CHAPTER 7

Polar Sets and Thermal Capacity

Sets on which a supertemperature can take the value $+\infty$ are called polar. Because any supertemperature is locally integrable, any polar set has Lebesgue measure zero. Indeed, polar sets play a role reminiscient of that played by null sets in measure theory, in that they are negligible in certain contexts. For example, in the boundary maximum principle for hypotemperatures, we can relax the standard limit superior condition on a polar subset of the boundary (cf. Theorem 7.9). As another example, a lower bounded supertemperature defined on $E\backslash Z$, where E is open and Z is polar and relatively closed, can be extended to one on the whole of E (cf. Theorem 7.14). A Borel polar set cannot support a non-null measure whose heat potential is bounded (cf. Theorem 7.52).

A measure gives a way of estimating the size of a set, and the thermal capacity does a similar job in a way the is especially appropriate to the current situation. The sets with thermal capacity zero are precisely the polar sets, and so are sometimes negligible, as outlined above. But more is true, because the cothermal capacity (the corresponding notion for the adjoint equation) of any set coincides with its thermal capacity, and so the copolar sets coincide with the polar sets (cf. Theorem 7.46). Moreover, for subsets of a hyperplane of the form $\mathbb{R}^n \times \{a\}$, the thermal capacity coincides with the n-dimensional Lebesgue measure on the hyperplane (cf. Theorem 7.55).

7.1. Polar Sets

DEFINITION 7.1. Let Z be any subset of $\mathbb{R}^{n+1}$. If there is an open set E containing Z, and a supertemperature w on E such that

$$Z \subseteq \{p \in E : w(p) = +\infty\},$$

then Z is called a *polar set*.

Obviously any subset of a polar set is itself polar. Since any supertemperature is locally integrable (by Theorem 3.56), any polar set has Lebesgue measure zero. Furthermore, Corollary 3.22 shows that, if B is an open ball in $\mathbb{R}^n$, then any set of the form $B \times \{a\}$ is not polar. Moreover, Corollary 6.40 shows that any polar subset of a heat sphere has surface area measure zero.

The polar sets are the negligible sets of heat potential theory. A statement about points of a set $S \subseteq \mathbb{R}^{n+1}$, which is true except for the points of some polar subset of S, is said to be true *quasi-everywhere* on S, or *q.e.* on S.

We already have an example of a polar set in $\mathbb{R}^2$, in Example 6.51. Here is a much simpler one, in $\mathbb{R}^{n+1}$ for any n.

EXAMPLE 7.2. Any singleton is a polar set. To show this, we take any point $(x_0, t_0) \in \mathbb{R}^{n+1}$, put $p_k = (x_0, t_0 - 2^{-k})$ for every positive integer k, and put $A = \{p_k : k \in \mathbb{N}\}$. We let μ be the measure supported by A such that $\mu(\{p_k\}) = k^{-2}$ for all k. Then $\mu(\mathbb{R}^{n+1}) = \sum_{k=1}^{\infty} k^{-2} < +\infty$, so that $G\mu$ is a heat potential, by Theorem 6.18. Moreover,

$$G\mu(x_0, t_0) = \int_A (4\pi(t_0 - s))^{-\frac{n}{2}} \, d\mu(y, s) = (4\pi)^{-\frac{n}{2}} \sum_{k=1}^{\infty} \left(2^{-k}\right)^{-\frac{n}{2}} = +\infty,$$

so that $\{(x_0, t_0)\}$ is polar.

In the definition of a polar set Z, the supertemperature w can be chosen to be the heat potential of a finite measure on any open superset of Z, and finite at any pre-assigned point outside Z, as we now show.

THEOREM 7.3. *Let Z be a polar subset of an open set E, and let $p_0 \in E \backslash Z$. Then there is a heat potental $G_E\mu$ of a finite measure μ, such that $G_E\mu(p) = +\infty$ for all $p \in Z$ and $G_E\mu(p_0) < +\infty$.*

PROOF. Because Z is polar, there exist an open set D containing Z, and a supertemperature w on D, such that $w(p) = +\infty$ for all $p \in Z$. Replacing D with $(D \cap E) \backslash \{p_0\}$, if necessary, we can assume that $D \subseteq E$ and $p_0 \notin D$. We choose a sequence $\{B_k\}$ of open balls, such that $\overline{B}_k \subseteq D$ for all k and $\bigcup_{k=1}^{\infty} B_k = D$. For each k, we define a finite measure ν_k by putting

$$\nu_k(A) = \frac{\mu_w(A \cap B_k)}{\mu_w(B_k) + 1},$$

for any Borel subset of $\mathbb{R}^{n+1}$, where μ_w is the Riesz measure associated with w. By Corollary 6.37, there is a temperature h_k on B_k such that

$$w(p) = \int_{B_k} G(p; q) \, d\mu_w(q) + h_k(p)$$

for all $p \in B_k$. Therefore the heat potential w_k, defined on $\mathbb{R}^{n+1}$ by

$$w_k(p) = \int_{\mathbb{R}^{n+1}} G(p; q) \, d\nu_k(q),$$

takes the value $+\infty$ at every point of $Z \cap B_k$. Furthermore, since $p_0 \notin D$ and $\overline{B}_k \subseteq D$, the point p_0 is outside the support of ν_k. Therefore Corollary 6.22 implies that $w_k(p_0) < +\infty$.

Now we put

$$\mu = \sum_{k=1}^{\infty} \frac{2^{-k} \nu_k}{1 + w_k(p_0)}.$$

Then $\mu(\mathbb{R}^{n+1}) \leq 1$, so that $G\mu$ is a heat potential by Theorem 6.18. Furthermore, $G\mu(p) = +\infty$ for all $p \in Z$, and $G\mu(p_0) < +\infty$. Since $\mu(\mathbb{R}^{n+1} \backslash E) = 0$, it follows from Theorem 6.24 that the restriction μ_E of μ to E satisfies $G\mu = G_E\mu_E + u$ on E, for some temperature u on E. It follows that $G_E\mu_E$ has all the properties stated in the theorem. □

THEOREM 7.4. *The union of any sequence of polar sets is itself a polar set.*

Proof. Let $\{Z_k\}$ be a sequence of polar sets. By Theorem 7.3, for each k we can find a heat potential $G\mu_k$ such that $G\mu_k = +\infty$ on Z_k and $\mu_k(\mathbb{R}^{n+1}) \leq 2^{-k}$. The measure $\mu = \sum_{k=1}^{\infty} \mu_k$ satisfies $\mu(\mathbb{R}^{n+1}) \leq 1$, so that $G\mu$ is a heat potential, by Theorem 6.18. Furthermore, $G\mu(p) = \sum_{k=1}^{\infty} G\mu_k(p)$ for all $p \in \mathbb{R}^{n+1}$, so that $G\mu(p) = +\infty$ for all $p \in \bigcup_{k=1}^{\infty} Z_k$. Hence $\bigcup_{k=1}^{\infty} Z_k$ is a polar set. □

Example 7.5. Any countable set is polar. For, by Example 7.2, any singleton is polar, and so Theorem 7.4 gives the result.

