 \in X$ with $t \geq t_0$, which satisfies the initial value and evolution properties

(i) $\phi(t_0, t_0, x_0) = x_0$ for all $t_0 \in \mathbb{T}$ and $x_0 \in X$,
(ii) $\phi(t_2, t_0, x_0) = \phi\big(t_2, t_1, \phi(t_1, t_0, x_0)\big)$ for all $t_0 \leq t_1 \leq t_2$ and $x_0 \in X$.

A process is often called a *two-parameter semi-group* on X in contrast with the *one-parameter semi-group* of an autonomous semi-dynamical system since it depends on both the initial time t_0 and the actual time t rather than just the elapsed time $t - t_0$.

1.2. Examples. The solution $x(t, t_0, x_0)$ of the nonautonomous differential equation (2.1) defines a continuous time process under the assumption of global existence and uniqueness of solutions. Indeed, this was the motivating example behind the definition of a process. Similarly, a nonautonomous difference equation generates a discrete time process.

Example 2.2 (Nonautonomous difference equations as processes). Let $f_n : \mathbb{R}^d \to \mathbb{R}^d$, $n \in \mathbb{Z}$, be continuous mappings. Then the nonautonomous difference equation

$$x_{n+1} = f_n(x_n) \tag{2.2}$$

generates a discrete time process ϕ which is defined for all $x_0 \in \mathbb{R}^d$ and $n, n_0 \in \mathbb{Z}$ with $n > n_0$ by

$$\phi(n_0, n_0, x_0) := x_0\,, \qquad \phi(n, n_0, x_0) := f_{n-1} \circ \cdots \circ f_{n_0}(x_0)\,.$$

In particular, note that ϕ is continuous, since the variation in the non-discrete variable $x_0 \mapsto \phi(n, n_0, x_0)$ is continuous as composition of finitely many continuous mappings.

Not all examples of processes involve either differential or difference equations.

Example 2.3 (Nonhomogeneous Markov chains as processes). Consider a nonhomogeneous Markov chain on a finite state space $\{1, \ldots, N\}$ with $d \times d$ probability transition matrices

$$P(t_0, t) = \big(p_{i,j}(t_0, t)\big)_{i,j=1,\ldots,d} \quad \text{for all } \ t_0, t \in \mathbb{T} \text{ with } t \geq t_0\,.$$

Such transition matrices satisfy $P(t_0, t_0) = \mathbb{1}$, the $d \times d$ identity matrix, for all $t_0 \in \mathbb{T}$. They also satisfy the so-called *Chapman–Kolmogorov property*

$$P(t_0, s)P(s, t) = P(t_0, t) \quad \text{for all } \ t_0 \leq s \leq t\,.$$

Let Σ_d denote the subset of $\mathbb{R}^d$ consisting of the N-dimensional probability row vectors, i.e., $p = (p_1, \ldots, p_d) \in \Sigma_d$ satisfies $\sum_{i=1}^d p_i = 1$ with $0 \le p_i \le 1$ for $i = 1, \ldots, d$. If the states of the Markov chain at time t_0 satisfy the probability vector $p(t_0) \in \Sigma_d$, then they are distributed according to a probability vector $p(t) = p(t_0)P(t_0, t)$ at time $t \ge t_0$.
Thus, the mapping ϕ defined by $\phi(t, t_0, p_0) := p_0 P(t_0, t)$ is a process on the state space Σ_d, which is in fact linear in the initial state component p_0 and thus continuous in this variable. Continuity in the time variables is trivial in the discrete time case and requires the additional assumption of continuity of the transition matrices in both of their variables in the continuous time case. The two-parameter semi-group property follows from the Chapman–Kolmogorov property.

1.3. Perturbed motions. In contrast to autonomous dynamical systems, every solution $\bar{x}(t) := \phi(t, \bar{t}_0, \bar{x}_0)$ of a process ϕ on a Banach space $(X, \|\cdot\|)$ can be transformed to a constant solution of a related process $\bar{\phi}$, which is defined by

$$\bar{\phi}(t, t_0, x_0) := \phi(t, t_0, x_0) - \bar{x}(t)\,.$$

The mapping $\bar{\phi}$ is also called the *process of perturbed motion*. In particular, such a description yields a notational advantage in the study of local properties of one fixed solution of a process. If the process ϕ is generated by a nonautonomous differential equation

$$\dot{x} = f(t, x)\,,$$

then $\bar{\phi}$ is generated by the differential equation

$$\dot{x} = \tilde{f}(t, x) := f\big(t, x + \bar{x}(t)\big) - f\big(t, \bar{x}(t)\big)\,,$$

which is called *differential equation of perturbed motion*. Analogously, let $(\bar{x}_n)_{n\in\mathbb{Z}}$ be a solution of the nonautonomous difference equation

$$x_{n+1} = f_n(x_n)$$

with continuous mappings $f_n : \mathbb{R}^d \to \mathbb{R}^d$, $n \in \mathbb{Z}$. Then the corresponding *difference equation of perturbed motion* is given by

$$x_{n+1} = \tilde{f}_n(x_n) := f_n(x_n + \bar{x}_n) - \bar{x}_{n+1}\,.$$

1.4. An interesting property of processes. A process can be reformulated as an autonomous semi-dynamical system, which has some interesting implications.

The extended phase space will be denoted by $\mathcal{X} := \mathbb{T} \times X$, and define a mapping $\pi : \mathbb{T}_0^+ \times \mathcal{X} \to \mathcal{X}$ by

$$\pi(t, (t_0, x_0)) := \big(t + t_0, \phi(t + t_0, t_0, x_0)\big) \quad \text{for all } (t, (t_0, x_0)) \in \mathbb{T}_0^+ \times \mathcal{X}\,.$$

Note that the variable t in $\pi(t, (t_0, x_0))$ is the time which has elapsed since starting at time t_0, while the actual time is $t + t_0$.

THEOREM 2.4. *π is an autonomous semi-dynamical system on $\mathcal{X}$.*

PROOF. It is obvious that π is continuous in its variables and satisfies the initial condition

$$\pi(0, (t_0, x_0)) = \big(t_0, \phi(t_0, t_0, x_0)\big) = (t_0, x_0)\,.$$

It also satisfies the (one-parameter) semi-group property

$$\pi(s + t, (t_0, x_0)) = \pi\big(s, \pi(t, (t_0, x_0))\big) \quad \text{for all } s, t \in \mathbb{T}_0^+\,,$$

since, by the evolution property (ii) of the process,

$$\begin{aligned}\pi(s+t,(t_0,x_0)) &= \big(s+t+t_0,\phi(s+t+t_0,t_0,x_0)\big)\\ &= \big(s+t+t_0,\phi\big(s+t+t_0,t+t_0,\phi(t+t_0,t_0,x_0)\big)\big)\\ &= \pi\big(s,(t+t_0,\phi(t+t_0,t_0,x_0)\big)\\ &= \pi\big(s,\pi(t,(t_0,x_0))\big)\,.\end{aligned}$$

This finishes the proof of this theorem. □

The autonomous semi-dynamical system π on the extended state space $\mathcal{X}$ generated by a process ϕ on the state space X has some unusual properties. In particular, π has no nonempty ω-limit sets and, indeed, no compact subset of $\mathcal{X}$ is π-invariant. This is a direct consequence of the fact that time is a component of the state.

This has significant implications and means that many concepts for autonomous systems need to be modified appropriately to be of any use in the nonautonomous context. For example, note that a π-invariant subset $\mathcal{A}$ of $\mathcal{X}$ has the form $\mathcal{A} = \bigcup_{t_0\in\mathbb{T}}(t_0, A_{t_0})$, where A_{t_0} is a nonempty subset of X for each $t_0 \in \mathbb{T}$. Then the invariance property $\pi(t,\mathcal{A}) = \mathcal{X}$ for $t \in \mathbb{T}_0^+$ is equivalent to

$$\phi\big(t+t_0,t_0,A_{t_0}\big) = A_{t+t_0} \quad \text{for all } t \in \mathbb{T}_0^+ \text{ and } t_0 \in \mathbb{T}\,.$$

This will be used in Chapter 3 to motivate the definition of ϕ-invariant sets for a process ϕ.

2. Skew product flow formulation

To motivate the concept of a skew product flow, first a triangular system of ordinary differential equations is considered in which the uncoupled component can be considered as the driving force in the equation for the other component.

Consider an autonomous system of ordinary differential equations of the form

$$\dot{p} = f(p)\,, \qquad \dot{x} = g(p,x)\,, \tag{2.3}$$

where $p \in \mathbb{R}^n$ and $x \in \mathbb{R}^m$, i.e., with a triangular structure.

Assuming global existence and uniqueness of solutions forwards in time, the system of differential equations (2.3) generates an autonomous semi-dynamical system π on $\mathbb{R}^{n+m}$ which will be written in component form as

$$\pi(t,p_0,x_0) = \big(p(t,p_0),x(t,p_0,x_0)\big)\,,$$

with initial condition $\pi(0,p_0,x_0) = (p_0,x_0)$.

There are two important points to observe here. Firstly, the p-component of the system is an independent autonomous system in its own right, i.e., its solution mapping $p = p(t,p_0)$ generates an autonomous semi-dynamical system on $\mathbb{R}^n$ and amongst other properties satisfies the semi-group property

$$p(s+t,p_0) = p(s,p(t,p_0)) \quad \text{for all } s,t \geq 0\,. \tag{2.4}$$

Secondly, the semi-group property for π on $\mathbb{R}^{n+m}$, i.e.,

$$\pi(s+t,p_0,x_0) = \pi(s,\pi(t,p_0,x_0))\,,$$

can be expanded out componentwise as

$$\begin{aligned}\pi(s+t,p_0,x_0) &= \big(p(s+t,p_0),x(s+t,p_0,x_0)\big)\\ &= \big(p(s,p(t,p_0)),x(s+t,p_0,x_0)\big)\,,\end{aligned}$$

using (2.4), and

$$\pi(s,\pi(t,p_0,x_0)) = \big(p(s,p(t,p_0)),x(s,p(t,p_0),x(t,p_0,x_0))\big)\,.$$

These are identical for all $s,t \geq 0$ and all $(p_0,x_0) \in \mathbb{R}^{n+m}$. Equating for the second components gives

$$x(s+t,p_0,x_0) = x\big(s,p(t,p_0),x(t,p_0,x_0)\big) \quad \text{for all } \ s,t \geq 0\,,$$

which is a generalization of the semi-group property and known as the *cocycle property*.

Given a solution $p = p(t,p_0)$ of the p-component of the triangular system (2.3), the x-component becomes a nonautonomous ordinary differential equation in the x variable on $\mathbb{R}^m$ of the form

$$\dot{x} = g(p(t,p_0),x)\,, \quad \text{where } \ t \geq 0 \text{ and } x \in \mathbb{R}^n\,. \tag{2.5}$$

The function $p = p(t,p_0)$ can be considered as "driving" the nonautonomous system here, i.e., as being responsible for the changes in the vector field with the passage of time.

The solution $x(t) = x(t,p_0,x_0)$ with initial value $x(0) = x_0$ (which also depends on the choice of p_0 as a parameter through the driving solution $p(t,p_0)$) then satisfies the following.

(i) *Initial condition.* $x(0,p_0,x_0) = x_0$.
(ii) *Cocycle property.* $x(s+t,p_0,x_0) = x\big(s,p(t,p_0),x(t,p_0,x_0)\big)$.
(iii) *Continuity condition.* $(t,p_0,x_0) \to x(t,p_0,x_0)$ is continuous.

The mapping $x : \mathbb{R}_0^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is called a *cocycle mapping.* It describes the evolution of the solution of the nonautonomous differential equation (2.5) with respect to the driving system. Note that the variable t here is the time since starting at the state x_0 with the driving system at state p_0.

As mentioned above, the product system π on $\mathbb{R}^n \times \mathbb{R}^m$ is an autonomous semi-dynamical system and is known as a *skew product flow* due to the asymmetrical roles of the two component systems. This motivates an alternative definition of a nonautonomous dynamical system, called the *skew product flow formalism*, where, for various reasons, the driving system p is usually taken to be a reversible dynamical system, i.e., forming a group rather than a semi-group. This will happen for example, if the driving differential equation is restricted to a compact invariant subset of $\mathbb{R}^n$. Driving systems which are only semi-groups or semi-dynamical systems will be considered in Chapter 10.

REMARK 2.5. Any nonautonomous differential equation $\dot{x} = f(t,x)$ can be written as the triangular autonomous system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ f(t,x) \end{pmatrix}\,,$$

with an increase of the dimension of the state space by one. Note, however, that this system has no equilibrium points and bounded solutions, and thus, the theory of autonomous dynamical systems is of no use here.

The process formulation of a nonautonomous dynamical system defined by the solution mapping of a nonautonomous differential equation is quite intuitive. In contrast, the skew product flow formulation is more abstract, but it contains more information about how the system evolves in time, especially when the driving system is on a compact space P.

2.1. Definition. Let (X, d_X) and (P, d_P) be metric spaces. A *nonautonomous dynamical system* (θ, φ) is defined in terms of a cocycle mapping φ on a state space X which is driven by an autonomous dynamical system θ acting on a base or parameter space P and the time set $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$.

Specifically, the *dynamical system* θ on P is a group of homeomorphisms $(\theta_t)_{t\in\mathbb{T}}$ under composition on P with the properties that

(i) $\theta_0(p) = p$ for all $p \in P$,
(ii) $\theta_{s+t} = \theta_s(\theta_t(p))$ for all $s,\, t \in \mathbb{T}$,
(iii) the mapping $(t, p) \mapsto \theta_t(p)$ is continuous,

and the *cocycle mapping* $\varphi : \mathbb{T}_0^+ \times P \times X \to X$ satisfies

(i) $\varphi(0, p, x) = x$ for all $(p, x) \in P \times X$,
(ii) $\varphi(t + s, p, x) = \varphi\big(t, \theta_s(p), \varphi(s, p, x)\big)$ for all $s, t \in \mathbb{T}_0^+$, $(p, x) \in P \times X$,
(iii) the mapping $(t, p, x) \mapsto \varphi(t, p, x)$ is continuous.

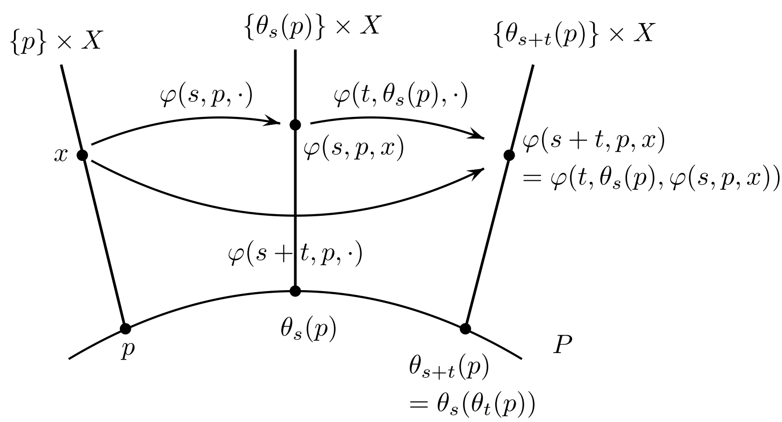

FIGURE 2.1. The cocycle property.

The mapping $\pi : \mathbb{T}_0^+ \times P \times X \to P \times X$ defined by

$$\pi(t, (p, x)) := \big(\theta_t(p), \varphi(t, p, x)\big) \tag{2.6}$$

forms an autonomous semi-dynamical system on $\mathcal{X} = P \times X$.

DEFINITION 2.6 (Skew product formulation). The autonomous semi-dynamical system π on $\mathcal{X} = P \times X$ defined by (2.6) is called the *skew product flow* associated with the nonautonomous dynamical system (θ, φ).

EXERCISE 2.7. Show that the mapping π defined by (2.6) defines a continuous time autonomous semi-dynamical system on $\mathcal{X}$.

2.2. Examples. Nonautonomous difference equations and differential equations provide a rich source of examples for skew product flows.

EXAMPLE 2.8. The solution mapping $x(t) = x(t, t_0, x_0)$ of a nonautonomous differential equation (2.1) with initial value $x(t_0, t_0, x_0) = x_0$ at time t_0 defines a process. Theorem 2.4 shows that such a process can be reformulated as a skew product flow with the cocycle mapping φ on the state space $X = \mathbb{R}^d$ defined by $\varphi(t, t_0, x_0) := x(t_0 + t, t_0, x_0)$ and the driving system θ on the (noncompact) base space $P = \mathbb{R}$ defined by the shift operators $\theta_t(t_0) := t - t_0$. The disadvantages of this representation were discussed above.

The advantages of the skew product flow formulation reveals itself, for instance, when the generating nonautonomous differential equation is periodic or almost periodic in time, because the base space is then compact.

EXAMPLE 2.9. The skew product formulation of a nonautonomous differential equation (2.1) is based on the fact that whenever $x(t)$ is a solution of the differential equation, then the time-shifted solution $x_\tau(t) := x(\tau + t)$ (for some fixed τ) satisfies the nonautonomous differential equation

$$\dot{x}_\tau(t) = f_\tau(t, x_\tau(t)) := f(\tau + t, x(\tau + t)) .$$

Consider the set of functions $\{f_\tau(\cdot, \cdot) := f(\tau + \cdot, \cdot) : \tau \in \mathbb{R}\}$. Its closure $\mathcal{F}$ in an appropriate topology is called the *hull* of the vector field given by the nonautonomous differential equation (2.1). See SELL [**218**] for examples and typical topologies. For example, $\mathcal{F}$ is a compact metric space for periodic or almost periodic differential equations (see Exercise 2.12 below).

Introduce a group of shift operators $\theta_\tau : \mathcal{F} \mapsto \mathcal{F}$ by $\theta_\tau(f) := f_\tau$ for each $\tau \in \mathbb{R}$, define $\mathcal{X} = \mathcal{F} \times \mathbb{R}^d$ and write $\varphi(t, f, x_0)$ for the solution of (2.1) with initial value x_0 at initial time $t_0 = 0$. Finally, define $\pi : \mathbb{R}^+ \times \mathcal{X} \to \mathcal{X}$ by $\pi(t, x_0, f) := (\theta_t(f), \varphi(t, f, x_0))$.

Then, $\pi = (\theta, \varphi)$ is a continuous-time skew product flow on the state space $\mathcal{X}$. To see this, observe that the second component of the semi-group identity $\pi(t + s, f, x_0) = \pi(t, \pi(s, f, x_0))$ expands out as the cocycle property

$$\varphi(t + s, f, x_0) = \varphi(t, \theta_s(f), \varphi(s, f, x_0)) .$$

Nonautonomous difference equations (2.2) generate discrete time skew product flows, the simplest coming from discrete time processes via Theorem 2.4 and have $\mathbb{Z}$ as their base space. When more is known about how the different mappings f_n vary with $n \in \mathbb{Z}$, it is often possible to have a compact base space.

EXAMPLE 2.10. Suppose that the mappings f_n in the nonautonomous difference equation (2.2) are chosen in some way from a finite number of continuous mappings $R_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i \in \{1, \dots, r\}$. Then the difference equation has the form

$$x_{n+1} = R_{i_n}(x_n) ,$$

where the $i_n \in \{1, \ldots, r\}$ for all $n \in \mathbb{N}$. It generates a discrete time skew product flow over the parameter set $P = \{1, \ldots, r\}^{\mathbb{Z}}$ of bi-infinite sequences $p = (i_n)_{n\in\mathbb{Z}}$ in $\{1, \ldots, r\}$ with respect to the group of left shift operators $(\theta_m)_{m\in\mathbb{N}}$, where $\theta_m\big((i_n)_{n\in\mathbb{Z}}\big) = (i_{n+m})_{n\in\mathbb{Z}}$. The cocycle mapping $\varphi(n, \cdot, \cdot)$ is defined by

$$\varphi(0, p, x) := x \quad \text{and} \quad \varphi(n, p, x) := \big(R_{i_{n-1}} \circ \cdots \circ R_{i_0}\big)(x)$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ and $p = (i_n)_{n\in\mathbb{N}} \in P$. The parameter space $P = \{1, \ldots, r\}^{\mathbb{Z}}$ here is a compact metric space with the metric

$$d(p, p') = \sum_{n=-\infty}^{\infty} (r+1)^{-|n|} \left|i_n - i'_n\right| ,$$

and the mappings $p \mapsto \theta_n(p)$ and $(p, x) \mapsto \varphi(n, p, x)$ are continuous for each $n \in \mathbb{N}$. To see this, note that $d(p, p') \leq \delta < 1$ requires $i_j = i_{j'}$ for $j = 0, \pm 1, \ldots, \pm N(\delta)$. Then take δ small enough corresponding to a given $\varepsilon > 0$ and fixed n.

