Preface

This book is about the representation theory of commutative local rings, specifically the study of maximal Cohen-Macaulay modules over Cohen-Macaulay local rings.

The guiding principle of representation theory, broadly speaking, is that we can understand an algebraic structure by studying the sets upon which it acts. Classically, this meant understanding finite groups by studying the vector spaces they act upon; the powerful tools of linear algebra can then be brought to bear, revealing information about the group that was otherwise hidden. In other branches of representation theory, such as the study of finite-dimensional associative algebras, sophisticated technical machinery has been built to investigate the properties of modules, and how restrictions on modules over a ring restrict the structure of the ring.

The representation theory of maximal Cohen-Macaulay modules began in the late 1970s and grew quickly, inspired by three other areas of algebra. Spectacular successes in the representation theory of finite-dimensional algebras during the 1960s and 70s set the standard for what one might hope for from a representation theory. In particular, this period saw: P. Gabriel's introduction of the representations of quivers and his theorem that a quiver has finite representation type if and only if it is a disjoint union of ADE Coxeter-Dynkin diagrams; M. Auslander's influential Queen Mary notes applying his work on functor categories to representation theory; Auslander and I. Reiten's foundational work on AR sequences; and key insights from the Kiev school, particularly Y. Drozd, L. A. Nazarova, and A. V. Roĭter. All these advances continued the work on finite representation type begun in the 1940s and 50s by T. Nakayama, R. Brauer, R. Thrall, and J. P. Jans. Secondly, the study of lattices over orders, a part of integral representation theory, blossomed in the late 1960s. Restricting attention to lattices rather than arbitrary modules allowed a rich theory to develop. In particular, the work of Drozd-Roĭter and H. Jacobinski around this time introduced the conditions we call "the Drozd-Roĭter conditions" classifying commutative orders with only a finite number of non-isomorphic indecomposable lattices. Finally, M. Hochster's study of the homological conjectures emphasized the importance of the maximal Cohen-Macaulay condition (even for non-finitely generated modules). The equality of the geometric invariant of dimension with the arithmetic one of depth makes this class of modules easy to work with, simultaneously ensuring that they faithfully reflect the structure of the ring.

The main focus of this book is on the problem of classifying Cohen-Macaulay local rings having only a finite number of indecomposable maximal Cohen-Macaulay modules, that is, having *finite CM type.* Notice that we wrote "the problem," rather than "the solution." Indeed, there is no complete classification to date. There are many partial results, however, including complete classifications in dimensions zero and one, a characterization in dimension two under some mild assumptions, and a complete understanding of the hypersurface singularities with this property. The tools developed to obtain these classifications have many applications to other problems as well, in addition to their inherent beauty. In particular there are applications to the study of other representation types, including countable type and bounded type.

This is not the first book about the representation theory of Cohen-Macaulay modules over Cohen-Macaulay local rings. The text [**Yos90**] by Y. Yoshino is a fantastic book and an invaluable resource, and has inspired us both on countless occasions. It has been the canonical reference for the subject for twenty years. In those years, however, there have been many advances. To give just two examples, we mention C. Huneke and Leuschke's elementary proof in 2002 of Auslander's theorem that finite CM type implies isolated singularity, and R. Wiegand's 2000 verification of F.-O. Schreyer's conjecture that finite CM type ascends to and descends from the completion. These developments alone might justify a new exposition. Furthermore, there are many facets of the subject not covered in Yoshino's book, some of which we are qualified to describe. Thus this book might be considered simultaneously an updated edition of [**Yos90**], a companion volume, and an alternative.

In addition to telling the basic story of finite CM type, our choice of material is guided by a number of themes.

(i) For a homomorphism of local rings $R \longrightarrow S$, which maximal Cohen-Macaulay modules over S "come from" R is a basic question. It is especially important when $S = \widehat{R}$, the completion of R, for then the Krull-Remak-Schmidt uniqueness theorem holds for direct-sum decompositions of $\widehat{R}$-modules.

(ii) The failure of the Krull-Remak-Schmidt theorem is often more interesting than its success. We can often quantify exactly how badly it fails.

(iii) A certain amount of non-commutativity can be useful even in pure commutative algebra. In particular, the endomorphism ring of a module, while technically a non-commutative ring, should be a standard object of consideration in commutative algebra.

(iv) An abstract, categorical point of view is not always a good thing in and of itself. We tend to be stubbornly concrete, emphasizing explicit constructions over universal properties.

The main material of the book is divided into 17 chapters. The first chapter contains some vital background information on the Krull-Remak-Schmidt Theorem, which we view as a version of the Fundamental Theorem of Arithmetic for modules, and on the relationship between modules over a local ring R and over its completion $\widehat{R}$. Chapter 2 is devoted to an analysis of exactly how badly the Krull-Remak-Schmidt Theorem can fail. Nothing here is specifically about Cohen-Macaulay rings or maximal Cohen-Macaulay modules.

Chapters 3 and 4 contain the classification theorems for Cohen-Macaulay local rings of finite CM type in dimensions zero and one. Here essentially everything is known. In particular Chapter 3 introduces an auxiliary representation-theoretic problem, the Artinian pair, which is then used in Chapter 4 to solve the problem of finite CM type over one-dimensional rings via the conductor-square construction.

The two-dimensional Cohen-Macaulay local rings of finite CM type are at a focal point in our telling of the theory, with connections to algebraic geometry, invariant theory, group representations, solid geometry, representations of quivers, and other areas, by way of the *McKay correspondence.* Chapter 5 sets the stage for this material, introducing (in arbitrary dimension) the necessary invariant theory and results of Auslander relating a ring of invariants to the associated skew group ring. These results are applied in Chapter 6 to show that two-dimensional rings of invariants have finite CM type. In particular this applies to the Kleinian singularities, also known as Du Val singularities, rational double points, or ADE hypersurface singularities. We also describe some aspects of the McKay correspondence, including the geometric results due to M. Artin and J.-L. Verdier. Finally Chapter 7 gives the full classification of complete local two-dimensional $\mathbb{C}$-algebras of finite CM type. This chapter also includes Auslander's theorem mentioned earlier that finite CM type implies isolated singularity.