Theorem 7.6. *If E is a connected open set, and Z is a relatively closed polar subset of E, then $E\backslash Z$ is connected.*

Proof. The set $E\backslash Z$ is open, and so its components are also open. Let D be a component of $E\backslash Z$. By Theorem 7.3, we can choose a heat potential u on $\mathbb{R}^{n+1}$ such that $u(p) = +\infty$ for all $p \in Z$. We define a function v on E by putting

$$v(p) = \begin{cases} u(p) & \text{if} \quad p \in D, \\ +\infty & \text{if} \quad p \in E\backslash D. \end{cases}$$

Then v is lower finite and lower semicontinuous on E. Moreover, the inequality $v(p) \geq \mathcal{V}(v; p; c)$ holds whenever $\overline{\Omega}(p; c) \subseteq D$, and whenever $\overline{\Omega}(p; c) \subseteq E\backslash D$, so that given any point $p \in E$ and $\epsilon > 0$, we can find a positive number $c < \epsilon$ such that the inequality holds. If $q \in D$, then $v(p) < +\infty$ on a dense subset of $\Lambda(q; D)$, and so for any point p in the larger set $\Lambda(q; E)$ we can find a point $r \in \Lambda(q; E)$ such that $p \in \Lambda(r; \Lambda(q; E)) = \Lambda(r; E)$ and $v(r) < +\infty$. Hence, by Theorem 3.56, v is a supertemperature on $\Lambda(q; E)$, which implies that $\Lambda(q; E) \subseteq D$.

Suppose that $D \neq E\backslash Z$. Then we can find a component C of $E \cap \partial D$ that forms part of the boundary between D and another component of $E\backslash Z$. Since $\Lambda(q; E) \subseteq D$ for all $q \in D$, the set C is contained in some hyperplane $\mathbb{R}^n \times \{a\}$, and is a relatively open subset because of how C was chosen. Therefore C is not polar, which contradicts the fact that $C \subseteq Z$. Hence $D = E\backslash Z$, which proves the theorem. □

Example 7.7. It follows from Theorem 7.6 that a nonempty, relatively open subset of *any* hyperplane in $\mathbb{R}^{n+1}$ (not necessarily of the form $\mathbb{R}^n \times \{a\}$) cannot be polar.

As an example of the negligibility of polar sets, we shall give a refinement of the boundary maximum principle for hypotemperatures (Theorem 3.13), using the following terminology.

Definition 7.8. A sequence $\{p_k\}$ of points in an open set E is called a Λ-*sequence* if it satisfies the condition $p_{k+1} \in \Lambda(p_k; E)$ for all k.

Theorem 7.9. *Let w be a hypotemperature on an open set E, and let Z be a polar subset of ∂E. Suppose that*

$$\limsup_{k \to \infty} w(p_k) \leq A$$

for every Λ-sequence $\{p_k\}$ in E that tends either to a point of $\partial E\backslash Z$ or to the point at infinity, and that

$$\limsup_{k \to \infty} w(p_k) < +\infty$$

for every Λ-sequence $\{p_k\}$ in E that tends to a point of Z. Then $w(p) \leq A$ for all $p \in E$.

PROOF. In view of Theorem 7.3, we can find a heat potential v on $\mathbb{R}^{n+1}$ such that $v(q) = +\infty$ for all $q \in Z$. Because v is lower semicontinuous, we have $\lim_{p \to q} v(p) = +\infty$ for all $q \in Z$. Therefore, for any $\epsilon > 0$, the function $w - \epsilon v$ is a hypotemperature on E such that $\limsup_{k \to \infty}(w - \epsilon v)(p_k) \leq A$ for every Λ-sequence $\{p_k\}$ in E that tends either to a point of ∂E or to the point at infinity. Hence $w(p) - \epsilon v(p) \leq A$ for all $p \in E$, by Theorem 3.13. Making $\epsilon \to 0$, we see that $w(p) \leq A$ for all $p \in E$ such that $v(p) < +\infty$, and hence almost everywhere on E by Theorem 3.56. Now Theorem 3.51 shows that whenever $\overline{\Omega}(p; c) \subseteq E$ we have

$$w(p) \leq \mathcal{V}(w; p; c) \leq \mathcal{V}(A; p; c) \leq A,$$

which completes the proof. □

EXAMPLE 7.10. The result of Theorem 7.9 cannot be obtained if we allow

$$\liminf_{k \to \infty} w(p_k) = +\infty$$

for Λ-sequences $\{p_k\}$ in E that tend to even a single point of Z. For example, if $w = G(\cdot; q_0)$, $E = \mathbb{R}^{n+1} \backslash \{q_0\}$, $Z = \{q_0\}$ and $A = 0$, then w is a temperature on E that satisfies the hypotheses of Theorem 7.9 except at the point q_0, but $w \geq A$ on E.

7.2. Families of Supertemperatures

By Theorem 3.63, if $\mathcal{F}$ is an upward-directed family of supertemperatures on E, then $\sup \mathcal{F}$ is usually also a supertemperature on E. However, we shall also need to consider more general families of supertemperatures, and whether their infima are supertemperatures. This is a more delicate matter, as the infimum of a family of lower semicontinuous functions is not usually lower semicontinuous. It transpires that the lack of lower semicontinuity is the only obstacle, and this can effectively be overcome by replacing the infimum by what is called its lower semicontinuous smoothing. We now define this concept, and proceed towards the result we require (Theorem 7.13).

DEFINITION 7.11. If u is an extended real-valued function on an open set E, then the *lower semicontinuous smoothing* $\widehat{u}$ of u is defined by

$$\widehat{u}(p) = u(p) \wedge \liminf_{q \to p} u(q),$$

for all $p \in E$.

As the name suggests, $\widehat{u}$ is lower semicontinuous on E, and clearly $\widehat{u} \leq u$ on E. Furthermore, if w is a lower semicontinuous minorant of u on E, then $w = \widehat{w} \leq \widehat{u}$ on E, so that $\widehat{u}$ is the greatest such minorant.

The following result is called *Choquet's topological lemma.* It plays a similar rôle to that of Lemma 3.62, but in the present more complex situation.

LEMMA 7.12. *Let $\{u_\alpha : \alpha \in I\}$ be a family of extended real-valued functions on an open set E, and for each subset J of I let $u_J(p) = \inf\{u_\alpha(p) : \alpha \in J\}$ for all $p \in E$. Then there is a countable set $K \subseteq I$ such that $\widehat{u}_K = \widehat{u}_I$ on E.*

Proof. The assertion is that, if g is lower semicontinuous and $g \leq u_K$ on E, then $g \leq u_I$ on E. Thus the result is concerned only with the order properties of functions, and so we can assume that $u_\alpha(E) \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}]$ for all $\alpha \in I$. This can be achieved by replacing each function u_α with $\tan^{-1} u_\alpha$, if necessary, because $\tan^{-1}$ is an order-preserving map of the extended real number system.

Let $\{B_k\}$ be a countable base for the Euclidean topology of E. For each k, we choose a point $p_k \in B_k$ such that

$$u_I(p_k) < \inf\{u_I(q) : q \in B_k\} + \frac{1}{k},$$

and then an index $\alpha_k \in I$ such that

$$u_{\alpha_k}(p_k) < u_I(p_k) + \frac{1}{k}.$$

Then we have

$$\inf\{u_{\alpha_k}(q) : q \in B_k\} \leq u_{\alpha_k}(p_k) < u_I(p_k) + \frac{1}{k} < \inf\{u_I(q) : q \in B_k\} + \frac{2}{k}. \tag{7.1}$$

We put $K = \{\alpha_k : k \in \mathbb{N}\}$, so that K is a countable subset of I, and show that $\widehat{u}_K = \widehat{u}_I$ on E. Let w be a lower semicontinuous minorant of u_K on E, let $p \in E$, and let $\epsilon > 0$. Then there is $l \in \mathbb{N}$ such that $p \in B_l$, $w(q) > w(p) - \frac{\epsilon}{2}$ for all $q \in B_l$, and $\frac{2}{l} < \frac{\epsilon}{2}$. Therefore

$$w(p) - \inf\{w(q) : q \in B_l\} \leq \frac{\epsilon}{2}.$$

Furthermore, because $w \leq u_K \leq u_{\alpha_l}$, we have

$$\inf\{w(q) : q \in B_l\} - \inf\{u_{\alpha_l}(q) : q \in B_l\} \leq 0.$$

Putting $k = l$ in (7.1), we also have

$$\inf\{u_{\alpha_l}(q) : q \in B_l\} - \inf\{u_I(q) : q \in B_l\} < \frac{2}{l} < \frac{\epsilon}{2}.$$