More generally, the difference equation may involve a parameter $q \in Q$, which varies from iterate to iterate, by choice or randomly,

$$x_{n+1} = f(x_n, q_n) \, . \tag{2.7}$$

Example 2.11. Consider the parametrically dependent difference equation (2.7) with the continuous mapping $f : \mathbb{R} \times [-1, 1] \to \mathbb{R}$, given by

$$f(x, q) = f_q(x) := \nu x + q \, ,$$

where $\nu \in [0, 1)$ and $q \in [-1, 1]$. Let $P = [-1, 1]^{\mathbb{Z}}$ be the space of bi-infinite sequences $p = (q_n)_{n\in\mathbb{Z}}$ taking values in $[-1, 1]$, which is a compact metric space with the metric

$$d(p, p') = \sum_{n=-\infty}^{\infty} 2^{-|n|}\left|q_n - q'_n\right| ,$$

and let $(\theta_n)_{n\in\mathbb{Z}}$ be the group of the left shift operators on this sequence space (cf. Example 2.10). Finally, define the mappings $\varphi(n, \cdot, \cdot)$ by

$$\varphi(0, p, x_0) := x \quad \text{and} \quad \varphi(n, p, x) := \big(f_{q_{n-1}} \circ \cdots \circ f_{q_0}\big)(x)$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $p = (q_n)_{n\in\mathbb{N}} \in P$. Specifically,

$$\varphi(n, p, x) = \nu^n x + \sum_{j=0}^{n-1} \nu^{n-1-j} q_j$$

for all $n \in \mathbb{N}$.

The mappings $p \mapsto \theta_n(p)$ and $(p, x) \mapsto \varphi(n, p, x)$ are obviously continuous here for each $n \in \mathbb{N}$. Thus, (θ, φ) is a discrete time skew product flow on $\mathbb{R}$ with the compact base space P.

Skew product flows need not be generated by either differential equations or difference equations. The reader is invited to invent an example.

Exercise 2.12. Show using the Theorem of Arzelà–Ascoli that the hull of the cosine function $\cos t$ is the compact subset $\big\{ \cos(\tau + \cdot) : \tau \in [0, 2\pi] \big\}$ of the Banach space $C(\mathbb{R}, \mathbb{R})$ of all uniformly continuous functions $f : \mathbb{R} \to \mathbb{R}$, which is equipped with the supremum norm $\|f\|_\infty = \sup_{t\in\mathbb{R}} |f(t)|$.

3. Entire solutions and invariant sets

The definition of an entire solution of a nonautonomous dynamical system is an obvious generalization of the autonomous case.

DEFINITION 2.13 (Entire solution of a process). An *entire solution* of a process ϕ on a metric space (X, d) with time set $\mathbb{T}$ is a mapping $\xi : \mathbb{T} \to X$ such that

$$\xi(t) = \phi(t, \tau, \xi(\tau)) \quad \text{for all } t, \tau \in \mathbb{T} \text{ with } t \geq \tau\,. \tag{2.8}$$

The discussion following Theorem 2.4, in which a process ϕ on X was formulated as a skew product flow on the extended state space $\mathbb{R} \times X$, suggests that it is more appropriate to consider the invariance of a family of time-dependent subsets rather than of a single set. This motivates the following definition.

DEFINITION 2.14 (Invariant families for processes). Let ϕ be a process on a metric space (X, d). A family $\mathcal{A} = (A_t)_{t\in\mathbb{T}}$ of nonempty subsets of X is said to be *invariant* with respect to ϕ, or *ϕ-invariant*, if

$$\phi\left(t, t_0, A_{t_0}\right) = A_t \quad \text{for all } t \geq t_0\,.$$

A simple example of a ϕ-invariant family $\mathcal{A} = (A_t)_{t\in\mathbb{T}}$ is given by an entire solution of ϕ, i.e., having the singleton subsets $A_t = \{\xi(t)\}$ for each $t \in \mathbb{T}$. In fact, ϕ-invariant families consist of entire solutions.

LEMMA 2.15. *Let $\mathcal{A} = (A_t)_{t\in\mathbb{T}}$ be a nonempty ϕ-invariant family of subsets of X of a process ϕ. Then for any $t_0 \in \mathbb{T}$ and $a_0 \in A_{t_0}$, there exists an entire solution ξ through a_0 which is contained in $\mathcal{A}$, i.e., with $\xi(t_0) = a_0$ and $\xi(t) \in A_t$ for all $t \in \mathbb{T}$.*

PROOF. This is easy to see in the forward direction by defining $\xi(t) := \phi(t, t_0, a_0)$ for $t \geq t_0$, so in particular $\xi(t_0) = a_0$. For negative times, note that, since $\phi(t_0, t_0-1, A_{t_0-1}) = A_{t_0}$, there is a point $a_{-1} \in A_{t_0-1}$ such that $\phi(t_0, t_0-1, a_{-1}) = a_0$ and define $\xi(t) := \phi(t, t_0-1, a_{-1})$ for $t \in [t_0-1, t_0] \cap \mathbb{T}$, which is in $\mathcal{A}$ by the invariance of $\mathcal{A}$. Then repeat this construction on each subinterval $[t_{t_0}-n-1, t_0-n]$ for $n \in \mathbb{N}$ by defining $\xi(t) := \phi(t, t_0-n-1, a_{-n+1})$ for $t \in [t_0-n-1, t_0-n]$, where $a_{-n-1} \in A_{t_0-n-1}$ is chosen so that $\phi(t_0-n, t_0-n-1, a_{-n-1}) = a_{-n}$. It follows from the semi-group property that ξ satisfies (2.8) and from the ϕ-invariance of $\mathcal{A}$ that $\xi(t) \in A_t$ for all $t \in \mathbb{T}$. □

When the subsets in a ϕ-invariant family are compact, it follows from the continuity of a continuous time process that the set-valued mapping $t \mapsto A_t$ is continuous in $t \in \mathbb{R}$ with respect to the Hausdorff metric h_X, since

$$h_X(A_t, A_{t_0}) = h_X\big(\phi(t, t_0, A_{t_0}), \phi(t_0, t_0, A_{t_0})\big) \to 0 \quad \text{as } t \to t_0$$

by the continuity of the process ϕ in its first variable. The proof requires the result of the following exercise (see also ROXIN [**205**]).

EXERCISE 2.16. Let $f : \mathbb{R} \times X \to X$ be continuous, and define

$$F(t) := f(t, A) := \bigcup_{a\in A} \{f(t, a)\} \quad \text{for all } t \in \mathbb{R}\,,$$

where A is a compact subset of a metric space (X,d). Show that the sets $F(t)$ are compact subsets of X and that the setvalued mapping $t \mapsto F(t)$ is continuous with respect to the Hausdorff distance h.

Similar definitions hold for positive and negative invariant families of sets.

DEFINITION 2.17 (Positive and negative invariance). Let ϕ be a process on a metric space (X,d). A family $\mathcal{A} = (A_t)_{t\in\mathbb{T}}$ of nonempty subsets of X is said to be *positive invariant* with respect to ϕ, or *ϕ-positive invariant*, if

$$\phi\left(t, t_0, A_{t_0}\right) \subset A_t \quad \text{for all } \ t \geq t_0\,,$$

and *negative invariant* with respect to ϕ, or *ϕ-negative invariant*, if this holds with "$\supset$" instead of "$\subset$".

The corresponding definitions for skew product flows are stated here for completeness and later use.

DEFINITION 2.18 (Entire solution of a skew product flow). An *entire solution* of a skew product flow (θ,φ) on a metric phase space (X,d) and a base set P with time set $\mathbb{T}$ is a mapping $\xi : P \to X$ such that

$$\xi(\theta_t(p)) = \varphi(t-s, \theta_s(p), \xi(\theta_s(p))) \quad \text{for all } \ p \in P \text{ and } s,t \in \mathbb{T} \text{ with } s \leq t\,.$$

DEFINITION 2.19 (Invariant families for skew product flows). Let (θ,φ) be a skew product flow on a metric phase space (X,d) and a base set P. A family $\mathcal{A} = (A_p)_{p\in P}$ of nonempty sets of X is said to be *invariant* with respect to (θ,φ), or *φ-invariant*, if

$$\varphi(t,p,A_p) = A_{\theta_t(p)} \quad \text{for all } \ t \geq 0 \text{ and } p \in P\,.$$

For positive and negative invariant families, replace "$=$" here by "$\subset$" or "$\supset$", respectively.

The compact set-valued mapping $t \mapsto A_{\theta_t(p)}$ induced by a φ-invariant family $(A_p)_{p\in P}$ of compact subsets is continuous in $t \in \mathbb{R}$ with respect to the Hausdorff metric for each fixed $p \in P$.

3.1. Invariant subfamily of positive invariant family of sets. Analogously to the autonomous case, there is an invariant family of sets of a positively invariant family of compact sets of nonautonomous dynamical systems, in the sense that its component sets are subsets of the component sets of the positive invariant family. It then follows from Lemma 2.15 that the positive invariant family contains entire solutions.

This will be proved here for processes, which is notationally simpler than for skew product flows.

LEMMA 2.20. *Let $\mathcal{A} = (A_t)_{t\in\mathbb{T}}$ be a family of nonempty compact subsets of X which is positively invariant for the process ϕ, i.e., $\phi(t,t_0,A_{t_0}) \subset A_t$ for all $t_0 \in \mathbb{T}$ and $t \geq t_0$. Then there exists a family of nonempty compact subsets $\mathcal{A}^\infty = (A_t^\infty)_{t\in\mathbb{T}}$ of $\mathcal{A}$, which is ϕ-invariant, i.e., $\phi(t,t_0,A_{t_0}^\infty) = A_t^\infty$ for all $t \geq t_0$.*

PROOF. Since A_{t_0} is compact and the process ϕ is continuous, the set $\phi(t, t_0, A_{t_0})$ is compact for all $t \geq t_0$. Moreover, by the two-parameter semi-group property

$$\phi(t, s_0, A_{s_0}) = \phi\big(t, t_0, \phi(t_0, s_0, A_{s_0})\big) \subset \phi(t, t_0, A_{t_0}) \subset A_t$$

for all $s_0 \leq t_0 \leq t$, so for fixed $t \in \mathbb{T}$, the sets $\phi(t, t_0, A_{t_0})$ for $t_0 \leq t$ are a nested family of nonempty compact subsets of A. Hence, the set defined by

$$A_t^\infty = \bigcap_{t_0 \leq t} \phi(t, t_0, A_{t_0})$$

is a nonempty compact subset of A_t for each $t \in \mathbb{T}$. It remains to prove that $\mathcal{A}^\infty = (A_t^\infty)_{t\in\mathbb{T}}$ is ϕ-invariant, i.e., $\phi(t, t_0, A_{t_0}^\infty) = A_t^\infty$.
($\subset$) Let $\bar{a} \in A_{t_0}^\infty$. Then $\bar{a} \in \phi(t_0, s_0, A_{s_0})$ for all $s_0 \leq t_0$. Hence,

$$\phi(t, t_0, \bar{a}) \in \phi(t, t_0, \phi(t_0, s_0, A_{s_0})) = \phi(t, s_0, A_{s_0})$$

for any $t \geq t_0$ and $s_0 \leq t_0$, and

$$\phi(t, t_0, A_{t_0}) = \phi(t, s_0, \phi(s_0, t_0, A_{t_0})) \subset \phi(t, s_0, A_{s_0})$$

for any $t_0 \leq s_0 \leq t$, so

$$\phi(t, t_0, \bar{a}) \in \bigcap_{s_0 \leq t_0} \phi(t, s_0, A_{s_0}) = \bigcap_{s_0 \leq t} \phi(t, s_0, A_{s_0}) = A_t^\infty\,.$$

It follows that $\phi(t, t_0, A_{t_0}^\infty) \subset A_t^\infty$.
($\supset$) Let $\bar{a} \in A_t^\infty$. Then $\bar{a} \in \phi(t, s_n, A_{s_n}) = \phi(t, t_0, \phi(t_0, s_n, A_{s_n}))$ for all $s_n \leq t_0 \leq t$. Hence, there exist $b_n \in \phi(t_0, s_n, A_{s_n}) \subset A_{t_0}$ for all $n \in \mathbb{N}$ such that $\phi(t, t_0, b_n) = \bar{a}$. Now $b_n \in A_{t_0}$ for all $n \in \mathbb{N}$, and A_{t_0} is compact, so there exists a convergent subsequence $b_{n_j} \to \bar{b}$ in A_{t_0}. Moreover, one can choose the s_n so that $s_n \to -\infty$. In fact, $\bar{b} \in A_{t_0}^\infty$, since

$$\operatorname{dist}\big(\bar{b}, A_{t_0}^\infty\big) \leq \operatorname{dist}\big(\bar{b}, b_{n_j}\big) + \operatorname{dist}\big(\phi\big(t_0, s_{n_j}, A_{s_{n_j}}\big), A_{t_0}^\infty\big) \to 0 \tag{2.9}$$

as $j \to \infty$. Finally, by continuity, $\bar{a} = \phi(t, t_0, b_{n_j}) \to \phi(t, t_0, \bar{b})$, so $\bar{a} = \phi(t, t_0, \bar{b})$, which means that $A_t^\infty \subset \phi(t, t_0, A_{t_0}^\infty)$.
The convergence of the second term in (2.9) to zero follows from Lemma 1.27. □

3.2. Invariant subfamily of negative invariant family of sets. A negative invariant family of sets also contains an invariant subfamily, but the proof is somewhat more complicated than in the positive invariant case just considered. It will be given first for discrete time processes.

LEMMA 2.21. *Let $\mathcal{A} = (A_n)_{n\in\mathbb{Z}}$ be a family of nonempty compact subsets of X which is ϕ-negatively invariant for a discrete time process ϕ, i.e., $A_n \subset \phi(n, n_0, A_{n_0})$ for all $n \geq n_0$. Then there exists a family of nonempty compact subsets $\mathcal{A}^\infty = (A_n^\infty)_{n\in\mathbb{Z}}$ of $\mathcal{A}$, which is ϕ-invariant, i.e., $\phi(n, n_0, A_{n_0}^\infty) = A_n^\infty$ for all $n \geq n_0$.*

PROOF. Define $B_{n,0} := A_n$ for all $n \in \mathbb{Z}$, and for a fixed $n \in \mathbb{Z}$, define $B_{n,-1}$ to be the maximal subset of $B_{n-1,0}$ such that

$$B_{n,0} = \phi\big(n, n-1, B_{n,-1}\big)\,.$$

To see that the set $B_{n,-1}$ is compact, consider a sequence $(b_k)_{k\in\mathbb{N}}$ in $B_{n,-1}$ and define $a_k = \phi(n, n-1, b_k)$ for all $k \in \mathbb{N}$. Since $B_{n,0}$ and $B_{n-1,0}$ are compact and both $a_k \in B_{n,0}$ and $b_k \in B_{n,-1} \subset B_{n-1,0}$, there are convergent subsequences

$a_{k_j} \to \bar{a} \in B_{n,0}$ and $b_{k_j} \to \bar{b} \in B_{n-1,0}$. Then by the continuity of the mapping $\phi(n, n-1, \cdot)$, one has $a_{k_j} = \phi(n, n-1, b_{k_j}) \to \phi(n, n-1, \bar{b})$, so $\phi(n, n-1, \bar{b}) = \bar{a}$. This means that $\bar{b} \in B_{n,-1}$, which implies compactness of $B_{n,-1}$. Repeating this procedure gives a sequence of nonempty compact subsets $B_{n,-j}$ for $j \in \mathbb{N}$ such that $B_{n,-j} = \phi\big(n-j, n-j-1, B_{n,-j-1}\big)$. This means that

$$B_{n,0} = \phi\big(n, n-j, B_{n,-j}\big) \quad \text{for all } j \in \mathbb{N}_0 \,.$$

It is proved now that $B_{n+k+1,-k-1} \subset B_{n+k,-k}$ for each $k \in \mathbb{N}$, which yields a nested family of nonempty compact subsets of A_n. To see this, consider the case $k = 1$ and recall that $\phi(n+1, n, B_{n+1,-1}) = A_{n+1}$. Using the two-parameter semi-group property, this implies that

$$\begin{aligned} A_{n+2} = \phi(n+2, n, B_{n+2,-2}) &\subset \phi(n+2, n+1, A_{n+1}) \\ &= \phi\big(n+2, n+1, \phi(n+1, n, B_{n+1,-1})\big) \\ &= \phi(n+2, n, B_{n+1,-1})\,. \end{aligned}$$

The set A_n^∞, defined by

$$A_n^\infty = \bigcap_{k \in \mathbb{N}} B_{n+k,-k} \quad \text{for all } n \in \mathbb{N}$$

is a nonempty compact subset of A_n. It remains to prove that the family of nonempty compact subsets $\mathcal{A}^\infty = (A_n^\infty)_{n\in\mathbb{Z}}$ is ϕ-invariant, i.e., $\phi\big(n, n_0, A_{n_0}^\infty\big) = A_n^\infty$ for all $n \geq n_0$.

($\subset$) Let $\bar{a} \in A_{n_0}^\infty$. Then $\bar{a} \in B_{n_0+k,-k}$ and $\phi(n, n_0, \bar{a}) \in \phi\big(n, n_0, B_{n_0+k,-k}\big)$ for all $k \in \mathbb{N}$. Moreover, for $k \geq n - n_0$ and $\ell = k - n + n_0 \geq 0$, one has

$$B_{n_0+k,-k} = B_{n_0+(\ell+n-n_0),-(\ell+n-n_0)} = B_{n+\ell,-\ell-n+n_0}\,.$$

However,

$$\phi\big(n, n_0, B_{n+\ell,-\ell-n+n_0}\big) = B_{n+\ell,-\ell}$$

by construction, which means that

$$\phi(n, n_0, \bar{a}) \in \phi\big(n, n_0, B_{n_0+k,-k}\big) = \phi\big(n, n_0, B_{n+\ell,-\ell-n+n_0}\big) = B_{n+\ell,-\ell}\,.$$

Hence,

$$\phi(n, n_0, \bar{a}) \in \bigcap_{\ell \in \mathbb{N}} B_{n+\ell,-\ell} = A_n^\infty\,,$$

which implies that $\phi\left(n, n_0, A_{n_0}^\infty\right) \subset A_n^\infty$.