In dimensions higher than two, our understanding of finite CM type is imperfect. We do, however, understand the Gorenstein case more or less completely. By a result of J. Herzog, a complete Gorenstein local ring of finite CM type is a hypersurface ring; these are completely classified in the equicharacteristic case. This classification is detailed in Chapter 9, including the theorem of R.-O. Buchweitz, G.-M. Greuel, and Schreyer which states that if a complete equicharacteristic hypersurface singularity over an algebraically closed field has finite CM type, then it is a simple singularity in the sense of V. I. Arnol′d. We also write down the matrix factorizations for the indecomposable MCM modules over the Kleinian singularities, from which the matrix factorizations in arbitrary dimension can be obtained. Our proof of the Buchweitz-Greuel-Schreyer result is by reduction to dimension two via the double branched cover construction and H. Knörrer's periodicity theorem. Chapter 8 contains these background results, after a brief presentation of the theory of matrix factorizations.

Chapter 10 addresses the critical questions of ascent and descent of finite CM type along ring extensions, particularly between a Cohen-Macaulay local ring and its completion, as well as passage to a local ring with a larger residue field. This allows us to extend the classification theorem for hypersurface singularities of finite CM type to non-algebraically closed fields.

Chapters 11 and 13 describe two powerful tools in the study of maximal Cohen-Macaulay modules over Cohen-Macaulay rings: MCM approximations and Auslander-Reiten sequences. We are not aware of another complete, concise and explicit treatment of Auslander and Buchweitz's theory of MCM approximations and hulls of finite injective dimension, which we believe deserves to be better known. The theory of Auslander-Reiten sequences and quivers, of course, is essential. Chapter 12 establishes some homological tools and introduces totally reflexive modules, whose homological behavior over general local rings mimics that of MCM modules over Gorenstein rings.

The last four chapters consider other representation types, namely countable and bounded CM type, and finite CM type in higher dimensions. Chapter 14 uses recent results of I. Burban and Drozd, based on a modification of the conductor-square construction, to prove Buchweitz-Greuel-Schreyer's classification of the hypersurface singularities with countable CM type. It also proves certain structural results for rings of countable CM type, due to Huneke and Leuschke. Chapter 15 contains a proof of the first Brauer-Thrall conjecture, that an excellent isolated singularity with bounded CM type necessarily has finite CM type. Our presentation follows the original proofs of E. Dieterich and Yoshino. The Brauer-Thrall theorem is then used, in Chapter 16, to prove that two three-dimensional examples have finite CM type. We also quote the theorem of D. Eisenbud and Herzog which classifies the homogeneous rings of finite CM type; in particular, their result says that there are no examples in dimension $\geqslant 3$ other than the ones we have described in the text. Finally, in Chapter 17, we consider the rings of bounded but infinite CM type. It happens that for hypersurface rings they are precisely the same as the rings of countable but infinite CM type. We also classify the one-dimensional rings of bounded CM type.

We include two Appendices. In Appendix A, we gather for ease of reference some basic definitions and results of commutative algebra that are prerequisites for the book. Appendix B, on the other hand, contains material that we require from ramification theory that is not generally covered in a general commutative algebra course. It includes the basics on unramified and étale homomorphisms, Henselian rings, ramification of prime ideals, and purity of the branch locus. We make essential use of these concepts, but they are peripheral to the main material of the book.

The knowledgeable reader will have noticed significant overlap between the topics mentioned above and those covered by Yoshino in his book [**Yos90**]. To a certain extent this is unavoidable; the basics of the area are what they are, and any book on Cohen-Macaulay representation types will mention them. However, the reader should be aware that our guiding principles are quite different from Yoshino's, and consequently there are few topics on which our presentation parallels that in [**Yos90**]. When it does, it is generally because both books follow the original presentation of Auslander, Auslander-Reiten, or Yoshino.

Early versions of this book have been used for advanced graduate courses at the University of Nebraska in Fall 2007 and at Syracuse University in Fall 2010. In each case, the students had had at least one full-semester course in commutative algebra at the level of Matsumura's book [**Mat89**]. A few more advanced topics are needed from time to time, such as the basics of group representations and character theory, properties of canonical modules and Gorenstein rings, Cohen's structure theory for complete local rings, the Artin-Rees Lemma, and the material on multiplicity and Serre's conditions in the Appendix. Many of these can be taken on faith at first encounter, or covered as extra topics.

The core of the book, Chapters 3 through 9, is already more material than could comfortably be covered in a semester course. One remedy would be to streamline the material, restricting to the case of complete local rings with algebraically closed residue fields of characteristic zero. One might also skip or sketch some of the more tangential material. We regard the following as essential: Chapter 3 (omitting most of the proof of Theorem 3.7); the first three sections of Chapter 4; Chapter 5; Chapter 6 (omitting the proof of Theorem 6.11, the calculations in §3, and §4); Chapters 7 and 8; and the first two sections of Chapter 9. Chapters 2 and 10 can each stand alone as optional topics, while the thread beginning with Chapters 11 and 13, continuing through Chapters 15 and 17 could serve as the basis of a completely separate course (though some knowledge of the first half of the book would be necessary to make sense of Chapters 14 and 16).

At the end of each chapter is a short section of exercises of varying difficulty, over 120 in all. Some are independent problems, while others ask the solver to fill in details of proofs omitted from the body of the text.

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