Adding these last three inequalities, we obtain

$$w(p) - \inf\{u_I(q) : q \in B_l\} < \epsilon.$$

The fact that $p \in B_l$ now implies that

$$w(p) < \inf\{u_I(q) : q \in B_l\} + \epsilon \leq u_I(p) + \epsilon.$$

It follows that $w \leq u_I$ whenever w is a lower semicontinuous minorant of u_K, so that $\widehat{u}_K \leq \widehat{u}_I$. Since $K \subseteq I$ we also have $\widehat{u}_I \leq \widehat{u}_K$, and so equality holds. □

Theorem 7.13. *Let $\mathcal{F} = \{u_\alpha : \alpha \in I\}$ be a family of supertemperatures on an open set E, and suppose that the function $u = \inf \mathcal{F}$ is locally lower bounded. Then the lower semicontinuous smoothing $\widehat{u}$ is a supertemperature on E, is equal to u almost everywhere on E, and satisfies*

$$\widehat{u}(p) = \liminf_{q \to p} u(q)$$

for all $p \in E$.

Proof. Since the family $\mathcal{F}$ may be uncountable, the function u may not be measurable. We therefore seek an appropriate countable family that still has $\widehat{u}$ as the lower semicontinuous smoothing of its infimum. To this end, we first let $\mathcal{G} = \{v_\alpha : \alpha \in J\}$ denote the family of all pointwise minima $u_{\alpha_1} \wedge ... \wedge u_{\alpha_l}$ $(l \geq 1)$ that can be formed using finitely many elements of $\mathcal{F}$. Corollary 3.18 shows that

$\mathcal{G}$ is a family of supertemperatures on E, and clearly $u = \inf \mathcal{G}$. By Lemma 7.12, there is a sequence $\{w_k\}$ of functions in $\mathcal{G}$ whose infimum has lower semicontinuous smoothing $\widehat{u}$. We put $w = \inf\{w_k : k \in \mathbb{N}\}$, and note that $u \leq w$ and $\widehat{u} = \widehat{w}$.

For each $\alpha \in J$ and $p \in E$, it follows from Lemma 3.16 that

$$v_\alpha(p) = \liminf_{q \to p} v_\alpha(q) \geq \liminf_{q \to p} u(q),$$

so that $u(p) \geq \liminf_{q\to p} u(q)$, and hence $\widehat{u}(p) = \liminf_{q\to p} u(q)$.

We now choose an integer $m \geq 5$. By Theorem 6.46, whenever $\overline{\Omega}_m(p;c) \subseteq E$ we have $\mathcal{V}_m(w_k;p;c) \leq w_k(p)$ for all k, so that

$$\mathcal{V}_m(\widehat{w};p;c) \leq \mathcal{V}_m(w;p;c) \leq w(p). \tag{7.2}$$

Because $w \geq u$, w is locally lower bounded. Since w is also majorized by each supertemperature w_k, Theorem 3.56 implies that w is locally integrable on E. Therefore Theorem 1.28 implies that the function $p \mapsto \mathcal{V}_m(w;p;c)$ is continuous on $\{p : \overline{\Omega}_m(p;c) \subseteq E\}$, and so it follows from (7.2) that

$$\mathcal{V}_m(\widehat{w};p;c) \leq \mathcal{V}_m(w;p;c) \leq \widehat{w}(p), \tag{7.3}$$

because $\widehat{w}$ is the greatest lower semicontinuous minorant of w on any open subset of E. It now follows from Theorem 6.46 that $\widehat{w}$ is a supertemperature on E.

Now Corollary 6.47 and (7.3) show that

$$\widehat{w}(p) = \lim_{c\to 0+} \mathcal{V}_m(\widehat{w};p;c) \leq \liminf_{c\to 0+} \mathcal{V}_m(w;p;c) \leq \limsup_{c\to 0+} \mathcal{V}_m(w;p;c) \leq \widehat{w}(p).$$

Thus $\lim_{c\to 0+} \mathcal{V}_m(w;p;c) = \widehat{w}(p)$ for all $p \in E$. In view of Theorem 1.28(b) and the local integrability of w, given any compact subset K of E we have

$$\lim_{c\to 0+} \int_K |\mathcal{V}_m(w;q;c) - w(q)|\, dq = 0,$$

and so we can find a null sequence $\{c_i\}$ such that $\lim_{i\to\infty} \mathcal{V}_m(w;q;c_i) = w(q)$ for almost every point $q \in K$. Therefore $\widehat{w}(p) = w(p)$ for almost every $p \in E$. Thus $\widehat{w}(p) = \widehat{u}(p) \leq u(p) \leq w(p) = \widehat{w}(p)$, and hence $\widehat{u}(p) = u(p)$, for almost all $p \in E$. □

Using Theorem 7.13, we now prove an extension theorem for supertemperatures across a relatively closed polar subset of E.

THEOREM 7.14. *Let E be an open set, let Z be a relatively closed polar subset of E, and let u be a supertemperature on $E\backslash Z$ that is locally lower bounded on E as a function defined almost everywhere on E. Then the function $\bar{u}$, defined for all $p \in E$ by*

$$\bar{u}(p) = \liminf_{q\to p,\, q\in E\backslash Z} u(q),$$

is the unique extension of u to a supertemperature on E.

PROOF. In view of Theorem 7.3, we can find a heat potential v on E such that $v(p) = +\infty$ for all $p \in Z$. Since v is lower semicontinuous on E, for all $p \in Z$ we have $+\infty = \lim_{q\to p} v(q)$. Given any positive integer k, we put

$$u_k(p) = \begin{cases} u(p) + \frac{1}{k}v(p) & \text{if} \quad p \in E\backslash Z, \\ +\infty & \text{if} \quad p \in Z. \end{cases}$$

Because u is locally lower bounded on E, we have $u_k(p) = +\infty = \lim_{q\to p} u_k(q)$ for all $p \in Z$. Therefore each function u_k is lower semicontinuous on E, and hence a supertemperature on E. We now put

$$w(p) = \inf\{u_k(p) : k \in \mathbb{N}\}$$

for all $p \in E$. For each k we have $u \le u_k$ on $E\backslash Z$, so that $u \le w$ there. Hence w is locally lower bounded on E. Theorem 7.13 now shows that the lower semicontinuous smoothing $\widehat{w}$ is a supertemperature on E. Since u is a lower semicontinuous as well as a minorant of w on $E\backslash Z$, we have $u \le \widehat{w}$ there. Moreover, $w(p) = u(p)$ for all $p \in E\backslash Z$ such that $v(p) < +\infty$, and hence almost everywhere on E, so that $u = \widehat{w} = w$ a.e. The functions u and $\widehat{w}$ are both supertemperatures on $E\backslash Z$, so that $u = \widehat{w}$ everywhere on $E\backslash Z$, by Theorem 3.59. Thus $\widehat{w}$ is an extension of u to a supertemperature on E. It is unique, because any other such extension would be equal to $\widehat{w}$ a.e. on E, and hence everywhere on E. Finally, because $w(p) = +\infty$ for all $p \in Z$, we have

$$\widehat{w}(p) = \liminf_{q\to p} w(q) = \liminf_{q\to p,\, q\in E\backslash Z} w(q) \ge \liminf_{q\to p,\, q\in E\backslash Z} \widehat{w}(q) \ge \widehat{w}(p)$$

for all $p \in E$, so that

$$\widehat{w}(p) = \liminf_{q\to p,\, q\in E\backslash Z} \widehat{w}(q) = \liminf_{q\to p,\, q\in E\backslash Z} u(q).$$