($\supset$) Let $\bar{a} \in A_n^\infty$. Then one obtains $\bar{a} \in B_{n+\ell,-\ell}$ for all $\ell \in \mathbb{N}$. However,

$$\begin{aligned} B_{n+\ell,-\ell} &= B_{n_0+(\ell+n-n_0),-\ell} \\ &= B_{n_0+k,-(k-n+n_0)} \quad \text{for all } k = \ell + n - n_0 \geq n - n_0\,. \end{aligned}$$

Moreover,

$$\phi\big(n, n_0, B_{n_0+k,-k}\big) = B_{n_0+k,-(k-n+n_0)}\,,$$

so $\bar{a} \in \phi\big(n, n_0, B_{n_0+k,-k}\big)$. Hence, there exist $b_k \in B_{n_0+k,-k} \subset A_{n_0}$ such that $\phi(n, n_0, b_k) = \bar{a}$. Note that $b_k \in A_{n_0}$ for all $k \in \mathbb{N}$, which is a compact set, so there exists a convergent subsequence $b_{k_j} \to \bar{b}$ in A_{n_0}. In fact, $\bar{b} \in A_{n_0}^\infty$, since

$$\operatorname{dist}\left(\bar{b}, A_{n_0}^\infty\right) \leq \operatorname{dist}\left(\bar{b}, b_{k_j}\right) + \operatorname{dist}\left(B_{n_0+k_j,-k_j}, A_{n_0}^\infty\right) \to 0 \quad \text{as } j \to \infty\,.$$

It follows by continuity that $\bar{a} = \phi(n, n_0, b_{k_j}) \to \phi(n, n_0, \bar{b})$ as $j \to \infty$, so $\bar{a} = \phi(n, n_0, \bar{b})$, which finally means that $A_n^\infty \subset \phi(n, n_0, A_{n_0}^\infty)$. $\square$

THEOREM 2.22. *Let $\mathcal{A} = (A_t)_{t\in\mathbb{R}}$ be a family of nonempty compact subsets of X which is ϕ-negatively invariant for a continuous time process ϕ, i.e., $A_t \subset \phi(t, t_0, A_{t_0})$ for all $t \geq t_0$. Then there exists a family of nonempty compact subsets $\mathcal{A}^\infty = (A_t^\infty)_{t\in\mathbb{R}}$ of $\mathcal{A}$, which is ϕ-invariant, i.e., $\phi(t, t_0, A_{t_0}^\infty) = A_t^\infty$ for all $t \geq t_0$.*

PROOF. To simplify the notation, the proof will be given only for the special case that $A_t \equiv A$ for all $t \in \mathbb{R}$. First consider the process restricted the dyadic numbers in $\mathbb{R}$. Let $\mathbb{T}_0 = \mathbb{Z}$ and $\mathbb{D}_n = \{j2^{-n} : j = 0, 1, \ldots, 2^n\}$, and define

$$\mathbb{T}_n := \mathbb{Z} + \mathbb{D}_n = \{k + t_j^n : k \in \mathbb{Z} \text{ and } t_j^n \in \mathbb{D}_n\} \quad \text{for all } n \in \mathbb{N}$$

and apply the result of Lemma 2.21 to the discrete time system formed by the restriction $\phi|_{\mathbb{T}_0}$ of the mapping ϕ to the time set $\mathbb{T}_0$. This gives a family $\mathcal{A}^0 = (A_t^0)_{t\in\mathbb{T}_0}$ of nonempty compact subsets of A which is the maximal $\phi|_{\mathbb{T}_0}$-invariant family of subsets of A, i.e., with $\phi(n+1, n, A_n^0) = A_{n+1}^0$ for all $n \in \mathbb{Z}$.
A difficulty is that the $\phi(n + t, n, A_n^0)$ may not be a subset of A for all $t \in (0, 1)$. Therefore, the procedure will be repeated for the discrete time system formed by the restriction $\phi|_{\mathbb{T}_1}$ of the mapping ϕ to the time set $\mathbb{T}_1$, and one obtains a family $\mathcal{A}^1 = (A_t^1)_{t\in\mathbb{T}_1}$ of nonempty compact subsets of A which is the maximal $\phi|_{\mathbb{T}_1}$-invariant family of subsets of A, i.e., with

$$\phi\left(t_{j+1}^1, t_j^1, A_{t_j^1}^1\right) = A_{t_{j+1}^1}^1$$

for every $t_j^1, t_{j+1}^1 \in \mathbb{T}_1$ with $t_{j+1}^1 - t_j^1 = \frac{1}{2}$. By this and the semi-group property, one has

$$\begin{aligned} A_{m+1}^1 &= \phi(m+1, m+\tfrac{1}{2}, A_{m+1/2}^1) \\ &= \phi(m+1, m+\tfrac{1}{2}, \phi(m+\tfrac{1}{2}, m, A_m^1)) \\ &= \phi(m+1, m, A_m^1) \quad \text{for all } m \in \mathbb{Z}, \end{aligned}$$

so $\mathcal{A}^1$ is also a $\phi|_{\mathbb{T}_0}$-invariant family of compact subsets of A. But since $\mathcal{A}^0$ is the maximal $\phi|_{\mathbb{T}_0}$-invariant family of compact subset of A, one has $A_t^1 \subset A_t^0$ for all $t \in \mathbb{T}_0$.
Now repeat this procedure with the discrete time system formed by the restriction $\phi|_{\mathbb{T}_n}$ of the mapping ϕ to the time set $\mathbb{T}_n$ and obtain a family $\mathcal{A}^n = (A_t^n)_{t\in\mathbb{T}_n}$ of nonempty compact subsets of A, which is the maximal $\phi|_{\mathbb{T}_n}$-invariant family of subsets of A. Note that this family is also $\phi|_{\mathbb{T}_{n-1}}$-invariant. Hence, $A_t^n \subset A_t^{n-1}$ for all $t \in \mathbb{T}_{n-1} \cap \mathbb{T}_n = \mathbb{T}_{n-1}$ and $n \in \mathbb{N}$.
Thus for each $t_\ell \in \mathbb{T}_\ell$ for an arbitrary $\ell \in \mathbb{N}$, the subsets $A_{t_\ell}^n$ for $n \geq \ell$ are nonempty, compact and nested. Hence, the set defined by

$$A_{t_\ell}^\infty = \bigcap_{n\geq\ell} A_{t_\ell}^n$$

is a nonempty compact subset of A. In this way, one obtains a family $\mathcal{A}^\infty = (A_{t_d}^\infty)_{t_d\in\bigcup_{\ell\geq0}\mathbb{T}_\ell}$ of nonempty compact subsets of A.
Moreover, by Lemma 2.21, the family $\mathcal{A}^\infty$ is $\phi|_{\mathbb{T}_n}$-invariant for each $n \in \mathbb{N}$, i.e.,

$$\phi\left(t_{j+1}^n, t_j^n, A_{t_j^n}^\infty\right) = A_{t_{j+1}^n}^\infty$$

for every $t_j^n, t_{j+1}^n \in \mathbb{T}_n$ with $t_{j+1}^n - t_j^n = 2^{-n}$. From this and the semi-group property, it follows that $\phi(t_1, t_0, A_{t_0}^\infty) = A_{t_1}^\infty$ for all dyadic numbers $t_0 \leq t_1$ in $\mathbb{R}$. Finally, for non-dyadic t, one defines A_t^∞ by

$$A_t^\infty = \phi\big(t, t_0, A_{t_0}^\infty\big)\,,$$

where $t_0 < t$ is an arbitrary dyadic number (note that this definition is independent of the choice of t_0 by the semi-group property for the dyadic numbers). It follows that A_t^∞ is a nonempty compact subset of A. By continuity of ϕ in its first variable,

$$h\big(A_t^\infty, A_{t_n}^\infty\big) \leq h\big(\phi(t, t_0, A_{t_0}^\infty), \phi(t_n, t_0, A_{t_0}^\infty)\big) \to 0 \quad \text{as} \;\; t_n \to t$$

for dyadic $t_n \geq t_0$ with $t_n < t$.
Finally, extend that definition of $\mathcal{A}^\infty$ by $\mathcal{A}^\infty = (A_t^\infty)_{t\in\mathbb{R}}$. It remains to show that $A_t^\infty = \phi(t, s, A_s^\infty)$ for all $s < t$ in $\mathbb{R}$. From above, the only remaining case to show is for s non-dyadic. The desired result follows from the definition of A_s^∞ and the semi-group property, i.e.,

$$\phi(t, s, A_s^\infty) = \phi\big(t, s, \phi\big(s, t_0, A_{t_0}^\infty\big)\big) = \phi\big(t, t_0, A_{t_0}^\infty\big) = A_t^\infty\,,$$

where $t_0 < s$ is dyadic but otherwise arbitrary. Thus, $\mathcal{A}^\infty$ is ϕ-invariant, which finishes the proof of this theorem. □

Endnotes. The concept of a process is due to Dafermos [**62**] and Hale [**90**]. Skew product flows originated in ergodic theory and were extensively studied in connection with ordinary differential equations by Sell [**217, 218**]. See also Wenxian Shen & Yingfei Yi [**221**] as well as the monographs Carvalho, Langa & Robinson [**35**], Cheban [**38**], Chepyzhov & Vishik [**43**], Fink [**77**] and Kato, Martynyuk & Shestakov [**104**]. Section 1.3 on entire solutions and invariant sets is based on Kloeden & Marín-Rubio [**132**], and see Kloeden & Rodrigues [**139**] for the dynamics of a class of differential equations which are more general than almost periodic. The figures in this chapter were made by van Geene [**231**].

CHAPTER 3

Attractors

Simple generalizations of concepts for autonomous dynamical systems to nonautonomous dynamical systems are not always adequate or appropriate. For instance, it was seen in the last chapter that for nonautonomous dynamical systems, it is often too restrictive to consider the invariance of just a single set and that instead a family of subsets is more appropriate. A similar situation also applies to attractors, which are the most important examples of invariant sets.

Attractors of autonomous dynamical systems are given by ω-limit sets, which are invariant sets. Since the solution of a process φ depends both on initial time t_0 and initial value x_0, ω-limit sets for a process will also depend on both of these two parameters, specifically

$$\omega(t_0, x_0) = \Big\{x \in X : \lim_{n\to\infty} \varphi(t_n, t_0, x_0) = x \text{ for some sequence } t_n \to \infty\Big\} .$$

As in the autonomous case, one can show that $\omega(t_0, x_0)$ is a nonempty compact set when, for example, the forward trajectory $\bigcup_{t\geq t_0} \big\{\varphi(t, t_0, x_0)\big\}$ is precompact. However, unlike its autonomous counterpart, a nonautonomous ω-limit set $\omega(t_0, x_0)$ may not be invariant for the process.

As an example, consider the nonautonomous scalar differential equation

$$\dot{x} = -x + e^{-t} ,$$

which can be solved with the variation of constants formula to give the explicit solution

$$x(t, t_0, x_0) = e^{-(t-t_0)}x_0 + (t - t_0)\, e^{-t} .$$

This implies that

$$\lim_{t\to\infty} x(t, t_0, x_0) = 0 \quad \text{for all } \ (t_0, x_0) \in \mathbb{R} \times \mathbb{R} ,$$

so the nonautonomous ω-limit set is given by

$$\omega(t_0, x_0) = \{0\} \quad \text{for all } \ (t_0, x_0) \in \mathbb{R} \times \mathbb{R} .$$

However, $x(t, t_0, 0) = (t - t_0)\, e^{-t} \neq 0$ for all $t > t_0$, i.e., the ω-limit set here is not invariant in the sense of autonomous systems.

Nonautonomous sets. Let ϕ be a process on a metric space (X, d), and consider a family $\tilde{\mathcal{M}} = (M_t)_{t\in\mathbb{T}}$ of subsets of X. For a more compact and elegant formulation, such families $\tilde{\mathcal{M}}$ will henceforth be viewed equivalently as subsets of the extended phase space $\mathbb{T} \times X$, and the translation is as follows. The family $\tilde{\mathcal{M}}$ induces a subset $\mathcal{M} \subset \mathbb{T} \times X$, defined by

$$\mathcal{M} := \big\{(t, x) : x \in M_t\big\}$$

and, conversely, a subset $\mathcal{M}$ of the extended phase space $\mathbb{T} \times X$ leads to a family $\tilde{\mathcal{M}} = (M_t)_{t\in\mathbb{T}}$ of subsets of X with

$$M_t := \{x \in X : (t,x) \in \mathcal{M}\} \quad \text{for all } t \in \mathbb{T}.$$

The advantage of the new formulation is that $\mathcal{M}$ is a set, which makes the direct use of set-valued operations possible, and the notation becomes easier to read. Such sets $\mathcal{M}$ are called *nonautonomous sets* in the following. It will become clear soon that all interesting objects in the nonautonomous context are nonautonomous sets, i.e., subsets of the *extended* phase space, whereas the main interest focusses on subsets of the phase space in an autonomous setting. The precise definition of a nonautonomous set is given as follows.

Definition 3.1 (Nonautonomous set of a process). Let ϕ be a process on a metric space (X,d). A subset $\mathcal{M}$ of the extended phase space $\mathbb{T} \times X$ is called a *nonautonomous set*, and for each $t \in \mathbb{T}$, the set

$$M_t := \{x \in X : (t,x) \in \mathcal{M}\}$$

is called the *t-fiber of* $\mathcal{M}$. A nonautonomous set $\mathcal{M}$ is said to be *invariant* if $\phi(t,t_0,M_{t_0}) = M_t$ for all $t \geq t_0$. In general, $\mathcal{M}$ is said to have a topological property (such as compactness or closedness) if each fiber of $\mathcal{M}$ has this property.

The notion of a nonautonomous set will also be used in the setting of skew product flows.

Definition 3.2 (Nonautonomous set of a skew product flow). Let (θ,φ) be a skew product flow on a base set P and a metric phase space (X,d). A subset $\mathcal{M}$ of the extended phase space $P \times X$ is called a *nonautonomous set*, and for each $p \in P$, the set

$$M_p := \{x \in X : (p,x) \in \mathcal{M}\}$$

is called *p-fiber of* $\mathcal{M}$. A nonautonomous set $\mathcal{M}$ is said to be *invariant* if $\varphi(t,p,M_p) = M_{\theta_t(p)}$ for all $t \geq 0$ and $p \in P$. In general, $\mathcal{M}$ is said to have a topological property (such as compactness or closedness) if each fiber of $\mathcal{M}$ has this property.

1. Attractors of processes

There are basically two ways to define attraction of a compact and invariant nonautonomous set $\mathcal{A}$ for a process ϕ on a metric space (X,d) with time set $\mathbb{T}$. The first, and perhaps more obvious, corresponds to the attraction in Lyapunov asymptotic stability, is called *forward attraction* and involves a moving target, while the latter, called *pullback attraction*, involves a fixed target set with progressively earlier starting time. In general, these two types of attraction are independent concepts, while for the autonomous case, they are equivalent.

Definition 3.3 (Nonautonomous attractivity). Let ϕ be a process. A nonempty, compact and invariant nonautonomous set $\mathcal{A}$ is said to be

(i) *forward attracting* if

$$\lim_{t\to\infty} \operatorname{dist}\big(\phi(t,t_0,x_0), A_t\big) = 0 \quad \text{for all } x_0 \in X \text{ and } t_0 \in \mathbb{T},$$

(ii) and *pullback attracting* if

$$\lim_{t_0 \to -\infty} \operatorname{dist}\big(\phi(t, t_0, x_0), A_t\big) = 0 \quad \text{for all } x_0 \in X \text{ and } t \in \mathbb{T}\,.$$

Moreover, if the forward attraction in (i) is uniform with respect to $t_0 \in \mathbb{T}$, or equivalently, if the pullback attraction in (ii) is uniform with respect to $t \in \mathbb{T}$, then $\mathcal{A}$ is called *uniformly attracting.*

Figure 3.1 and Figure 3.2 illustrate forward and pullback attraction, respectively, of a nonautonomous set with singleton sets as fibers $A_t = \{\rho(t)\}$, i.e., an entire solution of the process.

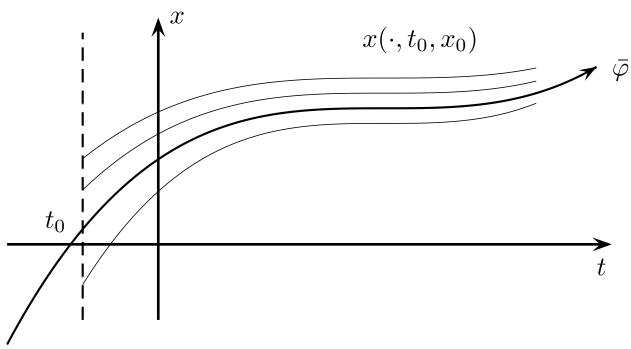

FIGURE 3.1. Forward attraction.

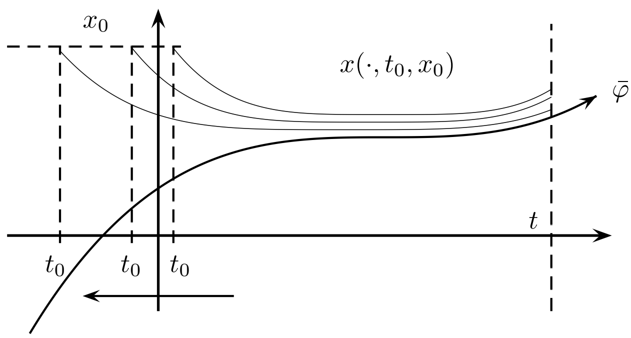

FIGURE 3.2. Pullback attraction.

In an autonomous system, the solutions depend only on the elapsed time $t - t_0$. Moreover, the limit relation $t - t_0 \to \infty$ either holds when $t \to \infty$ with t_0 fixed or as $t_0 \to -\infty$ with t fixed, so pullback and forward convergence are equivalent for an autonomous system.

Two types of nonautonomous attractors for processes are possible, depending which of the above types of attraction is used. It is required that the component subsets of such attractors are compact and that they attract bounded subsets D of initial values in X (rather than just individual points), in the sense that

$$\operatorname{dist}\big(\phi(t, t_0, D), A_t\big) \to 0 \begin{cases} \text{as } t \to \infty \text{ with } t_0 \text{ fixed (forward case),} \\ \text{as } t_0 \to -\infty \text{ with } t \text{ fixed (pullback case).} \end{cases}$$

Compare this with Definition 1.31 for the definition of an autonomous global attractor.

DEFINITION 3.4 (Nonautonomous attractors). Let ϕ be a process. A nonempty and invariant nonautonomous set $\mathcal{A}$ is called

(i) a *forward attractor* if it forward attracts bounded subsets of X,
(ii) a *pullback attractor* if it pullback attracts bounded subsets of X, and
(iii) a *uniform attractor* if it uniformly attracts bounded subsets of X.

Forward and pullback attractors will be discussed in more detail and generality in the context of skew product flows in the following sections. In general, they are independent concepts and one can exist without the other.

EXAMPLE 3.5. The nonautonomous set $\mathbb{R}\times\{0\}$, i.e., the trivial solution, is a forward attractor but not a pullback attractor of the system

$$\dot{x} = -2tx \tag{3.1}$$

with the general solution $x(t, t_0, x_0) = x_0 e^{-(t^2-t_0^2)}$, and a pullback attractor but not a forward attractor of the system

$$\dot{x} = 2tx \tag{3.2}$$

with the general solution $x(t, t_0, x_0) = x_0 e^{t^2-t_0^2}$.

This example demonstrates that forward attraction can be seen as an attraction concept for the *future* of the system, since the coefficient $-2t$ of (3.1) is negative for $t > 0$. On the other hand, pullback attraction means basically attraction for the *past* of the system, see the negativity of $2t$ for $t < 0$ in (3.2). The concept of a uniform attractor, however, is concerned with attractivity for the *entire time*.

EXAMPLE 3.6. Consider the nonautonomous scalar ordinary differential equation

$$\dot{x} = -x + 2\sin t\,. \tag{3.3}$$

If $x_1(t)$ and $x_2(t)$ are any two solutions, then their difference $z(t) = x_1(t) - x_2(t)$ satisfies the homogeneous linear differential equation

$$\dot{z} = -z$$

with the explicit solution $z(t) = z(t_0)e^{-(t-t_0)}$, so

$$\big|x_1(t) - x_2(t)\big| = \big|x_1(t_0) - x_2(t_0)\big|e^{-(t-t_0)} \to 0 \quad \text{as } t \to \infty\,,$$

from which it follows that all solutions converge to each other in time. What do they converge to?

The explicit solution of the nonautonomous differential equation (3.3) with initial value $x(t_0) = x_0$ is

$$\begin{aligned} x(t, t_0, x_0) &= x_0 e^{-(t-t_0)} + 2e^{-t}\int_{t_0}^{t} e^s \sin s\, \mathrm{d}s \\ &= \big(x_0 - (\sin t_0 - \cos t_0)\big)\, e^{-(t-t_0)} + (\sin t - \cos t)\,, \end{aligned}$$

from which it is clear that the forward limit $\lim_{t\to\infty} x(t, t_0, x_0)$ does not exist. On the other hand, the pullback limit does exist for all t and x_0, i.e.,

$$\lim_{t_0\to-\infty} x(t, t_0, x_0) = \sin t - \cos t =: \rho(t)\,,$$

and is independent of x_0, i.e.,

$$\lim_{t_0 \to -\infty} \left| x(t, t_0, x_0) - \rho(t) \right| = 0 \,.$$

Hence, the nonautonomous set $\mathcal{A}$ having the singleton fibers $A_t := \{\rho(t)\}$ is pullback attracting for the solution process.

Moreover, it is easily shown that $\rho(t)$ is a solution of the nonautonomous differential equation (3.3) and since all solutions converge to each other forward in time, the forward convergence

$$\lim_{t \to \infty} \left| x(t, t_0, x_0) - \rho(t) \right| = 0$$

also holds.

Exercise 3.7. The nonautonomous set $\mathcal{A}$ in Example 3.6 is both a pullback and forward attractor of the nonautonomous differential equation (3.3). Find other forward attractors of (3.3) which are not pullback attractors.

This exercise demonstrates that forward attractors can be nonunique, and this is quite typical for forward attractors. For pullback attractors, however, the following uniqueness result can be proved.

Proposition 3.8 (Uniqueness of pullback attractors). *Suppose that a process ϕ has two pullback attractors $\mathcal{A}$ and $\bar{\mathcal{A}}$ such that both $\bigcup_{t \le 0} A_t$ and $\bigcup_{t \le 0} \bar{A}_t$ are bounded. Then $\mathcal{A} = \bar{\mathcal{A}}$.*

Proof. The boundedness of $\bigcup_{t \le 0} A_t$ implies for all $t \in \mathbb{T}$ that

$$\begin{aligned} \operatorname{dist}\left(A_t, \bar{A}_t\right) &= \lim_{t_0 \to -\infty} \operatorname{dist}\left(\phi(t, t_0, A_{t_0}), \bar{A}_t\right) \\ &\le \lim_{t_0 \to -\infty} \operatorname{dist}\left(\phi(t, t_0, \textstyle\bigcup_{\tau \le 0} A_\tau), \bar{A}_t\right) = 0 \,. \end{aligned}$$

Analogously, one shows that $\operatorname{dist}\left(\bar{A}_t, A_t\right) = 0$, which finishes the proof, since both the sets A_t and $\bar{A}_t$ are compact. ☐

Exercise 3.9. Demonstrate by using a concrete example that an analogous statement of Proposition 3.8 for forward attractors is not possible.

Exercise 3.10. An invariant nonautonomous set $\mathcal{A}$ (such as a pullback attractor) consists of entire solutions, see Lemma 2.15. Give an example of a process ϕ which has entire solutions that are not contained in the pullback attractor.

2. Attractors of skew product flows

Let (θ, φ) be a skew product flow on a base space P and a state space X with time set $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, where (P, d_P) and (X, d_X) are metric spaces. Then $\pi = (\theta, \varphi)$ is an autonomous semi-dynamical system on the extended state space $\mathcal{X} := P \times X$. The definition of a global attractor for an autonomous semi-dynamical system π was given in Chapter 1. Specifically, a nonempty compact subset $\mathcal{A}$ of $\mathcal{X}$ which is π-invariant, i.e., which satisfies $\pi(t, \mathcal{A}) = \mathcal{A}$ for all $t \in \mathbb{T}_0^+$ is called a *global attractor* of π if

$$\lim_{t \to \infty} \operatorname{dist}_{\mathcal{X}}\left(\pi(t, D), \mathcal{A}\right) = 0$$

for every nonempty bounded subset D of $\mathcal{A}$. Suppose that P is compact. Then the global attractor $\mathcal{A}$ of π has the form

$$\mathcal{A} = \bigcup_{p \in P} \left\{(p, x) : x \in A_p\right\},$$

where A_p is a nonempty compact subset of X for each $p \in P$, and the π-invariance property $\pi(t, \mathcal{A}) = \mathcal{A}$ for $t \in \mathbb{T}_0^+$ is equivalent to the φ-invariance property $\varphi(t, p, A_p) = A_{\theta_t(p)}$ for all $t \in \mathbb{T}_0^+$ and $p \in P$.

A global attractor of the autonomous system π is a possible candidate for an attractor of the nonautonomous dynamical system described by the skew product flow. A disadvantage of this definition is that the extended state space $\mathfrak{X}$ includes the base space P as a component, which often does not have the same physical significance as the state space X.

Other types of attractors consisting of a family of nonempty compact subsets of the state space X have also been proposed for a skew product flow (θ, φ). These are analogues of the forward and pullback attractors of a process.

Definition 3.11 (Pullback and forward attractor for skew product flows). Let (θ, φ) be a skew product flow. A nonempty, compact and invariant nonautonomous set $\mathcal{A}$ is called a *pullback attractor* of (θ, φ) if the *pullback convergence*

$$\lim_{t \to \infty} \operatorname{dist}_X(\varphi(t, \theta_{-t}(p), D), A_p) = 0$$

holds for every nonempty bounded subset D of X and $p \in P$, and is called a *forward attractor* if the *forward convergence*

$$\lim_{t \to \infty} \operatorname{dist}_X\left(\varphi(t, p, D), A_{\theta_t(p)}\right) = 0$$

holds for every nonempty bounded subset D of X and $p \in P$.

As for processes, the concepts of forward and pullback attractors for skew products are generally independent of each other, and one can exist without the other existing. If the above limit is replaced by $\lim_{t \to \infty} \sup_{p \in P} \operatorname{dist}_X(\cdot, \cdot)$, then the attractors are called *uniform pullback* and *uniform forward* attractors, respectively. If either of the limits is uniform in this sense, then so is the other and the attractor is both a uniform pullback and a uniform forward attractor, which will be called simply a *uniform attractor.*

The relationship between these different kinds of nonautonomous attractors will be discussed in some detail in Section 4 of this chapter.

Exercise 3.12. Formulate and prove the corresponding statement about entire solutions and pullback attractors in Exercise 3.10 for skew product flows.

Example 3.13. Reconsider the nonautonomous scalar ordinary differential equation (3.3), now writing $p(t)$ instead of $\sin t$,

$$\dot{x} = -x + 2p(t),$$

with the initial condition $x(0) = x_0$. In the spirit of skew product flows as introduced in Section 2 of Chapter 2, the general solution of this equation depends on both p and x_0. The initial value problem has thus the explicit solution

$x(t) = x(t, p, x_0)$ given by

$$x(t) = x_0 e^{-t} + 2e^{-t} \int_0^t e^s p(s)\, \mathrm{d}s\,.$$

Introduce shift operators on the space $P = \{p(t + \cdot) : 0 \le t \le 2\pi\}$ defined by $\theta_t(p(\cdot)) = p(t + \cdot)$ and consider the solution corresponding to the driving term $\theta_{-\tau}(p(\cdot)) = p(-\tau + \cdot)$ at time τ, i.e.,

$$\begin{aligned} x\big(\tau, \theta_{-\tau}(p(\cdot)), x_0\big) &= x_0 e^{-\tau} + 2e^{-\tau} \int_0^\tau e^s \theta_{-\tau}(p(s))\, \mathrm{d}s \\ &= x_0 e^{-\tau} + 2e^{-\tau} \int_0^\tau e^s p(s - \tau)\, \mathrm{d}s \\ &= x_0 e^{-\tau} + 2 \int_0^\tau e^{s-\tau} p(s - \tau)\, \mathrm{d}s \\ &= x_0 e^{-\tau} + 2 \int_{-\tau}^0 e^t p(t)\, \mathrm{d}t\,, \end{aligned}$$

where the substitution $t := s - \tau$ has been used. The pullback limit as $\tau \to \infty$ gives

$$\lim_{\tau \to \infty} x\big(\tau, \theta_{-\tau}(p(\cdot)), x_0\big) = \alpha(p(\cdot)) := 2 \int_{-\infty}^0 e^t p(t)\, \mathrm{d}t$$

for x_0 in an arbitrary bounded subset D consists of singleton fibers $A_p = \{\alpha(p(\cdot))\}$, $p \in P$, and corresponds to the entire solution

$$\rho(t) := \sin t - \cos t$$

in the process version of this differential equation in Example 3.6, i.e., $\rho(t) = \alpha\big(\theta_t(p(\cdot))\big)$ for all $t \in \mathbb{R}$.

The pullback attraction in this example is uniform, and the pullback attractor is also a forward attractor, and hence a uniform attractor. Moreover, the autonomous semi-dynamical system $\pi = (\theta, \varphi)$ on the extended state space $P \times \mathbb{R}$ has a global attractor given by

$$\mathcal{A} = \bigcup_{p(\cdot) \in P} \left\{ \big(p(\cdot), \alpha(p(\cdot))\big) \right\}.$$

Remark 3.14. The analysis of this system as a skew product flow is somewhat more complicated and less transparent than its analysis as a process in Example 3.6. This is typical and is why the process formulation will often be used in subsequent examples (whenever it is possible).

Example 3.15. The difference equation in Example 2.11 generates a discrete time skew product flow with cocycle mapping

$$\varphi(n, p, x) = \nu^n x + \sum_{j=0}^{n-1} \nu^{n-1-j} q_j \quad \text{for all } n \in \mathbb{N}\,, \tag{3.4}$$

on the state space $X = \mathbb{R}$. The base space $P = [-1, 1]^{\mathbb{Z}}$ is the space of bi-infinite sequences $p = (q_n)_{n \in \mathbb{N}}$ taking values in $[-1, 1]$ and θ is the left shift operator on this sequence space.

Replacing p by $\theta_{-n}(p)$ in (3.4) implies

$$\varphi\big(n, \theta_{-n}(p), x\big) = \nu^n x + \sum_{j=0}^{n-1} \nu^{n-1-j} q_{-n+j} \quad \text{for all } n \in \mathbb{N},$$

which can be reindexed as

$$\varphi\big(n, \theta_{-n}(p), x\big) = \nu^n x + \sum_{k=-n}^{-1} \nu^{-k-1} q_k \quad \text{for all } n \in \mathbb{N}.$$

Taking pullback convergence gives

$$\lim_{n\to\infty} \varphi\big(n, \theta_{-n}(p), x\big) = \alpha(p) := \sum_{k=-\infty}^{-1} \nu^{-k-1} q_k .$$

The pullback attractor $\mathcal{A}$ thus consists of singleton fibers $A_p = \{\alpha(p)\}$ for $p = (q_n)_{n\in\mathbb{N}} \in P$. Since the pullback convergence here is in fact uniform in $p \in P$, the pullback attractor is also a uniform forward attractor, and hence a uniform attractor. Moreover, the corresponding subset $\mathcal{A}$ of $\mathcal{X}$ formed from $\mathcal{A}$ is also the global attractor of the autonomous semi-dynamical system $\pi = (\theta, \varphi)$ on the extended state space $P \times \mathbb{R}$.

3. Existence of pullback attractors

There are generalizations of Theorem 1.23 on the existence of an attractor for autonomous systems in Chapter 1 to pullback attractors for processes and skew product flows. These are also based on the supposed existence of an absorbing set, which is now absorbing in the pullback sense.

Definition 3.16 (Pullback absorbing set for processes). Let ϕ be a process on a metric space (X, d). A nonempty compact subset B of X is called *pullback absorbing* if for each $t \in \mathbb{T}$ and every bounded subset D of X, there exists a $T = T(t, D) > 0$ such that

$$\phi(t, t_0, D) \subset B \quad \text{for all } t_0 \in \mathbb{T} \text{ with } t_0 \leq t - T .$$

Definition 3.17 (Pullback absorbing set for skew product flows). Let (θ, ϕ) be a skew product flow on a metric space (X, d). A nonempty compact subset B of X is called *pullback absorbing* if for each $p \in P$ and every bounded subset D of X, there exists a $T = T(p, D) > 0$ such that

$$\varphi\big(t, \theta_{-t}(p), D\big) \subset B \quad \text{for all } t \geq T .$$

The existence theorems will be presented here under basic but restricted assumptions, which will then be relaxed and generalized.

3.1. Existence of pullback attractors for processes.

The following theorem is a simple generalization of Theorem 1.23 for attractors of autonomous semi-dynamical systems given in Chapter 1.

Theorem 3.18 (Existence of pullback attractors for processes). *Let ϕ be a process on a complete metric space X with a compact pullback absorbing set B such that*

$$\phi(t, t_0, B) \subset B \quad \textit{for all } t \geq t_0 .$$

Then there exists a pullback attractor $\mathcal{A}$ with fibers in B uniquely determined by

$$A_t = \bigcap_{\tau \geq 0} \overline{\bigcup_{t_0 \leq -\tau} \phi(t, t_0, B)} \quad \textit{for all } t \in \mathbb{T}\,. \tag{3.5}$$

A proof will not be given here since it is identical to that of Lemma 2.20, in which an invariant family of subsets contained in a positively invariant family was constructed, once the dynamics has entered the positively invariant pullback absorbing set. The proof is also similar to the proof of the corresponding theorem below, Theorem 3.20, for skew product flows.

The formula (3.5) is a kind of a nonautonomous ω-limit set of the set B. As seen in the introduction of this chapter, a naive definition of a nonautonomous ω-limit set leads to a set which is not positively invariant. However, the pullback construction used in (3.5) gives an invariant set and can be regarded as a proper version of a nonautonomous ω-limit set.

EXAMPLE 3.19. Consider a nonautonomous dynamical system in $\mathbb{R}^d$ given by

$$\dot{x} = f(t, x)\,, \tag{3.6}$$

where f is continuously differentiable and satisfies the uniform dissipative condition

$$\langle x, f(t, x)\rangle \leq K - L\|x\|^2 \quad \text{for all } x \in \mathbb{R}^d \text{ and } t \in \mathbb{R} \tag{3.7}$$

with positive constants K and L. These assumptions ensure that the differential equation (3.6) generates a process.

Moreover, any solution $x(t)$ of (3.6) satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 &= 2\langle x(t), \dot{x}(t)\rangle \\ &= 2\langle x(t), f(t, x(t))\rangle \leq 2K - 2L\|x(t)\|^2\,,\end{aligned}$$

from which, on integrating, it follows that

$$\|x(t)\|^2 \leq \|x(t_0)\|^2 e^{-2L(t-t_0)} + \frac{K}{L}\left(1 - e^{-2L(t-t_0)}\right)\,.$$

Suppose that for a bounded subset D of $\mathbb{R}^d$ with $\|D\| := \sup_{d\in D}\|d\| > 1$, one has $x(t_0) \in D$, and define

$$T := t_0 + \frac{1}{2L}\ln\left(L\|D\|^2\right)\,.$$

Then

$$\|x(t)\|^2 \leq \frac{1}{L} + \frac{K}{L} = \frac{K+1}{L}$$

for $x(t_0) \in D$ and $t_0 \leq t - T$. Thus, the closed ball

$$\overline{B_{\sqrt{(K+1)/L}}(0)} = \left\{x \in \mathbb{R}^d : \|x\|^2 \leq (K+1)/L\right\}$$

is pullback absorbing and positively invariant.

From Theorem 3.18, it follows that the process generated by the differential equation (3.6) has a pullback attractor in $\mathbb{R}^d$ with components subsets $A_t \subset B$.

3.2. Existence of pullback attractors for skew product flows. The counterpart of Theorem 3.18 for skew product flows is the first part of the following theorem. The second part provides some information about a form of forwards convergence of the cocycle mapping, which is different from that in the definition of a forward attractor.

THEOREM 3.20 (Existence of pullback attractors). *Let (θ, φ) be a skew product flow on a complete metric space X with a compact pullback absorbing set B such that*

$$\varphi(t, p, B) \subset B \quad \textit{for all } t \geq 0 \textit{ and } p \in P\,. \tag{3.8}$$

Then there exists a unique pullback attractor $\mathcal{A}$ with fibers in B uniquely determined by

$$A_p = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}(p), B)} \quad \textit{for all } p \in P\,. \tag{3.9}$$

If, in addition, (P, d_P) is a compact metric space, then

$$\lim_{t\to\infty} \sup_{p \in P} \operatorname{dist}\big(\varphi(t, p, D), A(P)\big) = 0 \tag{3.10}$$

for any bounded subset D of X, where $A(P) := \overline{\bigcup_{p \in P} A_p} \subset B$.

PROOF. The proof generalizes the proof of Theorem 1.23 for autonomous semi-dynamical systems. It will be divided into two parts, where in the first part, the existence of a pullback attractor is proved, and in the second part, the assertion concerning the compact base set P is treated.

Part 1. Let B be a pullback absorbing set satisfying (3.8), and let A_p be defined as in (3.9) for this absorbing set B.

(i) Firstly, it will be shown for any $p \in P$ that

$$\lim_{t\to\infty} \operatorname{dist}\big(\varphi(t, \theta_{-t}(p), B), A_p\big) = 0\,. \tag{3.11}$$

Assume to the contrary that there exist sequences $t_j \to \infty$ and $x_j \in \varphi\big(t_j, \theta_{-t_j}(p), B\big) \subset B$ such that $\operatorname{dist}(x_j, A_p) > \varepsilon$ for all $j \in \mathbb{N}$. The set $\{x_j : j \in \mathbb{N}\} \subset B$ is relatively compact, so there is a point $x_0 \in B$ and an index subsequence $j' \to \infty$ such that $x_{j'} \to x_0$. Now $x_{j'} \in \bigcup_{t\geq\tau} \varphi\big(t, \theta_{-t}(p), B\big)$ for all $\tau \geq 0$ with $t_{j'} \geq \tau$, which implies that

$$x_0 \in \overline{\bigcup_{t \geq \tau} \varphi\big(t, \theta_{-t}(p), B\big)} \quad \text{for all } \tau \geq 0\,.$$

Hence, $x_0 \in A_p$, and this contradiction proves the original assertion (3.11).