Taking $\bar{u} = \widehat{w}$, we obtain the result. □

Corollary 7.15. *Let E be an open set, let Z be a relatively closed polar subset of E, and let u be a temperature on $E\backslash Z$ that is locally bounded on E as a function defined almost everywhere on E. Then u has a unique extension to a temperature on E.*

Proof. By Theorem 7.14, the functions u and $v = -u$ can be extended to supertemperatures $\bar{u}$ and $\bar{v}$ on E. Since $\bar{u} = u = -v = -\bar{v}$ on $E\backslash Z$, we have $\bar{u} = -\bar{v}$ almost everywhere on E. Therefore, for all $p \in E$ we have

$$\bar{u}(p) = \lim_{c\to 0+} \mathcal{V}(\bar{u}; p; c) = \lim_{c\to 0+} \mathcal{V}(-\bar{v}; p; c) = -\bar{v}(p),$$

by Theorem 3.59. Hence $\bar{u} = -\bar{v}$ everywhere on E. Since $\bar{u}$ is a supertemperature and $-\bar{v}$ is a subtemperature, $\bar{u}$ is a temperature on E. □

As another application of Theorem 7.14, we show that the Green function with pole at p_0 is a minimal temperature for $E\backslash\{p_0\}$ (in the sense of Section 4.4).

Theorem 7.16. *Let E be an open set and let $p_0 \in E$. Then the restriction of $G_E(\cdot; p_0)$ to $E\backslash\{p_0\}$ is a minimal temperature for $E\backslash\{p_0\}$.*

Proof. Let u be a nonnegative temperature such that $u \le G_E(\cdot; p_0)$ on $E\backslash\{p_0\}$. Theorem 7.14 shows that u can be extended to a supertemperature $\bar{u}$ on E. By Theorem 6.34, if $\mu_{\bar{u}}$ denotes the Riesz measure associated with $\bar{u}$, then $G_E\mu_{\bar{u}}$ is a heat potential and $\bar{u} = G_E\mu_{\bar{u}} + h$ on E, where h is the greatest thermic minorant of $\bar{u}$ on E. Because $\bar{u}$ is a temperature on $E\backslash\{p_0\}$, the measure $\mu_{\bar{u}}$ is supported by $\{p_0\}$, in view of Theorem 6.25. Therefore $\bar{u} = cG_E(\cdot; p_0) + h$ for some nonnegative number c. Furthermore, $0 \le h \le \bar{u} = u \le G_E(\cdot; p_0)$ on $E\backslash\{p_0\}$, so that $h(p_0) = 0$ in view of Theorem 6.7 and the continuity of h. Hence $h \le G_E(\cdot; p_0)$ on E. It follows that $h = 0$ on E, so that $u = cG_E(\cdot; p_0)$ on $E\backslash\{p_0\}$, as required. □

7.3. The Natural Order Decomposition

The natural order decomposition theorem for nonnegative supertemperatures is an essential tool for proving basic properties of thermal capacity. In its proof we use the following characterization of supertemperatures, which shows, under certain conditions, that the lower semicontinuity requirement can be weakened in line with the result of Lemma 3.16.

THEOREM 7.17. *Let u be an extended real-valued function on an open set E. Then u is a supertemperature on E if and only if it satisfies the following four conditions:*

(a) u is locally lower bounded on E;

(b) for each point $(x_0, t_0) \in E$, the inequality

$$\liminf_{(x,t)\to(x_0,t_0-)} u(x,t) \geq u(x_0,t_0)$$

holds;

(c) u is finite on a dense subset of E;

(d) the inequality $u(p) \geq \mathcal{L}(u;p;c)$ holds whenever $\overline{\Delta}(p;c) \subseteq E$.

PROOF. If u is a supertemperature on E, then because it is lower finite and lower semicontinuous, it is locally lower bounded, by Lemma 3.4. Thus u satisfies conditions (a), (b) and (c), and Theorem 3.17 shows that it also satisfies (d).

To prove the converse, we have to prove that, if u satisfies the four conditions, then it is lower semicontinuous at each point of E. Let $p_0 = (x_0, t_0) \in E$. Given any subset S of E, we adopt the notation

$$S^+ = \{(x,t) \in S : t \geq t_0\}, \qquad S^- = \{(x,t) \in S : t < t_0\}.$$

Since u satisfies condition (b), given any number $A < u(x_0,t_0)$, we can find an open ball V, with centre (x_0,t_0), such that $\overline{V} \subseteq E$ and $u(y,s) > A$ for all $(y,s) \in V^-$. Let U be a neighbourhood of (x_0,t_0) such that $\overline{U} \subseteq V$. Then the distance between U and $\mathbb{R}^{n+1}\backslash V$ is positive, and so we can find $c_0 > 0$ such that $\overline{\Delta}(x,t;c_0) \subseteq V$ for all $(x,t) \in U$. Since u satisfies condition (a), we can find a number $M > 0$ such that $u(y,s) > -M$ for all $(y,s) \in V$.

For any point $p_1 = (x_1,t_1)$, we let μ_{p_1} denote the caloric measure at p_1 for the heat cylinder $\Delta(p_1;c_0)$. By Lemma 2.10, we have

$$\mu_{p_1}(\{(y,s) \in \partial_n\Delta(p_1;c_0) : s \geq t_1\}) = 0,$$

and so for any given $\epsilon > 0$ we can find a number $\delta_0 > 0$ such that

$$M\mu_{p_1}(\{(y,s) \in \partial_n\Delta(p_1;c_0) : s \geq t_1 - r\}) < \epsilon$$

whenever $0 < r < \delta_0$. Furthermore, because

$$\mu_{p_1}(\{(y,s) \in \partial_n\Delta(p_1;c_0) : s < t_1\}) = \int_{\partial_n\Delta(p_1;c_0)} d\mu_{p_1} = \mathcal{L}(1;p_1;c_0) = 1,$$

we can find a number δ_1, satisfying $0 < \delta_1 < \delta_0$, such that

$$\mu_{p_1}(\{(y,s) \in \partial_n\Delta(p_1;c_0) : s < t_1 - r\}) > 1 - \epsilon$$

whenever $0 < r < \delta_1$. It now follows from the translation invariance of caloric measure that, if the point $p = (x,t)$ is such that $t_0 < t < t_0 + \delta_1$, then

$$M \int_{\partial_n \Delta(p;c_0)^+} d\mu_p = M\mu_p(\{(y,s) \in \partial_n \Delta(p;c_0) : s \geq t_0\}) < \epsilon$$

and

$$\int_{\partial_n \Delta(p;c_0)^-} d\mu_p = \mu_p(\{(y,s) \in \partial_n \Delta(p;c_0) : s < t_0\}) > 1 - \epsilon.$$

Therefore, for every point $p = (x,t) \in U^+$ such that $t < t_0 + \delta_1$, condition (d) implies that

$$\begin{aligned} u(p) &\geq \int_{\partial_n \Delta(p;c_0)} u \, d\mu_p \\ &\geq -M \int_{\partial_n \Delta(p;c_0)^+} d\mu_p + A \int_{\partial_n \Delta(p;c_0)^-} d\mu_p \\ &> -\epsilon + (A - |A|\epsilon) \\ &= A - (1 + |A|)\epsilon. \end{aligned}$$

Hence

$$\liminf_{p \to p_0,\, p \in U^+} u(p) > A - (1 + |A|)\epsilon$$

for every $\epsilon > 0$ and number $A < u(p_0)$. Thus

$$\liminf_{p \to p_0,\, p \in U^+} u(p) \geq u(p_0),$$

which implies that u is lower semicontinuous at p_0, in view of condition (b). Hence u is a supertemperature on E. □

Corollary 7.18. *Let u be a locally bounded function on an open set E. Then u is a temperature on E if and only if, at every point $(x_0,t_0) \in E$, u satisfies*

(a) $u(x_0,t_0) = \lim_{(x,t) \to (x_0,t_0-)} u(x,t)$

and

(b) $u(x_0,t_0) = \mathcal{L}(w;x_0,t_0;c)$ whenever $\overline{\Delta}(x_0,t_0;c) \subseteq E$.