(ii) By (3.11), for every $\varepsilon > 0$ and $p \in P$, there exists a $T = T(\varepsilon, p) \geq 0$ such that

$$\operatorname{dist}\big(\varphi(T, \theta_{-T}(p), B), A_p\big) < \varepsilon\,.$$

Let D be a bounded subset of X. The fact that B is an absorbing set implies that $\varphi\big(t, \theta_{-t-T}(p), D\big) \subset B$ for all sufficiently large t. Hence, by the cocycle property, one has

$$\begin{aligned}\varphi\big(t+T, \theta_{-t-T}(p), D\big) &= \varphi\big(T, \theta_{-T}(p), \varphi(t, \theta_{-t-T}(p), D)\big)\\ &\subset \varphi\big(T, \theta_{-T}(p), B\big)\,.\end{aligned}$$

(iii) The φ-invariance of the nonautonomous set $\mathcal{A}$ will now be shown. By (3.8), the set $F_\tau(p) := \bigcup_{s\geq\tau} \varphi\big(s, \theta_{-s}(p), B\big)$ is contained in B for every $\tau \geq 0$, and by definition, $A_{\theta_{-t}(p)} = \bigcap_{\tau\geq 0} \overline{F_\tau\,(\theta_{-t}(p))}$. Firstly, it will be shown that

$$\varphi\Big(t, \theta_{-t}(p), \Big(\bigcap_{\tau\geq 0} \overline{F_\tau(\theta_{-t}(p))}\Big)\Big) = \bigcap_{\tau\geq 0} \varphi\big(t, \theta_{-t}(p), \overline{F_\tau(\theta_{-t}(p))}\big)\,, \tag{3.12}$$

and one sees directly that "$\subset$" holds. To prove "$\supset$", let x be contained in the set on the right side. Then for any $\tau \geq 0$, there exists an $x^\tau \in \overline{F_\tau(\theta_{-t}(p))} \subset B$ such that $x = \varphi\big(t, \theta_{-t}(p), x^\tau\big)$. Since the family $\overline{F_\tau(\theta_{-t}(p))}$ is monotonically decreasing with increasing τ, the set $\{x^\tau : \tau \geq 0\}$ has a limit point $\hat{x} \in \bigcap_{\tau\geq 0} \overline{F_\tau(\theta_{-t}(p))}$. By the continuity of $\varphi\big(t, \theta_{-t}(p), \cdot\big)$, it follows that $x = \varphi\big(t, \theta_{-t}(p), \hat{x}\big)$, and thus, $x \in \varphi\big(t, \theta_{-t}(p), \bigcap_{\tau\geq 0} \overline{F_\tau(\theta_{-t}(p))}\big) = \varphi\big(t, \theta_{-t}(p), A_{\theta_{-t}(p)}\big)$.
Hence, equation (3.12), the compactness of $\overline{F_\tau(\theta_{-t}(p))}$ and the continuity of $\varphi\big(t, \theta_{-t}(p), \cdot\big)$ imply that

$$\begin{aligned}
\varphi\big(t, \theta_{-t}(p), A_{\theta_{-t}(p)}\big) &= \bigcap_{\tau\geq 0} \varphi\big(t, \theta_{-t}(p), \overline{F_\tau(\theta_{-t}(p))}\big) \\
&\supset \bigcap_{\tau\geq 0} \overline{\varphi\big(t, \theta_{-t}(p), F_\tau(\theta_{-t}(p))\big)} \\
&= \bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} \varphi\big(t, \theta_{-t}(p), \varphi(s, \theta_{-t-s}(p), B)\big)} \\
&= \bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} \varphi\big(t+s, \theta_{-t-s}(p), B\big)} \\
&= \bigcap_{\tau\geq t} \overline{\bigcup_{s\geq\tau} \varphi\big(s, \theta_{-s}(p), B\big)} \supset A_p\,,
\end{aligned}$$

which means that

$$A_p \subset \varphi\left(t, \theta_{-t}(p), A_{\theta_{-t}(p)}\right) \quad \text{for all } t \geq 0 \text{ and } p \in P\,. \tag{3.13}$$

Replacing p by $\theta_{-\tau}(p)$ in (3.13) and using the cocycle property gives

$$\begin{aligned}
\varphi\big(\tau, \theta_{-\tau}(p), A_{\theta_{-\tau}(p)}\big) &\subset \varphi\big(\tau, \theta_{-\tau}(p), \varphi\big(t, \theta_{-\tau-t}(p), A_{\theta_{-\tau-t}(p)}\big)\big) \\
&= \varphi\big(t, \theta_{-t}(p), \varphi\big(\tau, \theta_{-\tau-t}(p), A_{\theta_{-\tau-t}(p)}\big)\big) \\
&\subset \varphi\big(t, \theta_{-t}(p), \varphi(\tau, \theta_{-\tau-t}(p), B)\big) \\
&\subset \varphi\big(t, \theta_{-t}(p), B\big) \subset U_\varepsilon(A_p)
\end{aligned}$$

for all ε-neighborhoods $U_\varepsilon(A_p)$ of A_p, $\varepsilon > 0$, provided that $t = t(\varepsilon)$ is sufficiently large. Hence,

$$\varphi\big(\tau, \theta_{-\tau}(p), A_{\theta_{-\tau}(p)}\big) \subset A_p \quad \text{for all } \tau \geq 0 \text{ and } p \in P\,.$$

With τ replaced by t, this yields with (3.13) the φ-invariance of the family $(A_p)_{p\in P}$.
(iv) It remains to observe that the sets A_p, $p \in P$, are uniformly bounded, because they are subsets of a common compact set B for all $p \in P$. They thus form a pullback attractor, the uniqueness of which follows by Proposition 3.8.
Part 2. Suppose now that the metric space (P, d_P) is compact, and assume to the contrary that the convergence (3.10) does not hold. Then there exist an $\varepsilon > 0$ and sequences $t_n \to \infty$, $\hat{p}_n \in P$ and $x_n \in B$ such that

$$\operatorname{dist}\big(\varphi(t_n, \hat{p}_n, x_n), A(P)\big) > \varepsilon\,. \tag{3.14}$$

Set $p_n = \theta_{t_n}(\hat{p}_n)$. By the compactness of P, there exists a convergent subsequence $p_{n'} \to p_0 \in P$. Because of the pullback attraction, there exists a $\tau > 0$ such that

$$\operatorname{dist}\left(\varphi(\tau, \theta_{-\tau}(p_0), B), A_{p_0}\right) < \frac{\varepsilon}{2}\,.$$

The cocycle property gives

$$\varphi\big(t_n, \theta_{-t_n}(p_n), x_n\big) = \varphi\big(\tau, \theta_{-\tau}(p_n), \varphi\big(t_n - \tau, \theta_{-t_n}(p_n), x_n\big)\big)$$

for any $t_n > \tau$. Now, by the positive invariance of B, it follows that

$$\varphi\big(t_n - \tau, \theta_{-t_n}(p_n), x_n\big) \subset B\,,$$

and since B is compact, there is also a further index subsequence n'' of n' (depending on τ) such that

$$s_{n''} := \varphi(t_{n''} - \tau, \theta_{-t_{n''}}(p_{n''}), x_{n''}) \to s_0 \in B.$$

The continuity of the skew product flow implies

$$\left\|\varphi\big(\tau, \theta_{-\tau}(p_{n''}), s_{n''}\big) - \varphi\big(\tau, \theta_{-\tau}(p_0), s_0\big)\right\| < \frac{\varepsilon}{2} \quad \text{when } n'' > n(\varepsilon)\,.$$

Therefore,

$$\begin{aligned}\varepsilon > \operatorname{dist}\left(\varphi\big(t_{n''}, \theta_{-t_{n''}}(p), x_{n''}\big), A_{p_0}\right) &= \operatorname{dist}\left(\varphi(t_{n''}, \hat{p}_{n''}, x_{n''}), A_{p_0}\right)\\ &\geq \operatorname{dist}\left(\varphi(t_{n''}, \hat{p}_{n''}, x_{n''}), A(P)\right),\end{aligned}$$

which contradicts (3.14), and thus, the asserted convergence (3.10) must be true. □

3.3. A continuous time example. Consider a nonautonomous dynamical system in $\mathbb{R}^d$ given by

$$\dot{x} = f(p, x) \tag{3.15}$$

with the driving system θ on a compact metric space (P, d_P). Suppose that f is regular enough to ensure that the differential equation (3.15) generates a skew product flow.

In addition, suppose that f satisfies the uniform dissipative condition

$$\big\langle x, f(p, x)\big\rangle \leq K - L\|x\|^2 \quad \text{for all } p \in P \text{ and } x \in \mathbb{R}^d \tag{3.16}$$

with positive constants K and L. Then, similarly to Example 3.19, a solution $x(t)$ satisfies the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|^2 \leq K - L\|x(t)\|^2\,,$$

which implies that the closed ball

$$B := B\big[0, \sqrt{(K+1)/L}\big] := \big\{x \in \mathbb{R}^d : \|x\|^2 \leq (K+1)/L\big\}$$

is pullback absorbing and positively invariant. From Theorem 3.20, it follows that the skew product flow has a pullback attractor in $\mathbb{R}^d$ with components subsets $A_p \subset B$, $p \in P$.

Suppose instead that the vector field f satisfies the uniform one-sided dissipative Lipschitz conditions

$$\big\langle x_1 - x_2, f(p, x_1) - f(p, x_2)\big\rangle \leq -L\|x_1 - x_2\|^2 \tag{3.17}$$

for all $p \in P$ and $x_1, x_2 \in \mathbb{R}^d$ with some constant $L > 0$. Then f satisfies the uniform dissipative condition (3.16) with constants

$$K' = \frac{2}{L} \sup_{p\in P} \|f(0)\| \quad \text{and} \quad L' = \frac{L}{2},$$

and the closed ball $B' := B_d\big(0, \sqrt{(K'+1)/L'}\big)$ is pullback absorbing and positively invariant. Thus, the skew product flow has a pullback attractor with component subsets A_p in this ball.

In fact, the fibers of the pullback attractor are singleton sets. The proof uses the fact that due to the uniform one-sided dissipative Lipschitz condition (3.17), the system satisfies

$$\|x_1(t) - x_2(t)\| \le e^{-Lt}\|x_{0,1} - x_{0,2}\| \tag{3.18}$$

for any pair of solutions with the same initial value $p \in P$ of the driving system. This follows from

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\big\|x_1(t) - x_2(t)\big\|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\big\langle x_1(t) - x_2(t), x_1(t) - x_2(t)\big\rangle \\
&= 2\big\langle x_1(t) - x_2(t), \dot{x}_1(t) - \dot{x}_2(t)\big\rangle \\
&= 2\big\langle x_1(t) - x_2(t), f\big(\theta_t p, x_1(t)\big) - f\big(\theta_t p, x_2(t)\big)\big\rangle \\
&\le -2L\big\|x_1(t) - x_2(t)\big\|^2,
\end{aligned}$$

which is integrated to give

$$\|x_1(t) - x_2(t)\|^2 \le e^{-2Lt}\|x_{0,1} - x_{0,2}\|^2.$$

Taking square roots yields the desired result.

THEOREM 3.21. *The pullback attractor $\mathcal{A}$ of the skew product flow (θ, φ) generated by the differential equation* (3.15) *consists of singleton fibers $A_p = \{a_p\}$ for each $p \in P$ when the vector field f satisfies the uniform one-sided dissipative Lipschitz condition* (3.17). *Moreover, $t \mapsto a_{\theta_t(p)}$, $t \in \mathbb{R}$, is an entire solution of* (3.15) *for each $p \in P$.*

PROOF. Since $A_p \subset B'$ for all $p \in P$, it follows that $\|A_p\| := \max_{a\in A_p} \|a\| \le R := (K'+1)/L'$ for each $p \in P$. Now consider a fixed $p \in P$, and suppose that there exists an $\varepsilon_0 > 0$ and points a_1, $a_2 \in A_p$ such that $\|a_1 - a_2\| = \varepsilon_0$. Moreover, choose $T > 0$ such that $2Re^{-LT} = \varepsilon_0$. The φ-invariance of the pullback attractor gives $\varphi\big(T, \theta_{-T}(p), A_{\theta_{-T}(p)}\big) = A_p$, which means that there exist $a_1', a_2' \in A_{\theta_{-T}p}$ such that

$$\varphi\big(T, \theta_{-T}(p), a_1'\big) = a_1 \quad \text{and} \quad \varphi\big(T, \theta_{-T}(p), a_2'\big) = a_2\,.$$

Then, from the inequality in (3.18), it follows that

$$\begin{aligned}
0 < \varepsilon_0 = \|a_1 - a_2\| &= \big\|\varphi\big(T, \theta_{-T}(p), a_1'\big) - \varphi\big(T, \theta_{-T}(p), a_2'\big)\big\| \\
&\le e^{-LT}\|a_1' - a_2'\| \le Re^{-LT} = \frac{1}{2}\varepsilon_0\,,
\end{aligned}$$

which is not possible. Hence, $a_1 = a_2$.

Finally, from the φ-invariance of the pullback attractor, $\varphi(t, p, a_p) = a_{\theta_t(p)}$ for all $t \in \mathbb{R}$ and $p \in P$, so the singleton sets forming the pullback attractor define an entire solution of the system. It follows from inequality (3.18) that this entire solution also forward attracts all other solutions with the same initial value of the driving system, so the pullback attractor is also a forward attractor. (There will be more than one such entire solution when P is not a minimal subset for the autonomous dynamical system θ). □

The above theorem generalizes the following autonomous result from STUART & HUMPHRIES [**226**].

COROLLARY 3.22. *An autonomous differential equation with a vector field f which satisfies a one-sided dissipative Lipschitz condition such as* (3.17) *(i.e., without the p-variable) has a unique globally asymptotically stable equilibrium point.*

3.4. A discrete time example. Consider the parametrically dependent difference equation

$$x_{n+1} = f(x_n, q_n)\,.$$

with the continuous mapping $f : \mathbb{R} \times \left[\frac{1}{2}, 1\right] \to \mathbb{R}$ given by

$$f(x, q) := f_q(x) := \frac{|x| + q^2}{1 + q}\,.$$

Let $P = \left[\frac{1}{2}, 1\right]^{\mathbb{Z}}$ be the space of bi-infinite sequences $p = (q_n)_{n\in\mathbb{Z}}$ taking values in $\left[\frac{1}{2}, 1\right]$, which is a compact metric space with the metric

$$d(p, p') = \sum_{n=-\infty}^{\infty} 2^{-|n|} \left|q_n - q_n'\right|,$$

and let $\{\theta_n : n \in \mathbb{Z}\}$ be the group generated by the left shift operator θ on this sequence space (analogously to Examples 2.10 and 2.11). Then the family of mappings $\varphi(n, \cdot, \cdot)$ defined by

$$\varphi(0, p, x) := \{x\} \quad \text{and} \quad \varphi(n, p, x) := f_{q_{n-1}} \circ \cdots \circ f_{q_0}(x)$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $p = (q_n)_{n\in\mathbb{Z}} \in P$ is a discrete time skew product flow on $\mathbb{R}$. Moreover, the mappings $p \mapsto \theta_n(p)$ and $(p, x) \mapsto \varphi(n, p, x)$ are continuous for each $n \in \mathbb{N}$.

Since

$$|f(x, q)| \le \frac{1}{1+q}|x| + \frac{q^2}{1+q} \le \frac{2}{3}|x| + \frac{2}{3}\,,$$

this discrete time skew product flow has an absorbing set $B = [-2, 2]$, which is positively invariant. Theorem 3.20 applies here which means that there exists a pullback attractor. Moreover, the sets A_p of the pullback attractor are singleton sets, since

$$\bigl|f(x, q) - f(y, q)\bigr| \le \frac{1}{1+q}|x - y| \le \frac{2}{3}|x - y|\,,$$

and it follows that solutions with the same p but different initial values converge to each other uniformly in the forward sense, so the pullback attractor is also a forward attractor. The sets $A_p = \{\alpha(p)\}$ are given by

$$\alpha(p) := \sum_{n=1}^{\infty} \frac{q_{-n}^2}{(1+q_{-1})(1+q_{-2}) \cdot \ldots \cdot (1+q_{-n})}\,,$$

where $p = (q_n)_{n\in\mathbb{Z}} \in \left[\frac{1}{2}, 1\right]^{\mathbb{Z}}$.

3.5. Pullback attractors for absorbing families and attraction universes. To take into account nonuniformities that are ubiquitous in nonautonomous dynamical systems, greater generality can be attained in the definition of a pullback attractor by considering arbitrary nonautonomous sets $\mathcal{B}$ and $\mathcal{D}$ instead just a single compact absorbing set B and single attracted bounded set D. This allows local as well as global attraction to be handled at the same time. The skew product flows (θ, φ) in this subsection are on a metric state space (X, d_X) with a metric base space (P, d_P) and a time set $\mathbb{T}$.

Definition 3.23 (Attraction universe). An *attraction universe* $\mathfrak{D}$ of a skew product flow (θ, φ) is a collection of bounded nonautonomous sets $\mathcal{D}$, which is closed in the sense that if $\emptyset \subsetneq \mathcal{D}' \subseteq \mathcal{D}$ for some $\mathcal{D}, \mathcal{D}' \in \mathfrak{D}$, then $\mathcal{D}' \in \mathfrak{D}$.

The definitions of pullback convergence and pullback attractor need to be extended accordingly.

Definition 3.24 (Pullback attractor with respect to an attraction universe). Let (θ, φ) be a skew product flow on $P \times X$. A nonempty, compact and invariant nonautonomous set $\mathcal{A}$ is called *pullback attractor* with respect to an attraction universe $\mathfrak{D}$ if the *pullback convergence*

$$\lim_{t\to\infty} \operatorname{dist}\big(\varphi\big(t, \theta_{-t}(p), D_{\theta_{-t}(p)}\big), A_p\big) = 0$$

holds for all $p \in P$ and $\mathcal{D} \in \mathfrak{D}$.

Exercise 3.25. Show that a pullback attractor is unique within a given attraction universe $\mathfrak{D}$.

The pullback absorbing property now depends on the attraction universe $\mathcal{D}$ under consideration.

Definition 3.26 (Pullback absorbing set with respect to an attraction universe). Let $\mathfrak{D}$ be an attraction universe of a skew product flow (θ, φ) on $P \times X$. A nonempty and compact nonautonomous set $\mathcal{B} \in \mathfrak{D}$ is called *pullback absorbing* with respect to $\mathfrak{D}$ if for each $\mathcal{D} \in \mathfrak{D}$ and $p \in P$, there exists a $T = T(p, D) > 0$ such that

$$\varphi\big(t, \theta_{-t}(p), D_{\theta_{-t}(p)}\big) \subset B_p \quad \text{for all } t \geq T\,.$$

Theorem 3.20 on the existence of a pullback attractor assuming that of a pullback absorbing set generalizes to attraction universes and pullback absorbing families.

Theorem 3.27 (Existence of a pullback attractor with respect to an attraction universe). *Let (θ, φ) be a skew product flow on $P \times X$, and suppose that the compact nonautonomous set $\mathcal{B}$ is pullback absorbing with respect to an attraction universe $\mathfrak{D}$. Then (θ, φ) has a pullback attractor $\mathcal{A}$ with respect to $\mathfrak{D}$, where the fibers A_p are defined for each $p \in P$ by*

$$A_p = \bigcap_{s>0} \overline{\bigcup_{t>s} \varphi\big(t, \theta_{-t}(p), B_{\theta_{-t}(p)}\big)}\,. \tag{3.19}$$

The proof is a direct modification of that of Theorem 3.20.

Remark 3.28. The assumption that the absorbing sets in Theorem 3.20 and Theorem 3.27 are compact is no restriction in a state space such as $\mathbb{R}^d$, which is locally compact, and thus, closed and bounded subsets are equivalently compact. This is not true for a general state space. In particular, for infinite-dimensional spaces, compact subsets are "thin" and it is much easier to determine an absorbing property for a closed and bounded subset, such as a unit ball, rather than a compact subset. Counterparts of Theorem 3.20 and Theorem 3.27 then hold, if the cocycle mapping is assumed to be compact, i.e., the mapping $\varphi(t, p, \cdot) : X \to X$ maps bounded subsets into precompact subsets for all $t > 0$ and $p \in P$, or more generally, asymptotically compact. These generalizations will be considered in Chapter 12 on infinite-dimensional dynamical systems, i.e., with an infinite-dimensional state space X.

EXERCISE 3.29. Formulate corresponding definitions of an attraction universe, a pullback absorbing family and a pullback attractor for a process.

4. Relationship between nonautonomous attractors

Simple examples show that a pullback attractor need not be a forward attractor and vice versa. However, Example 3.5 involves processes, and the associated skew product flows as defined in Theorem 2.4 of Chapter 2, when considered as autonomous semi-dynamical systems, do not have global attractors since the base space P of the driving system is the noncompact set $\mathbb{R}$. Much more can be said, however, about the relationships between the various kinds of nonautonomous attractors when the skew product flow has a compact base space P.

In this section, let (θ, φ) be a skew product flow on metric state space (X, d_X) with compact metric base space (P, d_P) and let the metric on the extended phase space $\mathcal{X} = P \times X$ be defined as

$$d_{\mathcal{X}}\big((p_1, x_1), (p_2, x_2)\big) = d_P(p_1, p_2) + d_X(x_1, x_2).$$

PROPOSITION 3.30. *Suppose that $\mathcal{A}$ is a uniform attractor (i.e., uniform in both the forward and pullback senses) of a skew product flow (θ, φ) and that $\bigcup_{p \in P} A_p$ is precompact in X. Then $\mathcal{A} := \bigcup_{p \in P} \{p\} \times A_p$ is the global attractor of the autonomous semi-dynamical system π associated with a skew product flow (θ, φ).*

PROOF. The π-invariance of $\mathcal{A}$ follows from the φ-invariance of $\mathcal{A}$, and the θ-invariance of P via

$$\begin{aligned} &\pi(t, \mathcal{A}) \\ &= \bigcup_{p \in P} \{\theta_t(p)\} \times \varphi(t, p, A_p) = \bigcup_{p \in P} \{\theta_t(p)\} \times A_{\theta_t(p)} = \bigcup_{q \in P} \{q\} \times A_q = \mathcal{A}\,. \end{aligned}$$

Since $\mathcal{A}$ is also a pullback attractor and $\bigcup_{p \in P} A_p$ is precompact in X (and P is compact too) by Theorem 3.34, the set-valued mapping $p \mapsto A_p$ is upper semi-continuous, which means that $p \mapsto F(p) := \{p\} \times A_p$ is also upper semi-continuous. Hence, $F(P) = \mathcal{A}$ is a compact subset of $\mathcal{X}$, cf. Example 2.16. Moreover, the definition of the metric $d_{\mathcal{X}}$ on $\mathcal{X}$ implies that

$$\begin{aligned} d_{\mathcal{X}}\big(\pi(t,(p,x)),\mathcal{A}\big) &= d_{\mathcal{X}}\big((\theta_t(p),\varphi(t,p,x)),\mathcal{A}\big) \\ &\le d_{\mathcal{X}}\big((\theta_t(p),\varphi(t,p,x)),\{\theta_t(p)\}\times A_{\theta_t(p)}\big) \\ &= d_P\big(\theta_t(p),\theta_t(p)\big) + \mathrm{dist}_X\left(\varphi(t,p,x),A_{\theta_t(p)}\right) \\ &= \mathrm{dist}_X\left(\varphi(t,p,x),A_{\theta_t(p)}\right), \end{aligned}$$

where $\pi(t,(p,x)) = \big(\theta_t(p),\varphi(t,p,x)\big)$. The desired attraction to $\mathcal{A}$ with respect to π then follows from the forward attraction of $\mathcal{A}$ with respect to φ. □

Without uniform attraction as in Proposition 3.30 a pullback attractor need not give a global attractor, see Example 3.33 below, but the following result does hold.

PROPOSITION 3.31. *If $\mathcal{A}$ is a pullback attractor for a skew product flow (θ,φ) and $\bigcup_{p\in P} A_p$ is precompact in X, then $\mathcal{A} := \bigcup_{p\in P}\{p\}\times A_p$ is the maximal invariant compact set of the associated autonomous semi-dynamical system π.*

PROOF. The compactness and π-invariance of $\mathcal{A}$ are proved in the same way as in first part of the proof of Proposition 3.30. To prove that the compact invariant set $\mathcal{A}$ is maximal, let $\mathcal{C}$ be any other compact invariant set of the autonomous semi-dynamical system π associated with the skew product flow. Then $\mathcal{A}$ is a compact and invariant nonautonomous set, and by pullback attraction, one has

$$\begin{aligned} \mathrm{dist}_X(C_p,A_p) &= \mathrm{dist}_X\left(\varphi\big(t,\theta_{-t}(p),C_{\theta_{-t}(p)}\big),A_p\right) \\ &\le \mathrm{dist}_X\left(\varphi\big(t,\theta_{-t}(p),K\big),A_p\right) \to 0 \quad \text{as } t\to\infty, \end{aligned}$$

where $K := \overline{\bigcup_{p\in P} C_p}$ is compact. Hence, $C_p \subseteq A_p$ for every $p\in P$, i.e., $\mathcal{C}\subseteq\mathcal{A}$, which finally means that $\mathcal{A}$ is a maximal π-invariant set. □

The set $\mathcal{A}$ here need not be the global attractor of π. In the opposite direction, the global attractor of the associated autonomous semi-dynamical system always forms a pullback attractor of the skew product flow.

PROPOSITION 3.32. *If $\mathcal{A}$ is the global attractor of the associated autonomous semi-dynamical system π, then $\mathcal{A}$ is a pullback attractor for the skew product flow (θ,φ).*

PROOF. The set $K = \bigcup_{p\in P} A_p$ is compact by the compactness of $\mathcal{A}$. Moreover, $\mathcal{A}\subset P\times K$, which is a compact set. Now

$$\begin{aligned} \mathrm{dist}_X\left(\varphi(t,p,x),K\right) &= \mathrm{dist}_P\left(\theta_t(p),P\right) + \mathrm{dist}_X\left(\varphi(t,p,x),K\right) \\ &= \mathrm{dist}_{\mathcal{X}}\left((\theta_t(p),\varphi(t,p,x)),P\times K\right) \\ &\le \mathrm{dist}_{\mathcal{X}}\left(\pi(t,(p,x)),P\times K\right) \\ &\le \mathrm{dist}_{\mathcal{X}}\left(\pi(t,P\times D),\mathcal{A}\right) \to 0 \quad \text{as } t\to\infty \end{aligned}$$

for all $(p,x)\in P\times D$ and every arbitrary compact subset D of X, since $\mathcal{A}$ is the global attractor of π. Hence, replacing p by $\theta_{-t}(p)$ implies

$$\lim_{t\to\infty} \mathrm{dist}_X\left(\varphi(t,\theta_{-t}(p),D),K\right) = 0.$$

Then the system is pullback asymptotic compact (see Definition 12.10) and by Theorem 12.14 in Chapter 12, this is a sufficient condition for the existence of a pullback attractor $\mathcal{A}'$ with $\bigcup_{p\in P} A'_p \subset K$. From Proposition 3.31, $\mathcal{A}'$ is the maximal π-invariant subset of $\mathcal{X}$, but so is the global attractor $\mathcal{A}$, which means that $\mathcal{A}' = \mathcal{A}$. Thus, $\mathcal{A}$ is a pullback attractor of the skew product flow (θ, φ). □

The following counterexample is taken from Cheban, Kloeden & Schmalfuss [**41**].

Example 3.33. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) := -\left(\frac{1+t}{1+t^2}\right)^2 \quad \text{for all } t \in \mathbb{R}\,,$$

and let θ be the autonomous dynamical system on $P = H(f)$, the hull of f in $C(\mathbb{R},\mathbb{R})$, formed by the shift operators $\theta_t f(\cdot) := f(t+\cdot)$ for $t \in \mathbb{R}$. Note that P is compact with respect to the supremum norm on $C(\mathbb{R},\mathbb{R})$. Moreover,

$$P = H(f) = \bigcup_{h\in\mathbb{R}} \{f(\cdot + h)\} \cup \{0\}.$$

Finally, let E be the *evaluation functional* on $C(\mathbb{R},\mathbb{R})$, i.e., $E(p) := p(0)$ for $p \in C(\mathbb{R},\mathbb{R})$. From a straightforward calculation, it follows that the functional

$$\gamma(p) = -\int_0^\infty e^{-\tau} E(\theta_\tau(p))\,\mathrm{d}\tau = -\int_0^\infty e^{-\tau} p(\tau)\,\mathrm{d}\tau$$

is well-defined and continuous on P, and that the function

$$t \mapsto \gamma(\theta_t(p)) = -e^t \int_t^\infty e^{-\tau} p(\tau)\,\mathrm{d}\tau = \begin{cases} \dfrac{1}{1+(t+h)^2} & : \quad p = \theta_h(f)\,, \\ 0 & : \quad p = 0\,, \end{cases}$$

is the unique solution of the scalar ordinary differential equation

$$\dot{x} = x + E(\theta_t(p)) = x + p(t)\,,$$

which exists and is bounded for all $t \in \mathbb{R}$. Now consider the nonautonomous scalar ordinary differential equation

$$\dot{x} = g(\theta_t(p), x)\,, \tag{3.20}$$

where

$$g(p,x) := \begin{cases} -x - E(p)x^2 & : \quad p \neq 0\,,\ 0 \le x\gamma(p) \le 1\,, \\ -\frac{1}{\gamma(p)}\left(1 + \frac{E(p)}{\gamma(p)}\right) & : \quad p \neq 0\,,\ 1 < x\gamma(p)\,, \\ -x & : \quad p = 0\,,\ 0 \le x\,. \end{cases}$$

It is easily shown that this differential equation has a unique solution defined on $\mathbb{R}$ passing through any point $x_0 \in X = \mathbb{R}^+$ at time $t = 0$. These solutions define a cocycle mapping

$$\varphi(t,p,x_0) = \begin{cases} \frac{x_0}{e^t(1-x_0\gamma(p)) + x_0\gamma(\theta_t p)} & : \quad p \neq 0\,,\ 0 \le x_0\gamma(p) \le 1\,, \\ x_0 + \frac{1}{\gamma(\theta_t(p))} - \frac{1}{\gamma(p)} & : \quad p \neq 0\,,\ 1 < x_0\,\gamma(p)\,, \\ e^{-t}x_0 & : \quad p = 0\,,\ 0 \le x_0\,. \end{cases} \tag{3.21}$$

From this construction, it follows that $\mathcal{A} := P \times \{0\}$ is the only compact invariant nonautonomous set. To see that $\mathcal{A}$ is a pullback attractor, observe that

$$\varphi\big(t, \theta_{-t}(p), x_0\big) = \begin{cases} \frac{x_0}{e^t(1 - x_0\gamma(\theta_{-t}(p))) + x_0\gamma(p)} & : \ p \neq 0\,,\ 0 \leq x_0\gamma(\theta_{-t}(p)) \leq 1\,, \\ x_0 + \frac{1}{\gamma(p)} - \frac{1}{\gamma(\theta_{-t}(p))} & : \ p \neq 0\,,\ 1 < x_0\,\gamma(\theta_{-t}(p))\,, \\ e^{-t}x_0 & : \ p = 0\,,\ 0 \leq x_0\,. \end{cases}$$

In particular, note that $\gamma(\theta_t(p))^{-1}$ is a solution of the nonautonomous differential equation (3.20). Since $\gamma(\theta_{-t}(p))^{-1}$ tends to ∞ subexponentially fast for $t \to \infty$, it follows that

$$\varphi\big(t, \theta_{-t}(p), x_0\big) \leq \frac{1}{2} L\, e^{-\frac{1}{2}t}$$

for any $x_0 \in [0, L]$ for any $L \geq 0$ and $p \in P$, provided that t is taken sufficiently large. Consequently, $\mathcal{A}$ is a pullback attractor for the skew product flow (θ, φ). In view of (3.21), the stable set

$$W^s(\mathcal{A}) := \Big\{(p, x) \in \mathfrak{X} : \lim_{t\to\infty} \operatorname{dist}_{\mathfrak{X}} \big(\pi(t, (p, x)), \mathcal{A}\big) = 0\Big\}$$

of $\mathcal{A} = P \times \{0\}$, i.e., the set of all points in in $\mathfrak{X} = P \times X$ that are attracted to $\mathcal{A}$ by the associated semi-dynamical system π, is given by

$$W^s(\mathcal{A}) = \big\{(p, x) \in \mathfrak{X} : p \in P \text{ and } x \geq 0 \text{ with } x\gamma(p) < 1\big\} \subsetneq \mathfrak{X}\,.$$

In summary, the skew product flow (θ, φ) in the above example has a pullback attractor which is not a forward attractor and also not a global attractor of the associated autonomous semi-dynamical system. One can show, however, that it is a local attractor. The situation of local attractivity in the nonautonomous case is discussed in Section 8.

5. Upper semi-continuous dependence on parameters

Analogously to autonomous attractors, pullback attractors also depend, in general, upper semi-continuously on parameters. This also holds for the base space "parameter" $p \in P$. Throughout this subsection, it is assumed that the state and base spaces of the skew product flow are metric spaces (X, d_X) and (P, d_P).

Theorem 3.34 (Upper semi-continuity of pullback attractors). *Let (θ, φ) be a skew product flow with a pullback attractor $\mathcal{A}$ such that $A(P) := \overline{\bigcup_{p\in P} A_p}$ is compact. Then the setvalued mapping $p \mapsto A_p$ is upper semi-continuous.*

Proof. Suppose that this is not true. Then there exists an $\varepsilon_0 > 0$ and a sequence $p_n \to p_0$ in P such that $\operatorname{dist}_X\big(A_{p_n}, A_{p_0}\big) \geq 3\varepsilon_0$ for all $n \in \mathbb{N}$. Since the sets A_{p_n} are compact, there exists a sequence $a_n \in A_{p_n}$ such that

$$\operatorname{dist}_X(a_n, A_{p_0}) = \operatorname{dist}_X(A_{p_n}, A_{p_0}) \geq 3\varepsilon_0 \quad \text{for all } n \in \mathbb{N}\,. \tag{3.22}$$

By pullback attraction, $\operatorname{dist}_X\big(\varphi(\tau, \theta_{-\tau}(p_0), B), A_{p_0}\big) \leq \varepsilon_0$ for $\tau > 0$ large enough, and by the φ-invariance of the pullback attractor, there exist $b_n \in A_{\theta_{-\tau}(p_n)} \subset A(P)$, $n \in \mathbb{N}$ such that $\varphi\big(\tau, \theta_{-\tau}(p_n), b_n\big) = a_n$. Since $A(P)$ is compact, there is a convergent subsequence $b_{n'} \to \bar{b} \in A(P)$. Finally, by the continuity of $\theta_{-\tau}(\cdot)$ and of the cocycle mapping,

$$d_X\big(\varphi\big(\tau, \theta_{-\tau}(p_{n'}), b_{n'}\big), \varphi\big(\tau, \theta_{-\tau}(p_0), \bar{b}\big)\big) \leq \varepsilon_0 \quad \text{for } n' \text{ large enough.}$$

Thus,

$$\begin{aligned}\operatorname{dist}_X\left(a_{n'}, A_{p_0}\right) &= \operatorname{dist}_X\left(\varphi(\tau, \theta_{-\tau}(p_{n'}), b_{n'}), A_{p_0}\right)\\ &\leq d_X\left(\varphi\big(\tau, \theta_{-\tau}(p_{n'}), b_{n'}\big), \varphi\big(\tau, \theta_{-\tau}(p_0), \bar{b}\big)\right)\\ &\qquad + \operatorname{dist}_X\left(\varphi\big(\tau, \theta_{-\tau}(p_0), \bar{b}\big), A_{p_0}\right) \leq 2\varepsilon_0\,,\end{aligned}$$

which contradicts (3.22). □

The upper semi-continuous dependence of the component subsets A_p cannot, in general, be strengthened to continuous dependence as the following simple counterexample shows.

EXAMPLE 3.35. Consider the autonomous scalar differential equation

$$\dot{x} = -x\big(x^4 - 2x^2 + 1 - p\big)\,, \quad \text{where } p \in P := [-1, 1]\,, \tag{3.23}$$

for which there are three parameter regimes for equilibrium solutions $\bar{x}_p$:

(i) $\bar{x}_p = 0$ for $p < 0$
(ii) $\bar{x}_p = 0, \pm\sqrt{1+\sqrt{\nu}}, \pm\sqrt{1-\sqrt{p}}$ for $0 \leq p < 1$, and
(iii) $\bar{x}_p = 0, \pm\sqrt{1+\sqrt{p}}$ for $p \geq 1$.

The zero solution here loses linear stability at $p = 1$ in a subcritical bifurcation to the nonlocal solutions $\pm\sqrt{1+\sqrt{p}}$. Note, however, that these equilibria, as well as $\pm\sqrt{1-\sqrt{p}}$, first appear at $p = 0$. The equilibria $\pm\sqrt{1+\sqrt{p}}$ are asymptotically stable for $p > 0$, whereas the equilibria $\pm\sqrt{1-\sqrt{p}}$ are unstable in their existence interval $0 \leq p < 1$. The global (autonomous) attractors here are $A_p = \{0\}$ for $p < 0$ and

$$A_p = \left[-\sqrt{1+\sqrt{p}}, \sqrt{1+\sqrt{p}}\right] \quad \text{for } p \geq 0\,.$$

In particular, the set-valued mapping $p \mapsto A_p$ is not continuous at $p = 0$ (being only upper semi-continuous there), but is continuous elsewhere, for example, at $p = 1$.

Now consider (3.23) as a nonautonomous differential equation with a driving system θ on $P = [-1, 1]$ such that $\theta_t(p) \equiv p$ for all $t \in \mathbb{R}$ and all $p \in P$, i.e., the driving system just remains at its initial value. Then $\mathcal{A}$ is a pullback (and forward) attractor for the resulting nonautonomous dynamical system for which the set-valued mapping $p \mapsto A_p$ is upper semi-continuous but not continuous at $p = 0$.

A similar result holds when the cocycle mappings of a family of skew product flows and their pullback attractors depend on a parameter, and the proof of this result is similar.

THEOREM 3.36. *Let $(\theta, \varphi^\lambda)$ be a family of skew product flows on a common state space X and base space P for which the cocycle mappings depend on a parameter $\lambda \in \Lambda$, where (Λ, d_Λ) is a compact metric space and (X, d_X) and (P, d_P) are metric spaces. Suppose that each skew product flow has a pullback attractor $\mathcal{A}^\lambda$ and that there is a common compact subset B of X such that $A_p^\lambda \subset B$ for all $p \in P$ and $\lambda \in \Lambda$. If the mapping $(t, p, x, \lambda) \mapsto \varphi^\lambda(t, p, x)$ is continuous, then the set-valued mapping $\lambda \mapsto A_p^\lambda$ is upper semi-continuous for each $p \in P$.*

PROOF. Suppose that this is not true. Then there exists $p \in P$, $\varepsilon_0 > 0$ and a sequence $\lambda_n \to \lambda_0$ in P such that $\mathrm{dist}_X\left(A_p^{\lambda_n}, A_p^{\lambda_0}\right) \geq 3\varepsilon_0$ for all $n \in \mathbb{N}$. Since the sets $A_p^{\lambda_n}$ are compact, there exists a sequence $a_n \in A_p^{\lambda_n}$ such that

$$\mathrm{dist}_X\left(a_n, A_p^{\lambda_0}\right) = \mathrm{dist}_X\left(A_p^{\lambda_n}, A_p^{\lambda_0}\right) \geq 3\varepsilon_0 \quad \text{for all } n \in \mathbb{N}\,. \tag{3.24}$$

By pullback attraction, $\mathrm{dist}_X\left(\varphi(\tau, \theta_{-\tau}(p), B), A_p^{\lambda_0}\right) \leq \varepsilon_0$ for $\tau > 0$ large enough, and by the φ-invariance of the pullback attractor, there exists $b_n \in A_{\theta_{-\tau}(p)}^{\lambda_n} \subset B$ for $n \in \mathbb{N}$ such that $\varphi\big(\tau, \theta_{-\tau}(p), b_n\big) = a_n$. Since B is compact, there is a convergent subsequence $b_{n'} \to \bar{b} \in B$. In addition, by the continuity of the cocycle mapping in x and λ, one obtains

$$d_X\left(\varphi^{\lambda_n}\big(\tau, \theta_{-\tau}(p), b_{n'}\big), \varphi^{\lambda_0}\big(\tau, \theta_{-\tau}(p), \bar{b}\big)\right) \leq \varepsilon_0 \quad \text{for } n' \text{ large enough.}$$

Thus,

$$\begin{aligned}\mathrm{dist}_X\left(a_{n'}, A_p^{\lambda_0}\right) &= \mathrm{dist}_X\left(\varphi^{\lambda_{n'}}(\tau, \theta_{-\tau}(p), b_{n'}), A_p^{\lambda_0}\right)\\ &\leq d_X\left(\varphi^{\lambda_{n'}}\big(\tau, \theta_{-\tau}(p), b_{n'}\big), \varphi^{\lambda_0}\big(\tau, \theta_{-\tau}(p), \bar{b}\big)\right)\\ &\quad + \mathrm{dist}_X\left(\varphi^{\lambda_0}\big(\tau, \theta_{-\tau}(p), \bar{b}\big), A_p^{\lambda_0}\right) \leq 2\varepsilon_0\,,\end{aligned}$$

which contradicts (3.24). □

REMARK 3.37. An analogue of Theorem 1.52 on the equivalence of equi-attraction and continuity with respect to parameters also holds for pullback attractors, see LI & KLOEDEN [**68**].