Proof. If u is a temperature on E, then condition (a) is obviously satisfied, and Theorem 2.14 shows that condition (b) is too.

Conversely, if conditions (a) and (b) are satisfied, then Theorem 7.17 shows that u is both a supertemperature and a subtemperature on E, so that u is a temperature on E. □

Remark 7.19. Results similar to Theorem 7.17 and Corollary 7.18, can be obtained using any of the other mean values $\mathcal{M}$, $\mathcal{V}$ or $\mathcal{V}_m$.

The proof of the Natural Order Decomposition Theorem depends also on the following result about pieceing together two supertemperatures.

Lemma 7.20. *Let u be a supertemperature on an open set E, and let v be a supertemperature on an open subset V of E. If*

$$\liminf_{p \to q,\, p \in V} v(p) \geq u(q) \qquad \text{for all} \quad q \in E \cap \partial V, \tag{7.4}$$

and w is defined on E by

$$w(p) = \begin{cases} (u \wedge v)(p) & \text{if} \quad p \in V, \\ u(p) & \text{if} \quad p \in E \backslash V, \end{cases}$$

then w is a supertemperature on E.

PROOF. It is clear that w is a supertemperature on $E \backslash \partial V$, that $w(p) > -\infty$ for all $p \in E$, and that $w < +\infty$ on a dense subset of E. Condition (7.4) ensures that, for each point $q \in E \cap \partial V$,

$$\liminf_{p \to q} w(p) = \big(\liminf_{p \to q,\, p \in V} v(p) \big) \wedge \big(\liminf_{p \to q} u(p) \big) \geq u(q) = w(q),$$

so that w is lower semicontinuous on E. It remains only to check that the inequality $w(p) \geq \mathcal{L}(w;p;c)$ holds whenever $p \in E \cap \partial V$ and $\overline{\Delta}(p;c) \subseteq E$. In this case we have, by Theorem 3.17,

$$w(p) = u(p) \geq \mathcal{L}(u;p,c) \geq \mathcal{L}(w;p,c).$$

Hence w is a supertemperature on E. □

We now come to the Natural Order Decomposition Theorem.

THEOREM 7.21. *If u, u_1 and u_2 are nonnegative supertemperatures such that $u \leq u_1 + u_2$ on an open set E, then there are nonnegative supertemperatures v_1 and v_2 which satisfy*

$$v_1 \leq u_1, \quad v_2 \leq u_2, \quad u = v_1 + v_2$$

on E.

PROOF. Let $\mathcal{F}$ denote the class of nonnegative supertemperatures v such that $u \leq v + u_2$ on E, and let w_1 denote the lower semicontinuous smoothing of $\inf \mathcal{F}$. Then $u \leq \inf \mathcal{F} + u_2$ on E, so that $u \leq w_1 + u_2$ almost everywhere on E, by Theorem 7.13. Since w_1 is a supertemperature on E, it follows from Theorem 3.59 that $u \leq w_1 + u_2$ everywhere on E. Furthermore, because $u_1 \in \mathcal{F}$, we have $w_1 \leq u_1$ on E.

Now we let $\mathcal{G}$ denote the class of nonnegative supertemperatures v such that $u \leq w_1 + v$ on E, and w_2 denote the lower semicontinuous smoothing of $\inf \mathcal{G}$. Then, by a similar argument, w_2 is a supertemperature such that $u \leq w_1 + w_2$ on E. Furthermore, because $u_2 \in \mathcal{G}$, we have $w_2 \leq u_2$ on E.

We now show that $u - w_2$, if defined appropriately on the set where $w_2 = +\infty$, is a supertemperature on E. Clearly $u \in \mathcal{G}$, so that $w_2 \leq u$ on E. We put $F = \{p \in E : w_2(p) < +\infty\}$, and define

$$(u - w_2)(p) = \begin{cases} u(p) - w_2(p) & \text{if} \quad p \in F, \\ \liminf_{q \to p,\, q \in F} \big(u(q) - w_2(q) \big) & \text{if} \quad p \in E \backslash F. \end{cases}$$

Then $u - w_2 \geq 0$ on E, and if $p \in E \backslash F$ then $u - w_2$ is lower semicontinuous at p.

For any $p \in E$, we let $\Delta = \Delta(p;c)$ be a heat cylinder with closure in E, and let π_Δ be the operator in Theorem 3.21. The inequality $u \leq w_1 + w_2$ implies that $\pi_\Delta u \leq \pi_\Delta w_1 + \pi_\Delta w_2 \leq w_1 + \pi_\Delta w_2$ on E, so that

$$u = \pi_\Delta u + u - \pi_\Delta u \leq w_1 + (\pi_\Delta w_2 + u - \pi_\Delta u)$$

on $\overline{\Delta}\backslash\partial_n\Delta$. The function $w_3 = \pi_\Delta w_2 + u - \pi_\Delta u = u - \big(\pi_\Delta u - \pi_\Delta w_2\big)$ is the difference of u and a nonnegative temperature on $\overline{\Delta}\backslash\partial_n\Delta$, and the inequality $u \geq \pi_\Delta u$ implies that $w_3 \geq \pi_\Delta w_2$ there. We now put

$$w_4(r) = \begin{cases} (w_2 \wedge w_3)(r) & \text{if} \quad r \in \overline{\Delta}\backslash\partial_n\Delta, \\ w_2(r) & \text{if} \quad r \in E\backslash(\overline{\Delta}\backslash\partial_n\Delta). \end{cases}$$

Since $\pi_\Delta w_2$ is lower semicontinuous on E, and is equal to w_2 on $\partial_n\Delta$, we have

$$\liminf_{r\to q,\, r\in\Delta} w_3(r) \geq \liminf_{r\to q,\, r\in\Delta} \pi_\Delta w_2(r) \geq w_2(q)$$

for all $q \in \partial_n\Delta$. It now follows from Lemma 7.20 that w_4 is a supertemperature on $E\backslash\overline{\Xi}$, where $\Xi = \partial\Delta\backslash\partial_n\Delta$. If $q \in \Xi$, then

$$\liminf_{r\to q} w_4(r) = \big(\liminf_{r\to q} w_2(r)\big) \wedge \big(\liminf_{r\to q,\, r\in\Delta} w_3(r)\big) \geq w_2(q) \wedge w_3(q) = w_4(q),$$

so that w_4 is lower semicontinuous at q. Moreover, if $q \in \overline{\Xi}\backslash\Xi$ then

$$\liminf_{r\to q} w_4(r) \geq \big(\liminf_{r\to q} w_2(r)\big) \wedge \big(\liminf_{r\to q,\, r\in\Delta} \pi_\Delta w_2(r)\big) \geq w_2(q) = w_4(q),$$

so that w_4 is lower semicontinuous at q. Hence w_4 is lower semicontinuous on E. Furthermore, whenever $q \in \Xi$ and $\overline{\Delta}(q;d) \subseteq \overline{\Delta}\backslash\partial_n\Delta$, we have

$$w_4(q) = w_2(q) \wedge w_3(q) \geq \mathcal{L}(w_2;q;d) \wedge \mathcal{L}(w_3;q;d) \geq \mathcal{L}(w_4;q;d).$$

Moreover, whenever $q \in \overline{\Xi}\backslash\Xi$ and $\overline{\Delta}(q;d) \subseteq E$, we have

$$w_4(q) = w_2(q) \geq \mathcal{L}(w_2;q;d) \geq \mathcal{L}(w_4;q;d).$$

It follows that w_4 is a supertemperature on E.

Moreover, because $u \leq w_1 + w_2$ on E and $u \leq w_1 + w_3$ on $\overline{\Delta}\backslash\partial_n\Delta$, we have $u \leq w_1 + w_4$ on E. Therefore $w_4 \in \mathcal{G}$, so that $w_4 \geq w_2$ on E, and hence

$$\pi_\Delta w_2 + u - \pi_\Delta u = w_3 \geq w_2$$

on $\overline{\Delta}\backslash\partial_n\Delta$. Thus

$$u - w_2 \geq \pi_\Delta u - \pi_\Delta w_2$$

on $\overline{\Delta}\backslash\partial_n\Delta$. It follows that $(u - w_2)(p) \geq \mathcal{L}(u - w_2;p;c)$, and that

$$\begin{aligned} \liminf_{r\to p,\, r\in\Delta} (u - w_2)(r) &\geq \lim_{r\to p,\, r\in\Delta} \big(\pi_\Delta u - \pi_\Delta w_2\big)(r) \\ &= \big(\pi_\Delta u - \pi_\Delta w_2\big)(p) \\ &= \mathcal{L}(u;p;c) - \mathcal{L}(w_2;p;c). \end{aligned}$$

Because u and w_2 are supertemperatures on E, Corollary 3.20 shows that

$$\mathcal{L}(u;p;c) - \mathcal{L}(w_2;p;c) \to u(p) - w_2(p)$$

as $c \to 0+$, if $p \in F$. Hence the inequality

$$\liminf_{r\to p,\, r\in\Delta} (u - w_2)(r) \geq (u - w_2)(p)$$

holds whenever $p \in F$, and it holds by definition whenever $p \in E\backslash F$. It now follows from Theorem 7.17 that $u - w_2$ is a supertemperature on E.

The relations $(u - w_2) \leq w_1$ and $u = (u - w_2) + w_2$ are valid on F, and therefore almost everywhere on E. Hence, because $u - w_2$, w_1, u and w_2 are all supertemperature on E, these relations hold everywhere on E. Taking $v_1 = u - w_2$ and $v_2 = w_2$, we obtain the result of the theorem. □

7.4. Reductions and Smoothed Reductions

DEFINITION 7.22. Let u be a nonnegative supertemperature on an open set E. If $L \subseteq E$, then the *reduction of* u *over* L (relative to E), denoted by R_u^L, is the infimum of the family of nonnegative supertemperatures on E that majorize u on L.

Note that $u = R_u^L$ on L.

By Theorem 7.13, the lower semicontinuous smoothing $\widehat{R}_u^L$ of the reduction of u over L is a supertemperature on E, is equal to R_u^L almost everywhere on E, and satisfies the equality

$$\widehat{R}_u^L(p) = \liminf_{q \to p} R_u^L(q)$$

at every point $p \in E$.

DEFINITION 7.23. We call $\widehat{R}_u^L$ the *smoothed reduction of* u *over* L (relative to E).

Note that $u \geq R_u^L \geq \widehat{R}_u^L \geq 0$ on E, and that $R_u^L = \widehat{R}_u^L$ on the interior of L.

Smoothed reductions allow the following characterization of polar sets.

THEOREM 7.24. *For any subset L of the open set E, the following statements are equivalent:*

(a) L is polar.

(b) There is a positive supertemperature u on E such that $\widehat{R}_u^L = 0$.

(c) $\widehat{R}_u^L = 0$ for every nonnegative supertemperature u on E.

PROOF. We suppose first that L is a polar subset of E, and take any point $p_0 \in E \backslash L$. In view of Theorem 7.3, we can find a heat potential v on E such that $v(p) = +\infty$ for all $p \in L$ and $v(p_0) < +\infty$. For any nonnegative supertemperature u on E, we have $\epsilon v \geq u$ on L for each $\epsilon > 0$, so that $R_u^L(p_0) \leq \epsilon v(p_0)$ and hence $R_u^L(p_0) = 0$. Since p_0 is arbitrary, we have $R_u^L(p) = 0$ for all $p \in E \backslash L$. The set L has no interior points, so that $\liminf_{q \to p} R_u^L(q) = 0$ for all $p \in E$, and hence $\widehat{R}_u^L = 0$. Thus (a) implies (c).

Clearly (c) implies (b), so it remains only to prove that (b) implies (a). Let u be a positive supertemperature on E such that $\widehat{R}_u^L = 0$. By Theorem 7.13, $R_u^L = 0$ almost everywhere on E. We put $S = \{p \in E : R_u^L(p) = 0\}$, and let $p_0 \in S$. Then for each positive integer k, we can find a supertemperature v_k on E such that $v_k \geq u$ on L and $v_k(p_0) \leq 2^{-k}$. We put $v = \sum_{k=1}^{\infty} v_k$ on E, so that $v(p) = +\infty$ for all $p \in L$, and $v(p_0) \leq 1$. By Corollary 3.57 and Theorem 3.60, v is a supertemperature on $\Lambda(p_0; E)$, and so $L \cap \Lambda(p_0; E)$ is polar. The set S is dense in E, and so $\bigcup_{p \in S} \Lambda(p; E) = E$. By Lindelöf's theorem, we can find a sequence $\{q_j\}$ of points of S, such that $\bigcup_{j=1}^{\infty} \Lambda(q_j; E) = E$. Since p_0 is an arbitrary point of S, it follows that $L \cap \Lambda(q_j; E)$ is polar for all j. Therefore, by Theorem 7.4, the set $L = \bigcup_{j=1}^{\infty} \big(L \cap \Lambda(q_j; E)\big)$ is polar. □

LEMMA 7.25. *Let E be an open set, and let $G_E\mu$ be the heat potential of a measure μ whose support F is a compact subset of E.*

(a) If D is an open superset of F such that $\overline{D} \subseteq E$, and v is a nonnegative supertemperature on E such that $v \geq G_E\mu$ on ∂D, then $v \geq G_E\mu$ on $E \backslash \overline{D}$ also. In

particular, if D is bounded, then $G_E\mu$ is bounded on $E\backslash D$.

(b) If L is a neighbourhood of F such that $\overline{L} \subseteq E$, then $R^L_{G_E\mu} = G_E\mu = \widehat{R}^L_{G_E\mu}$ on E.