6. Parametrically inflated pullback attractors

A pullback attractor $\mathcal{A}$ of a skew product flow (θ, φ) on $P \times X$ need not, in general, be forward attracting. However, by Theorem 3.20, one has under the assumption that (P, d_P) is a compact metric space that

$$\lim_{t\to\infty} \sup_{p\in P} \mathrm{dist}\left(\varphi(t, p, D), A(P)\right) = 0$$

for any bounded subset D of $\mathbb{R}^d$, where $A(P) := \overline{\bigcup_{p\in P} A_p}$. This is a form of forward attraction, but $A(P)$ is not φ-invariant and is thus not an attractor. The concept of a parametrically inflated pullback attractor, which was introduced in WANG, LI & KLOEDEN [**237**], provides a more precise description of this forward attraction.

DEFINITION 3.38 (Parametrically inflated pullback attractor). Let $\mathcal{A}$ be a pullback attractor of a skew product flow (θ, φ) on $P \times X$, and let $\varepsilon_0 > 0$. Then the nonautonomous set $\mathcal{A}(\varepsilon_0) = (A_p[\varepsilon_0])_{p\in P}$ with fibers defined by

$$A_p[\varepsilon_0] := \bigcup_{d_P(q,p)\leq\varepsilon_0} A_q$$

is called the *parametrically inflated pullback attractor.*

It is assumed throughout this section that (X, d_X) is a complete metric space and (P, d_P) is a compact metric space.

LEMMA 3.39. *The fiber sets $A_p[\varepsilon_0]$ of a parametrically inflated pullback attractor $\mathcal{A}(\varepsilon_0)$ are nonempty compact subsets of X.*

PROOF. The set $A_p[\varepsilon_0]$ is nonempty since it contains the nonempty set A_p. Compactness follows from the compactness of the subsets A_p, the upper semi-continuity of the mapping $p \mapsto A_p$ and, using Exercise 2.16, the fact that the closed ball $\overline{B_{\varepsilon_0}(p)}$ is compact in the compact space P. □

A parametrically inflated pullback attractor is not an attractor since it is not invariant but only φ-negative invariant, i.e., satisfies

$$A_{\theta_t(p)}[\varepsilon_0] \subset \varphi\big(t, p, A_p[\varepsilon_0]\big) \quad \text{for all } t \geq 0 \text{ and } p \in P\,. \tag{3.25}$$

EXERCISE 3.40. Prove (3.25).

However, in view of Theorem 3.41 below, a parametrically inflated pullback attractor is a forward attracting set. This was proved in [**237**] under weaker assumptions. It uses the fact that a skew product flow is an autonomous semi-dynamical system π on the product space $\mathcal{X} = P \times X$ defined by $\pi(t, (p, x)) = (\theta_t(p), \varphi(t, p, x))$ with respect to the metric $d_{\mathcal{X}}$ on $\mathcal{X}$ defined by

$$d_{\mathcal{X}}\big((p, x), (q, y)\big) = d_P(p, q) + d_X(x, y) \quad \text{for all } (p, x), (q, y) \in \mathcal{X}\,.$$

Clearly, $(\mathcal{X}, d_{\mathcal{X}})$ is complete.

THEOREM 3.41. *Assume that a skew product flow (θ, φ) with compact base space P has a compact positively invariant set $B \subset X$ which is* forward absorbing uniformly *in $p \in P$, i.e., for all bounded sets $D \subset X$, there exists a $T = T(D) > 0$ independently of $p \in P$ such that*

$$\varphi(t, p, D) \subset B \quad \textit{for all } t \geq T \textit{ and } p \in P\,, \tag{3.26}$$

and let $\mathcal{A}$ be the corresponding pullback attractor given by (3.19)*. Then for any fixed $\varepsilon_0 > 0$, the parametrically inflated pullback attractor $\mathcal{A}[\varepsilon_0] = (A_p[\varepsilon_0])_{p \in P}$ forward attracts each bounded subset D of X uniformly in $p \in P$, i.e., for any $\gamma > 0$, there is a $\tau = \tau(D, \gamma) > 0$ independent of $p \in P$ such that*

$$\operatorname{dist}_X\big(\varphi(t, p, D), A_{\theta_t p}[\varepsilon_0]\big) < \gamma \quad \textit{for all } t \geq \tau \textit{ and } p \in P\,.$$

REMARK 3.42. Since $T = T(D)$ is independent of $p \in P$, (3.26) is equivalent to $\varphi\big(t, \theta_{-t}(p), D\big) \subset B$ for all $t \geq T$ and $p \in P$, which corresponds to pullback absorption.

PROOF. Note that Theorem 3.20 holds here, so the skew product flow (θ, φ) has a unique pullback attractor $\mathcal{A}$, and let $\varepsilon_0 > 0$. It needs to be shown that for any bounded set $D \subset X$ and $\gamma > 0$ (it can be assumed that $\gamma < \varepsilon_0$), there is a $\tau = \tau(D, \varepsilon) > 0$ which is independent of $p \in P$ such that

$$\operatorname{dist}_X\big(\varphi(t, p, D), A_{\theta_t p}[\varepsilon_0]\big) < \gamma \quad \text{for all } t \geq \tau \text{ and } p \in P\,.$$

It suffices to show this for the compact set B since any bounded set D is absorbed by B in finite time and remains there since B is positively invariant. Because of the uniformity in $p \in P$, the compact subset $P \times B$ is a positively invariant absorbing set for the autonomous semi-dynamical system π on the product space $P \times X$ associated with the skew product flow (θ, φ). Thus, π has a global attractor $\mathcal{A}$ in $P \times X$, which has the form

$$\mathcal{A} = \bigcup_{p \in P} \{p\} \times \tilde{A}_p\,,$$

where $\tilde{A}_p$ is a nonempty compact subset of B. Due to Proposition 3.31, the so-generated nonautonomous set $\tilde{\mathcal{A}}$ is a pullback attractor for the skew product flow (θ, φ). Then $\tilde{A}_p = A_p$ for all $p \in P$, since the pullback attractor $\mathcal{A}$ of this skew product flow is unique. Thus,

$$\mathcal{A} = \bigcup_{p\in P} \{p\} \times A_p \,.$$

Since $\mathcal{A}$ attracts $P \times B$ under π, there exists a $T_1 = T_1(P \times B, \gamma) > 0$ such that

$$\mathrm{dist}_{\mathcal{X}}(\pi_t(p,x), \mathcal{A}) < \frac{\gamma}{2} \quad \text{for all } \ t \geq T_1 \text{ and } (p,x) \in P \times B\,,$$

i.e.,

$$\inf_{v\in\mathcal{A}} d_{\mathcal{X}}\big((\theta_t(p), \varphi(t,p,x)), v\big) < \frac{\gamma}{2} \quad \text{for all } \ t \geq T_1 \text{ and } (p,x) \in P \times B\,. \tag{3.27}$$

Replacing $\theta_t(p) = q$ in (3.27) yields

$$\inf_{v\in\mathcal{A}} d_{\mathcal{X}}\big((q, \varphi(t,\theta_{-t}(q),x), v\big) < \frac{\gamma}{2} \quad \text{for all } \ t \geq T_1 \text{ and } (p,x) \in P \times B\,.$$

For each q, partition $\mathcal{A}$ into two parts $\mathcal{A} = \mathcal{A}_q^1 \cup \mathcal{A}_q^2$, where

$$\mathcal{A}_q^1 = \bigcup_{\substack{p\in P,\\ d_P(p,q)\leq\varepsilon_0}} \{p\} \times A_p \quad \text{and} \quad \mathcal{A}_q^2 = \bigcup_{\substack{p\in P,\\ d_P(p,q)>\varepsilon_0}} \{p\} \times A_p\,.$$

Now suppose that $(q,x) \in P \times B$ and $t \geq T_1$. Then by the definition of $d_{\mathcal{X}}$, one has

$$d_{\mathcal{X}}\big((q, \varphi(t,\theta_{-t}(q),x)), v\big) \geq d_P(q,p) > \varepsilon_0 > \gamma \quad \text{for all } \ v = (p,y) \in \mathcal{A}_q^2\,,$$

so

$$\inf_{v\in\mathcal{A}_q^1} d_{\mathcal{X}}\big((q, \varphi(t,\theta_{-t}(q),x)), v\big) < \frac{\gamma}{2}\,.$$

Thus, there exists a point $v' = (p', y') \in \mathcal{A}_q^1$ such that

$$d_{\mathcal{X}}\big((q, \varphi(t,\theta_{-t}(q),x)), v'\big) \leq \frac{2}{3}\gamma\,.$$

Then $y' \in A_{p'} \subset A_q[\varepsilon_0]$, since $d_P(p',q) \leq \varepsilon_0$. From this and the definition of $d_{\mathcal{X}}$, it follows that

$$\begin{aligned}\mathrm{dist}_X\big(\varphi(t,\theta_{-t}(q),x), A_{p'}\big) &\leq d_X\big(\varphi(t,\theta_{-t}(q),x), y'\big)\\ &\leq d_{\mathcal{X}}\big((q, \varphi(t,\theta_{-t}(q),x)), (p',y')\big) < \frac{2}{3}\gamma\,,\end{aligned}$$

and hence, one has

$$\mathrm{dist}_X\big(\varphi(t,\theta_{-t}(q),x), A_q[\varepsilon_0]\big) \leq \mathrm{dist}_X\big(\varphi(t,\theta_{-t}(q),x), A_{p'}\big) < \frac{2}{3}\gamma\,.$$

Since $q \in P$, $x \in B$ and $t \geq T_1$ are otherwise arbitrary, this means that

$$\mathrm{dist}_X\big(\varphi(t,\theta_{-t}(q),B), A_q[\varepsilon_0]\big) < \frac{2}{3}\gamma < \gamma \quad \text{for all } \ t \geq T_1 \text{ and } q \in P\,.$$

Finally, writing $p = \theta_{-t}(q)$, this gives

$$\mathrm{dist}_X\big(\varphi(t,p,B), A_{\theta_t p}[\varepsilon_0]\big) < \gamma \quad \text{for all } \ t \geq T_1 \text{ and } p \in P\,,$$

since T_1 is independent of $p \in P$. ☐

The following theorem shows the robust stability of pullback attraction of each fiber A_p of the pullback attractor $\mathcal{A}$ with respect to perturbations in p. It will be used in the next section.

THEOREM 3.43. *Suppose that a skew product flow (θ, φ) satisfies the assumptions of Theorem 3.41 and let $\mathcal{A}$ be its pullback attractor. Then for any $p_0 \in P$ and $\varepsilon > 0$, there exists $\delta = \delta(p_0, \varepsilon) > 0$ such that for any bounded set $D \subset X$, one can find a $T > 0$ such that*

$$\varphi(t, \theta_{-t}(p), D) \subset B_\varepsilon(A_{p_0}) \quad \textit{for all } t \geq T \textit{ and } p \in B_\delta(p_0)\,. \tag{3.28}$$

PROOF. Since the compact set B is forward absorbing uniformly in $p \in P$, it is also pullback absorbing, i.e., for any bounded subset D of X, there exists a $T_0 = T_0(D) > 0$ independent of $p \in P$ such that

$$\varphi(t, \theta_{-t}(p), D) \subset B \quad \text{for all } t \geq T_0 \text{ and } p \in P\,.$$

To prove the theorem, it thus suffices to check that (3.28) holds true for $D = B$. Firstly, one has

$$\varphi(t, \theta_{-t}(p), B) \subset B \quad \text{for all } t \geq 0 \text{ and } p \in P \tag{3.29}$$

by the positive invariance of B. Secondly, by the definition of pullback attraction, there is a time $T_1 \geq 0$, such that

$$\varphi(t, \theta_{-t}(p_0), B) \subset B_\varepsilon(A_{p_0}) \quad \text{for all } t \geq T_1\,.$$

Then, since $p \mapsto \varphi(T_1, p, x)$ is continuous uniformly in $x \in B$, given $\varepsilon > 0$, there exists a $\delta = \delta(p_0, \varepsilon) > 0$ such that

$$\varphi(T_1, \theta_{-T_1}(p), B) \subset B_\varepsilon(A_{p_0}) \quad \text{for all } \in B_\delta(p_0)\,. \tag{3.30}$$

Finally, each $t \geq T = T_1$ can be rewritten as $t = nT_1 + s$, where $s \in [0, T_1]$, so, by (3.29) and (3.30), it follows that

$$\begin{aligned}\varphi\big(t, \theta_{-t}(p), B\big) &= \varphi\big(T_1, \theta_{-T_1}(p), \varphi\big((n-1)T_1 + s, \theta_{-nT_1 - s}(p), B\big)\big) \\ &\subset \varphi\big(T_1, \theta_{-T_1}(p), B\big) \subset B_\varepsilon(A_{p_0})\end{aligned}$$

for all $p \in B_\delta(p_0)$. □

7. Pullback attractors with continuous fibers

In general, the fibers of a pullback attractor $\mathcal{A}$ of a skew product flow (θ, φ) are only upper semi-continuous in the parameter p. However, in special cases, they are also lower semi-continuous and hence continuous with respect to the Hausdorff metric. This happens, for example when the fibers are singleton sets.

As in the previous section, it is also assumed in this section that (X, d) is a complete metric space and (P, d_P) is a compact metric space.

THEOREM 3.44. *Suppose that a skew product flow (θ, φ) satisfies the assumptions of Theorem 3.41, and let $\mathcal{A}$ be its pullback attractor. In addition, suppose that the set-valued mapping $p \mapsto A_p$ is lower semi-continuous in p, i.e.,*

$$\lim_{p \to p_0} \operatorname{dist}_X\left(A_{p_0}, A_p\right) = 0 \quad \textit{for all } p_0 \in P\,.$$

Then $\mathcal{A}$ is a uniform attractor of (θ, φ).

PROOF. It suffices to show that the rate of pullback attraction of $\mathcal{A}$ is uniform with respect to $p \in P$, i.e., for any bounded set $D \subset X$ and $\varepsilon > 0$, there exists a time $T = T(D, \varepsilon) > 0$, which is independent of $p \in P$, such that

$$\operatorname{dist}_X\left(\varphi(t, \theta_{-t}(p), D), A_p\right) < \varepsilon \quad \text{for all } t \geq T \text{ and } p \in P.$$

Assume on the contrary that this is not true. Then there are sequences $t_n \in \mathbb{R}^+$, $x_n \in D$, and $p_n \in P$ with $t_n \to \infty$ as $n \to \infty$ such that

$$d\big(\varphi(t_n, \theta_{-t_n}(p_n), x_n), A_{p_n}\big) \geq \varepsilon.$$

Since P is compact, there is a subsequence of p_n which will also be labeled as p_n such that $p_n \to p_0$ as $n \to \infty$. By lower semi-continuity of A_p in p, one has

$$\operatorname{dist}_X\left(A_{p_0}, A_{p_n}\right) < \frac{\varepsilon}{2} \tag{3.31}$$

for n sufficiently large. On the other hand, by Theorem 3.43,

$$\operatorname{dist}_X\left(\varphi\big(t_n, \theta_{-t_n}(p_n), D\big), A_{p_0}\right) < \frac{\varepsilon}{2} \tag{3.32}$$

for n sufficiently large. Combining (3.31) and (3.32), it follows that

$$\operatorname{dist}_X\left(\varphi\big(t_n, \theta_{-t_n}(p_n), x_n\big), A_{p_n}\right) < \varepsilon$$

for n sufficiently large, which is a contradiction. This finishes the proof of this theorem. □

7.1. Periodic and almost periodic driving systems. When the driving system is periodic or almost periodic, the fibers of the pullback attractor are continuous in the parameter. This is easy to show in the periodic case.

LEMMA 3.45. *Suppose that the driving system θ of a skew product flow (θ, φ) is periodic with minimal period $T > 0$ and $P = \{\theta_t(\bar{p}) : 0 \leq t \leq T\}$ for some $\bar{p} = \theta_T(\bar{p}) \in P$. Then for any pullback attractor $\mathcal{A}$, the set-valued mapping $p \mapsto A_p$ is continuous in the Hausdorff metric.*

PROOF. Let $q_n \to \bar{q}$ in P. Then $q_n = \theta_{t_n}(\bar{p})$ and $\bar{q} = \theta_{\bar{t}}(\bar{p})$ for some $t_n, \bar{t} \in [0, T]$. Assume that $t_n - \bar{t} \in [0, T]$ (otherwise replace t_n by $t_n + T$) and that $t_n \to \bar{t}$ as $n \to \infty$. Then $q_n = \theta_{t_n}(\bar{p}) = \theta_{t_n - \bar{t}}(\theta_{\bar{t}}(\bar{p})) = \theta_{t_n - \bar{t}}(\bar{q})$ and

$$A_{q_n} = A_{\theta_{t_n - \bar{t}}(\bar{q})} = \varphi(t_n - \bar{t}, \bar{q}, A_{\bar{q}}) \to \varphi(0, \bar{q}, A_{\bar{q}}) = A_{\bar{q}} \quad \text{as } n \to \infty,$$

since the set-valued mapping $t \mapsto A_{\theta_t(p)} = \varphi(t, p, A_p)$ is continuous in the Hausdorff metric. □

The almost periodic case is similar, but the proofs are more complicated. See the Appendix for the definition and basic properties of almost periodic functions. Let M be a complete metric space with metric d_M and let $C_b(\mathbb{R}, M)$ be the space of uniformly continuous bounded functions $f : \mathbb{R} \to M$, which is complete under the metric $d_\infty(f, g) = \sup_{t \in \mathbb{R}} d_M(f(t), g(t))$. Recall that the hull of a function $f \in C_b(\mathbb{R}, M)$ is defined as the closure of the subset $\{\theta_t(\bar{p}) : t \in \mathbb{R}\}$, where θ_t is defined to be the left shift operator, $\theta_t(f(\cdot)) = f(\cdot + t)$. By classical results, the hull of an almost periodic function is a compact subset of $C_b(\mathbb{R}, M)$. Similar results hold for other function spaces and topologies, see SELL [**218**]. In addition, let $(\mathcal{K}, h_X)$ be the complete metric space of all nonempty compact subsets of X with the Hausdorff metric h_X.

THEOREM 3.46. *Let P be the hull of an almost periodic function $\bar{p} \in C_b(\mathbb{R}, M)$ in the space $\big(C_b(\mathbb{R}, M), d_\infty\big)$, and let $\mathcal{A}$ be the pullback attractor of a (θ, φ) on $P \times X$. If $p \mapsto A_p$ is continuous, then $t \mapsto A(t) := A_{\theta_t(\bar{p})}$ is almost periodic as a mapping in $C(\mathbb{R}, \mathcal{K})$.*

PROOF. Since $p \mapsto A_p$ is continuous and P is compact due to the assumed almost periodicity, the set-valued mapping $p \mapsto A_p$ is uniformly continuous in p. Therefore, for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$h_X(A_p, A_q) < \varepsilon \quad \text{for all } p, q \in P \text{ with } d_P(p, q) < \delta\,.$$

Since $\bar{p}$ is almost periodic, there is a relatively dense set $E \subset \mathbb{R}$ such that

$$d_P\big(\theta_{t+\tau}(\bar{p}), \theta_t(\bar{p})\big) < \delta \quad \text{for all } \tau \in E \text{ and } t \in \mathbb{R}\,.$$

Then

$$h_X\big(A_{\theta_{t+\tau}(\bar{p})}, A_{\theta_t(\bar{p})}\big) < \varepsilon \quad \text{for all } t \in \mathbb{R} \text{ and } \tau \in E\,.$$

Hence, the set-valued mapping $t \mapsto A(t)$ is almost periodic. □

A simple and interesting case is when the A_p are singleton sets for each p. Then, by Theorem 3.41 and Theorem 3.46, $\mathcal{A}$ is both a uniform pullback attractor and a uniform forward attractor of (θ, φ), hence a uniform attractor. Moreover, $t \mapsto A_{\theta_t(p)}$ is almost periodic. Singleton fibers occur in the following case.

THEOREM 3.47. *Let P be the hull of an almost periodic function $\bar{p} \in C_b(\mathbb{R}; M)$. Suppose that the skew product flow (θ, φ) has a pullback attractor $\mathcal{A}$ and is uniformly asymptotically stable, i.e., for any bounded subset D of X and $\varepsilon > 0$, there exists a $T = T(D, \varepsilon) > 0$ such that*

$$d_X\big(\varphi(t, p, x), \varphi(t, p, y)\big) < \varepsilon \quad \textit{for all } t \geq T\,, p \in P \textit{ and } x, y \in D\,.$$

Then the following statements hold.

(i) *A_p is a singleton set for each $p \in P$,*
(ii) *$p \mapsto A_p$ is continuous,*
(iii) *the single-valued function $\gamma_p(t) := A_{\theta_t(p)}$ is almost periodic.*

PROOF. Only the first conclusion (i) needs to be verified. This follows easily from the uniform asymptotic stability of (θ, φ) and the φ-invariance of $\mathcal{A}$. □

8. Local attractors and repellers

The attractors discussed in this chapter so far are *global* in the sense that they attract all bounded subsets of the phase space. The aim of this section is to provide suitable concepts of *local* attractivity for both forward and pullback convergences, meaning that only sets within a certain neighborhood of the attractor need to be attracted.

DEFINITION 3.48 (Local attractivity). Let ϕ be a process on a metric space (X, d). A compact and invariant nonautonomous set $\mathcal{A}$ is called

(i) a *local forward attractor* if there exists an $\eta > 0$ such that

$$\lim_{t\to\infty} \operatorname{dist}\big(\phi(t, t_0, B_\eta(A_{t_0})), A_t\big) = 0 \quad \text{for all } t_0 \geq 0\,,$$

(ii) a *local pullback attractor* if there exists an $\eta > 0$ such that

$$\lim_{t_0 \to -\infty} \operatorname{dist}\big(\phi\big(t, t_0, B_\eta(A_{t_0})\big), A_t\big) = 0 \quad \text{for all } t \leq 0\,,$$

(iii) a *local uniform attractor* if it is a local forward or local pullback attractor such that the attraction is uniform with respect to $t_0 \in \mathbb{T}$ or $t \in \mathbb{T}$, respectively, i.e., there exists an $\eta > 0$ such that

$$\lim_{t \to \infty} \sup_{t_0 \in \mathbb{T}} \operatorname{dist}\big(\phi\big(t_0 + t, t_0, B_\eta(A_{t_0})\big), A_{t_0+t}\big) = 0\,.$$

The supremum over all $\eta > 0$ for which the above relations hold is called the *forward (pullback, uniform, respectively) radius of attraction* of $\mathcal{A}$.

These definitions allow the empty set to be a local uniform attractor, and hence, also a local forward and pullback attractor. In addition, if the phase space X is compact, then the entire extended phase space $\mathcal{A} = \mathbb{T} \times X$ is also a local uniform attractor.

Exercise 3.49. Show that the concept of a local pullback attractor fits into Definition 3.24, i.e., a local pullback attractor is a pullback attractor with respect to an appropriate attraction universe.

Example 3.50. Consider again the linear inhomogeneous differential equation (3.3) from the Examples 3.6 and 3.13, which is given by

$$\dot{x} = -x + 2\sin t\,.$$

It was shown that the entire solution $t \mapsto \rho(t) = \sin t - \cos t$ gives rise to both a pullback and forward attractor $\mathcal{A}$ with $A_t = \{\rho(t)\}$, which is also a uniform attractor. It follows from the exercise below that the set $\mathcal{A}$ is also a local uniform attractor, and hence, also a local pullback and forward attractor.

Exercise 3.51. Consider a process ϕ with a (global) pullback or uniform attractor $\mathcal{A}$, respectively, and assume that $\bigcup_{t \in \mathbb{R}} A_t$ is compact. Show that $\mathcal{A}$ is also a local pullback or uniform attractor, respectively. Formulate and prove a corresponding statement for local forward attractors under weaker assumptions.

In addition to attractivity, also corresponding concepts of repulsivity will be treated in the following, where repulsivity means attraction in backward time. This means that the notion of repulsivity requires that of invertibility for processes, which needs to be defined. Let (X, d) be a complete metric state space, and consider a time set $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$. A process $(t, t_0, x_0) \mapsto \phi(t, t_0, x_0)$ is said to be *invertible* if it is not only defined for all $t \geq t_0$, but also for $t < t_0$, so an invertible process satisfies both the initial value and evolution property

(i) $\phi(t_0, t_0, x_0) = x_0$ for all $t_0 \in \mathbb{T}$ and $x_0 \in X$,
(ii) $\phi(t_2, t_0, x_0) = \phi\big(t_2, t_1, \phi(t_1, t_0, x_0)\big)$ for all $t_0, t_1, t_2 \in \mathbb{T}$ and $x_0 \in X$.

Invertibility is satisfied if the process comes from an ordinary differential equation restricted to an invariant subset in contrast to the discrete case of difference equations.

Definition 3.52 (Local repulsivity). Let ϕ be an invertible process on a metric space (X, d). A compact and invariant nonautonomous set $\mathcal{R}$ is called

(i) a *local forward repeller* if there exists an $\eta > 0$ such that

$$\lim_{t\to-\infty} \operatorname{dist}\big(\phi(t,t_0,B_\eta(R_{t_0})),R_t\big) = 0 \quad \text{for all } t_0 \le 0\,,$$

(ii) a *local pullback repeller* if there exists an $\eta > 0$ such that

$$\lim_{t_0\to\infty} \operatorname{dist}\big(\phi(t,t_0,B_\eta(R_{t_0})),R_t\big) = 0 \quad \text{for all } t \ge 0\,,$$

(iii) a *local uniform repeller* if it is a local forward or pullback repeller such that the repulsion is uniform with respect to $t_0 \in \mathbb{T}$ or $t \in \mathbb{T}$, respectively, i.e., there exists an $\eta > 0$ such that

$$\lim_{t\to\infty} \sup_{t_0\in\mathbb{T}} \operatorname{dist}\big(\phi(t_0-t,t_0,B_\eta(R_{t_0})),R_{t_0-t}\big) = 0\,.$$

The supremum over all $\eta > 0$ for which the above relations hold is called the *forward (pullback, uniform, respectively) radius of repulsion* of $\mathcal{R}$.

It was already seen in Section 1 that forward attraction concerns the *future* of the system, pullback attraction the *past* and uniform attraction the *entire time*. The situation is different for the notions of repulsivity. It can be seen directly from the definitions that a local forward repeller is a repeller for the past, a local pullback repeller concerns the future, and uniform repulsivity is a concept for the entire time. In particular, this point of view will be important in the next chapter, where the interplay of attractor and repeller is discussed with respect to the different time domains.

The above notions of attractivity and repulsivity will be used in particular also for invariant nonautonomous sets with singleton fibers. These are given as graphs of solutions $t \mapsto x(t) := \phi(t,t_0,x_0)$ for fixed initial time t_0 and initial value x_0. A solution x is called

(i) *locally pullback attractive* if $\operatorname{graph} x$ is a local pullback attractor,
(ii) *locally forward attractive* if $\operatorname{graph} x$ is a local forward attractor,
(iii) *locally uniformly attractive* if $\operatorname{graph} x$ is a local uniform attractor,
(iv) *locally pullback repulsive* if $\operatorname{graph} x$ is a local pullback repeller,
(v) *locally forward repulsive* if $\operatorname{graph} x$ is a local forward repeller,
(vi) *locally uniformly repulsive* if $\operatorname{graph} x$ is a local uniform repeller.

These concepts will be discussed in the following exercise.

EXERCISE 3.53. Consider the process ϕ generated by the nonautonomous ordinary differential equation

$$\dot{x} = a(t)x + b(t)x^3$$

with continuous functions $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to (\gamma,\infty)$ for some $\gamma > 0$. Prove that the trivial solution is a

(i) locally forward attractive if $\liminf_{t\to\infty} -a(t)/b(t) > 0$,
(ii) locally pullback attractive if $\liminf_{t\to-\infty} -a(t)/b(t) > 0$,
(iii) locally uniformly attractive if $\inf_{t\in\mathbb{R}} -a(t)/b(t) > 0$,
(iv) locally forward repulsive if $\liminf_{t\to-\infty} a(t) \ge 0$,
(v) locally pullback repulsive if $\liminf_{t\to\infty} a(t) \ge 0$.
(vi) locally uniformly repulsive if $a(t) \ge 0$ for all $t \in \mathbb{R}$.

In particular, this example demonstrates the relationships of the different notions to the time domains, since the conditions which have to be imposed on the equation indicate the relevant time region.

Attractivity and repulsivity of solutions can be obtained by an exponential condition on the linearization along the solution. Because of the concept of the equation of perturbed motion (see Subsection 1.3 of Chapter 2), it suffices to consider the trivial solution

THEOREM 3.54 (Linearized attractivity and repulsivity). *Consider an unbounded interval* $\mathbb{I}$ *of the form* $\mathbb{R}_0^-$, $\mathbb{R}_0^+$ *or* $\mathbb{R}$, *respectively, and let*

$$\dot{x} = B(t)x + F(t,x) \tag{3.33}$$

be a nonautonomous differential equation with continuous functions $B : \mathbb{I} \to \mathbb{R}^{d\times d}$ *and* $F : \mathbb{I} \times U \to \mathbb{R}^d$, $U \subset \mathbb{R}^d$ *a neighborhood of* 0, *such that* $F(t,0) = 0$ *for all* $t \in \mathbb{I}$. *Let* ϕ *denote the process induced by* (3.33) *and* $\Phi : \mathbb{I}\times\mathbb{I} \to \mathbb{R}^{d\times d}$ *denote the transition operator of the linearized equation* $\dot{x} = B(t)x$. *Then the following statements are fulfilled:*

(i) *If there exist* $\beta < 0$, $K \ge 1$ *and* $\delta > 0$ *such that*

$$\|\Phi(t,s)\| \le Ke^{\beta(t-s)} \quad \text{for all } t \ge s$$

and

$$\|F(t,x)\| \le \frac{-\beta}{2K}\|x\| \quad \text{for all } t \in \mathbb{I} \text{ and } x \in B_\delta(0)\,, \tag{3.34}$$

then one has

$$\operatorname{dist}\big(\phi\big(t,t_0,B_{\delta/K}(0)\big),\{0\}\big) \le \delta\, e^{\frac{\beta}{2}(t-t_0)} \quad \text{for all } t_0, t \in \mathbb{I} \text{ with } t_0 \le t\,,$$

i.e., the trivial solution of (3.33) *is locally pullback (forward, uniformly, respectively) attractive.*

(ii) *If there exist* $\beta > 0$, $K \ge 1$ *and* $\delta > 0$ *such that*

$$\|\Phi(t,s)\| \le Ke^{\beta(t-s)} \quad \text{for all } t \le s$$

and

$$\|F(t,x)\| \le \frac{\beta}{2K}\|x\| \quad \text{for all } t \in \mathbb{I} \text{ and } x \in B_\delta(0)\,,$$

then one has

$$\operatorname{dist}\big(\phi\big(t,t_0,B_{\delta/K}(0)\big),\{0\}\big) \le \delta e^{\frac{\beta}{2}(t-t_0)} \quad \text{for all } t_0, t \in \mathbb{I} \text{ with } t \le t_0\,,$$

i.e., the trivial solution of (3.33) *is locally forward (pullback, uniformly, respectively) repulsive.*

PROOF. It suffices to prove (i), since (ii) can be shown analogously. Given $t_0 \in \mathbb{I}$ and $x_0 \in B_\delta(0)$, an estimate for the process ϕ is proved under the additional assumption

$$\phi(t,t_0,x_0) \in B_\delta(0) \quad \text{for all } t \ge t_0\,. \tag{3.35}$$

The solution $\phi(\cdot,t_0,x_0)$ of (3.33) is also a solution of inhomogeneous linear differential equation

$$\dot{x} = B(t)x + F(t,\phi(t,t_0,x_0))\,.$$

Thus, the variation of constants formula implies that

$$\phi(t,t_0,x_0) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,s)F(s,\phi(s,t_0,x_0))\,\mathrm{d}s \quad \text{for all } t \geq t_0\,,$$

and hence,

$$\begin{aligned} \|\phi(t,t_0,x_0)\| &\leq \|\Phi(t,t_0)\|\,\|x_0\| + \int_{t_0}^{t} \|\Phi(t,s)\|\,\|F(s,\phi(s,t_0,x_0))\|\,\mathrm{d}s \\ &\overset{(3.34)}{\leq} Ke^{\beta(t-t_0)}\|x_0\| + \int_{t_0}^{t} Ke^{\beta(t-s)}\frac{-\beta}{2K}\|\phi(s,t_0,x_0)\|\,\mathrm{d}s \end{aligned}$$

for all $t \geq t_0$ is fulfilled. This implies

$$e^{-\beta t}\|\phi(t,t_0,x_0)\| \leq Ke^{-\beta t_0}\|x_0\| + \frac{-\beta}{2}\int_{t_0}^{t} e^{-\beta s}\|\phi(s,t_0,x_0)\|\,\mathrm{d}s$$

for all $t \geq t_0$. Hence, Gronwall's inequality (see, e.g., Abraham, Marsden & Ratiu [**1**, Theorem 4.1.7, p. 242]) yields the estimate

$$\|\phi(t,t_0,x_0)\| \leq Ke^{\frac{\beta}{2}(t-t_0)}\|x_0\| \quad \text{for all } t \geq t_0\,. \tag{3.36}$$

Define $\eta := \frac{\delta}{K}$. Since $\frac{\beta}{2} < 0$, the assumption (3.35) is fulfilled for all $t_0 \in \mathbb{I}$ and $x_0 \in B_\eta(0)$, and thus, (3.36) holds for such t_0 and x_0. This implies that

$$\operatorname{dist}\big(\phi(t,t_0,B_\eta(0)),\{0\}\big) \leq K\eta e^{\frac{\beta}{2}(t-t_0)} \quad \text{for all } t_0,t \in \mathbb{I} \text{ with } t_0 \leq t\,.$$

From this inequality, the required conditions for the local attractivity are easily obtained. ☐

Repellers can be seen as attractors of the process under time reversal as the following proposition shows.

Proposition 3.55 (Process under time reversal). *Let ϕ be an invertible process on a metric space (X,d), and consider the* process under time reversal ϕ^{-1} *which is defined by*

$$\phi^{-1}(t,t_0,x_0) = \phi(-t,-t_0,x_0) \quad \textit{for all } t,t_0 \in \mathbb{T} \textit{ and } x_0 \in X\,.$$

It follows that ϕ^{-1} is also an invertible process, and if one defines

$$\mathcal{M}^{-1} := \{(t,x) \in \mathbb{T} \times X : (-t,x) \in \mathcal{M}\}$$

for a given nonautonomous $\mathcal{M}$, then the following statements are fulfilled:

(i) *$\mathcal{M}$ is a local forward attractor of ϕ if and only if $\mathcal{M}^{-1}$ is a local forward repeller of ϕ^{-1}.*
(ii) *$\mathcal{M}$ is a local pullback attractor of ϕ if and only if $\mathcal{M}^{-1}$ is a local pullback repeller of ϕ^{-1}.*
(iii) *$\mathcal{M}$ is a local uniform attractor of ϕ if and only if $\mathcal{M}^{-1}$ is a local uniform repeller of ϕ^{-1}.*

Proof. To show that ϕ^{-1} is an invertible process, first note the initial value property

$$\phi^{-1}(t_0,t_0,x_0) = \phi(-t_0,-t_0,x_0) = x_0 \quad \text{for all } t_0 \in \mathbb{T} \text{ and } x_0 \in X\,,$$

and the evolution property is proved by

$$\begin{aligned}\phi^{-1}(t_2, t_0, x_0) &= \phi(-t_2, -t_0, x_0)\\ &= \phi\big(-t_2, -t_1, \phi(-t_1, -t_0, x_0)\big)\\ &= \phi^{-1}\big(t_2, t_1, \phi^{-1}(t_1, t_0, x_0)\big)\end{aligned}$$

for all $t_0, t_1, t_2 \in \mathbb{T}$ and $x_0 \in X$. Now let $\mathcal{M}$ be a local forward attractor of ϕ, i.e., there exists an $\eta > 0$ such that

$$\lim_{t\to\infty} \operatorname{dist}\big(\phi\big(t, t_0, B_\eta(M_{t_0})\big), M_t\big) = 0 \quad \text{for all } t_0 \geq 0\,.$$

This is equivalent to

$$\lim_{t\to\infty} \operatorname{dist}\big(\phi^{-1}\big(-t, -t_0, B_\eta(M_{-(-t_0)})\big), M_{-(-t)})\big) = 0 \quad \text{for all } t_0 \geq 0\,,$$

which also reads as

$$\lim_{t\to-\infty} \operatorname{dist}\big(\phi^{-1}\big(t, t_0, B_\eta(M_{-t_0})\big), M_{-t})\big) = 0 \quad \text{for all } t_0 \leq 0\,.$$

This means that $\mathcal{M}^{-1}$ is a local forward repeller of ϕ^{-1}. The other assertions can be shown analogously. □

This theorem shows that in contrast to the autonomous case (see Proposition 1.19), the notions of a local pullback attractor and local forward repeller are not dual. The same holds for local forward attractors and pullback repellers, however, duality is fulfilled for local uniform attractors and repellers by (iii). The lack of duality in case of pullback attraction and repulsion has far reaching consequences which will be discussed in the following chapters. One important consequence is that both concepts do not have the same uniqueness properties.

Exercise 3.56. Show that local pullback attractors are locally unique and local pullback repellers are intrinsically non-unique.

Endnotes. For review articles, see Caraballo, Langa & Kloeden [**28**] and Kloeden [**116**]. Nonautonomous sets have a long history in the literature and under that name in, e.g., Aulbach, Rasmussen & Siegmund [**11**] and Rasmussen [**194**]. The definition of a pullback attractor was motivated by that of a random attractor, see the references under Chapter 14 below. In some earlier papers such as Kloeden & Schmalfuß [**140, 142, 141**] and Kloeden & Stonier [**145**], they were called cocycle attractors and the name pullback attractor was later introduced, apparently first in Kloeden [**116**], to distinguish them from forward attractors. See Crauel & Flandoli [**61**] and Pötzsche [**188**] for nonautonomous ω-limit sets. Theorem 3.20 on the existence of a pullback attractor has appeared in many versions in the literature with the original proofs being based on those for the existence of random attractors. Attraction universes were introduced in Schmalfuß [**214**] for random dynamical systems. Similar existence theorems to Theorem 3.27 can be found in Flandoli & Schmalfuß [**78**], Kloeden & Schmalfuß [**140, 142**] and Schmalfuß [**213**], see also Crauel, Debussche & Flandoli [**59**]. Section 4 on the relationship between the different types of nonautonomous attractors is based on Cheban, Kloeden & Schmalfuß [**41**]. The upper semi-continuous dependence of pullback attractors on parameters in Section 5 has been considered in many papers, including explicitly Caraballo & Langa [**32**] and Kloeden [**122**]. Li Desheng & Kloeden [**68, 69**] considered the equi-attraction and the continuous dependence of pullback attractors on parameters. Section 6 on parametrically inflated pullback attractors is based on Wang Yejuan, Li Desheng & Kloeden [**237**] and Section 7 on pullback attractors with continuous fibers on Cheban, Kloeden & Schmalfuß [**41**] and Wang Yejuan, Li Desheng & Kloeden [**237**]. Section 8 on local attractors and repellers is from Rasmussen [**194**]. The figures in this chapter were made by van Geene [**231**] and Storck [**224**].