Proof. (a) Since F is compact, we have $\mu(F) < +\infty$. Therefore $G\mu$ is a heat potential, by Theorem 6.18. Let h denote the greatest thermic minorant of $G\mu$ on E. Theorem 6.31 shows that the Riesz measure associated with $G\mu$ is μ itself, and so the Riesz Decomposition Theorem shows that $G\mu = G_E\mu + h$ on E. Therefore $G\mu - h = G_E\mu \leq v$ on ∂D. We now put

$$w = \begin{cases} h \vee (G\mu - v) & \text{on} \quad E\backslash\overline{D}, \\ h & \text{on} \quad \overline{D}. \end{cases}$$

The function $G\mu$ is a temperature on $\mathbb{R}^{n+1}\backslash F$, by Corollary 6.22, so that $G\mu - v$ is a subtemperature on $E\backslash F$. Because $F \subseteq D$, D is open and $\overline{D} \subseteq E$, we have $\partial D \subseteq E\backslash F$, and therefore

$$\limsup_{p \to q,\, p \in E\backslash\overline{D}} (G\mu - v)(p) \leq (G\mu - v)(q) \leq h(q)$$

for all $q \in \partial D$. Hence Lemma 7.20 shows that w is a subtemperature on E. Moreover, $w = h \leq G\mu$ on $\overline{D}$ and $w \leq h \vee G\mu = G\mu$ on $E\backslash\overline{D}$, so that $w \leq G\mu$ on E. Thus w is a subtemperature which minorizes $G\mu$ on E, which implies that $w \leq h$. Therefore $G\mu - v \leq h$ on $E\backslash\overline{D}$, and hence $G_E\mu = G\mu - h \leq v$ there.

If D is bounded, then $G_E\mu$ is bounded on ∂D because it is a temperature on $E\backslash F$. So we can take v to be any upper bound for $G_E\mu$ on ∂D.

(b) Let v be a nonnegative supertemperature on E such that $v \geq G_E\mu$ on L. Let D be an open set such that $F \subseteq D$ and $\overline{D} \subseteq L$. Then $v \geq G_E\mu$ on $\overline{D}$, so that it follows from part (a) that $v \geq G_E\mu$ on $E\backslash\overline{D}$ also, and hence on the whole of E. Thus $R^L_{G_E\mu} = G_E\mu$ on E, and the lower semicontinuity of $G_E\mu$ now implies that $\widehat{R}^L_{G_E\mu} = G_E\mu$ on E. □

Example 7.26. Let E be an open set, let $p_0 \in E$, and let L be a neighbourhood of p_0 such that $\overline{L} \subseteq E$. Then

$$R^L_{G_E(\cdot;p_0)} = G_E(\cdot;p_0) = \widehat{R}^L_{G_E(\cdot;p_0)}$$

on E. To see this, take μ to be the unit mass at p_0 in Lemma 7.25(b).

Our next result gives some of the basic properties of reductions and smoothed reductions.

Theorem 7.27. *Let u and v be nonnegative supertemperatures on an open set E, and let L, M, Z, be subsets of E.*

(a) If $u \leq v$ on L, then $R^L_u \leq R^L_v$ and $\widehat{R}^L_u \leq \widehat{R}^L_v$ on E.

(b) If $L \subseteq M$, then $R^L_u \leq R^M_u$ and $\widehat{R}^L_u \leq \widehat{R}^M_u$ on E.

(c) If α is a positive number, then $R^L_{\alpha u} = \alpha R^L_u$ and $\widehat{R}^L_{\alpha u} = \alpha \widehat{R}^L_u$ on E.

(d) On $E\backslash\overline{L}$, the functions R^L_u and $\widehat{R}^L_u$ are equal and are temperatures.

(e) If Z is polar, then $\widehat{R}^{L \cup Z}_u = \widehat{R}^L_u$ on E.

Proof. The proofs of (a) and (b) are elementary.

To prove (c), we observe that v is a nonnegative supertemperature on E that majorizes αu on L, if and only if v is α times a nonnegative supertemperature on

E that majorizes u on L. Therefore, if v and w are nonnegative supertemperatures on E, we have

$$R_{\alpha u}^L = \inf\{v : v \geq \alpha u \quad \text{on} \quad L\} = \inf\{\alpha w : w \geq u \quad \text{on} \quad L\} = \alpha R_u^L.$$

It now follows from Theorem 7.13 that the result for smoothed reductions holds almost everywhere on E, and therefore Theorem 3.59 shows that it holds everywhere on E because both sides of the equation are supertemperatures.

For (d), we let $\mathcal{F}$ denote the family of all restrictions to $E\backslash\overline{L}$ of nonnegative supertemperatures on E that majorize u on L. Then $\mathcal{F}$ is a saturated family (in the sense of Section 3.3), and so its infimum (which is the restriction to $E\backslash\overline{L}$ of R_u^L) is a temperature on $E\backslash\overline{L}$, by Theorem 3.26. Hence R_u^L is continuous on $E\backslash\overline{L}$, which implies that $\widehat{R}_u^L = R_u^L$ there.

If Z is a polar set, we can find a heat potential w on E such that $w(p) = +\infty$ for all $p \in Z$, by Theorem 7.3. If v is a nonnegative supertemperature on E that majorizes u on L, then for each $\epsilon > 0$ the function $v + \epsilon w$ is a nonnegative supertemperature on E that majorizes u on $L \cup Z$, and so $v + \epsilon w \geq R_u^{L\cup Z}$. Making $\epsilon \to 0+$, we deduce that $v \geq R_u^{L\cup Z}$ on the set $F = \{p \in E : w(p) < +\infty\}$. It follows that $R_u^L \geq R_u^{L\cup Z}$ on F, and (b) shows that $R_u^L \leq R_u^{L\cup Z}$ on E. Therefore $R_u^L = R_u^{L\cup Z}$ almost everywhere on E, so that $\widehat{R}_u^L = \widehat{R}_u^{L\cup Z}$ almost everywhere (by Theorem 7.13), and hence everywhere (by Theorem 3.59), on E. □

Our next two theorems contain some consequences of Theorem 7.27(d) that involve heat potentials.

THEOREM 7.28. *If u is a nonnegative supertemperature on an open set E, and $\overline{L}$ is a compact subset of E, then $\widehat{R}_u^L$ is a heat potential.*

PROOF. We show that the greatest thermic minorant of $\widehat{R}_u^L$ is zero. Let U and V be bounded open sets such that $\overline{L} \subseteq U$, $\overline{U} \subseteq V$, and $\overline{V} \subseteq E$. We put $v = \widehat{R}_u^U$, and note that v is a temperature on $E\backslash\overline{U} \supseteq \partial V$, by Theorem 7.27(d). The collection of open sets $\{\Lambda^*(p; E) : p \in E\}$ covers the compact set ∂V, and so we can find a finite set $\{p_1, ..., p_l\}$ of points of E such that $\partial V \subseteq \bigcup_{i=1}^l \Lambda^*(p_i; E)$. By Theorem 6.7, $G_E(\cdot; p_i) > 0$ on $\Lambda^*(p_i; E)$ for each i, and so the supertemperature

$$g = \sum_{i=1}^l G_E(\cdot; p_i) > 0$$

on an open superset of ∂V. We can therefore choose a positive number α such that $\alpha g \geq v$ on ∂V, since $v \in C(\partial V)$. We now define

$$w(p) = \begin{cases} v(p) & \text{if} \quad p \in \overline{V}, \\ (\alpha g(p)) \wedge v(p) & \text{if} \quad p \in E\backslash\overline{V}. \end{cases}$$

Since

$$\liminf_{p \to q} \alpha g(p) \geq \alpha g(q) \geq v(q)$$

for all $q \in \partial V$, it follows from Lemma 7.20 that w is a supertemperature on E. Moreover $w \geq 0$ on E, and $w = v = u$ on $U \supseteq L$, so that $w \geq R_u^L \geq \widehat{R}_u^L$ on E. If h is a thermic minorant of $\widehat{R}_u^L$ on E, then $h \leq w \leq \alpha g$ on $E\backslash\overline{V}$, and so $h \leq \alpha g$ on E by the minimum principle. By Theorem 3.66, the greatest thermic minorant of

αg on E is the sum of the greatest thermic minorants of $\alpha G_E(\cdot;p_i)$ $(i \in \{1,...,l\})$ on E, and hence is zero. Thus $h \leq 0$, and so $\widehat{R}_u^L$ is a heat potential by Corollary 6.39. □

THEOREM 7.29. *Let L be a relatively closed subset of the open set E. If L is not polar, then there is a bounded heat potential $G_E\mu$ such that μ is not null and is supported by a compact subset of L.*

PROOF. We can write L as the union of a sequence of compact subsets of E. If every set in that sequence was polar, then L would also be polar, by Theorem 7.4. Thus we can find a compact subset K of L such that K is not polar. We put $1(p) = 1$ for all $p \in E$. Then $\widehat{R}_1^K$ is a heat potential by Theorem 7.28, and is not identically 0 by Theorem 7.24. Moreover, $\widehat{R}_1^K$ is bounded on E, and is a temperature on $E \backslash K$ by Theorem 7.27(d). Therefore we can write $\widehat{R}_1^K = G_E\mu$, where $\mu(E\backslash K) = 0$ by Theorem 6.25. □

The following corollary reflects on the sharpness of Corollary 7.15.

COROLLARY 7.30. *If L is a compact subset of the open set E, and is not polar, then there is a bounded temperature on $E\backslash L$ that cannot be extended to a supertemperature on E.*

PROOF. By the theorem, we can find a bounded heat potential $G_E\mu$ such that μ is not null and is supported by a compact subset K of L. If $u = -G_E\mu$, then u is a bounded subtemperature on E, and a temperature on $E\backslash K \supseteq E\backslash L$ by Theorem 7.27(d). Suppose that there is a supertemperature v on E such that $v = u$ on $E\backslash L$. Then $v-u$ is a supertemperature on E, and $v-u = 0$ on $E\backslash L$. Since L is compact, the minimum principle shows first that $v - u \geq 0$ on E, then that $v = u$ on E. Therefore $G_E\mu$ is a temperature on E, and Corollary 6.20 shows that μ is null, a contradiction. □

We now present some properties of reductions over open subsets of E.

THEOREM 7.31. *Let u and v be nonnegative supertemperatures on an open set E, and let C, D be open subsets of E.*

(a) The equality $R_u^D = \widehat{R}_u^D$ holds on E, and so R_u^D is a supertemperature on E.

(b) The equality $R_{u+v}^D = R_u^D + R_v^D$ holds on E.

(c) If $C \subseteq D$, then the equalities

$$R^D_{R^C_u} = R^C_{R^D_u} = R^C_u$$

hold on E.

(d) If L is any subset of E, and $u \in C(D)$ for some open superset D of L, then

$$R_u^L = \inf\{R_u^C : C \text{ is an open superset of } L\}.$$

PROOF. Since D is an open set, we have $u = R_u^D = \widehat{R}_u^D$ on D. Since $\widehat{R}_u^D$ is itself a nonnegative supertemperature on E, it follows that $\widehat{R}_u^D \geq R_u^D$ on E. The reverse inequality is always true, and so (a) holds.

Since $u = R_u^D$ and $v = R_v^D$ on D, the nonnegative supertemperature $R_u^D + R_v^D$ majorizes $u + v$ on D, and so $R_u^D + R_v^D \geq R_{u+v}^D$ on E. Therefore, by the natural order decomposition (Theorem 7.21), there are nonnegative supertemperatures u^* and v^* on E such that

$$u^* \le R_u^D, \quad v^* \le R_v^D, \quad R_{u+v}^D = u^* + v^*.$$

On D, we have $u = R_u^D \ge u^*$, $v = R_v^D \ge v^*$, and $u + v = R_{u+v}^D = u^* + v^*$, so that $u^* = u$ and $v^* = v$. Hence $u^* \ge R_u^D$ and $v^* \ge R_v^D$ on E, and so equality holds. This proves (b).

For any nonnegative supertemperatures v and w such that $v \le w$ on E, we have $R_v^C \le R_v^D \le v$ and $R_v^C \le R_w^C$, by Theorem 7.27(a),(b). Therefore, because R_u^C is a supertemperature on E by (a), we have

$$R_{R_u^C}^C \le R_{R_u^C}^D \le R_u^C \quad \text{and} \quad R_{R_u^C}^C \le R_{R_u^D}^C \le R_u^C,$$

so that the result of (c) will follow if we prove the special case $D = C$. Since $R_u^C = u$ on C, a supertemperature w majorizes u on C if and only if it majorizes R_u^C on C, from which the case $D = C$ follows.

For (d), let $v \ge u$ on L and let $\epsilon > 0$. Since $u \in C(D)$, the function $v - u$ is lower semicontinuous on D, and so the set $V = \{p \in D : v(p) - u(p) > -\epsilon\}$ is an open superset of L. Because $v + \epsilon \ge u$ on V, we have $v + \epsilon \ge R_u^V$ on E, so that

$$v + \epsilon \ge \inf\{R_u^C : C \text{ is an open superset of } L\}.$$

Therefore

$$R_u^L + \epsilon \ge \inf\{R_u^C : C \text{ is an open superset of } L\}$$

for every $\epsilon > 0$, so that

$$R_u^L \ge \inf\{R_u^C : C \text{ is an open superset of } L\}.$$

The reverse inequality follows from Theorem 7.27(b). □

Our next result is the important *Strong Subadditivity Property* of reductions and smoothed reductions. We shall eventually be able to show, in Theorem 9.32, that the condition $v \in C(E)$ is unnecessary.

THEOREM 7.32. *If v is a nonnegative supertemperature on E, and a member of $C(E)$, then for any subsets L, M of E we have*

$$R_v^{L\cup M} + R_v^{L\cap M} \le R_v^L + R_v^M, \tag{7.5}$$

and

$$\widehat{R}_v^{L\cup M} + \widehat{R}_v^{L\cap M} \le \widehat{R}_v^L + \widehat{R}_v^M \tag{7.6}$$

on E. Moreover, if $L_+ = \{p \in L : v(p) > 0\}$ then $R_v^L = R_v^{L_+}$ and $\widehat{R}_v^L = \widehat{R}_v^{L_+}$ on E.

PROOF. We first consider the case of open subsets of E, with a view to using Theorem 7.31(d). Let A and B be open subsets of E, and put $w = R_v^A \wedge R_v^B$. On A, we have $R_v^A + R_v^B = v + R_v^B$ and $R_v^A = v \ge R_v^B$. Similarly, on B we have $R_v^A + R_v^B = R_v^A + v$ and $R_v^B = v \ge R_v^A$. Hence, on $A \cup B$ we have

$$R_v^A + R_v^B = v + R_v^A \wedge R_v^B = v + w.$$

It therefore follows from Theorem 7.31(b),(c) that

$$R_v^{A\cup B} + R_w^{A\cup B} = R_{v+w}^{A\cup B} = R_{R_v^A + R_v^B}^{A\cup B} = R_{R_v^A}^{A\cup B} + R_{R_v^B}^{A\cup B} = R_v^A + R_v^B \tag{7.7}$$

on E. Furthermore, on $A \cap B$ we have $R_w^{A\cup B} = w = R_v^A \wedge R_v^B = v$. Therefore, because $R_w^{A\cup B}$ is a nonnegative supertemperature on E by Theorem 7.31(a), we have $R_w^{A\cup B} \geq R_v^{A\cap B}$ on E. Now (7.7) implies that

$$R_v^{A\cup B} + R_v^{A\cap B} \leq R_v^A + R_v^B,$$

for open sets A and B.

By Theorem 7.31(d), we have

$$R_v^L = \inf\{R_v^C : C \text{ is an open superset of } L\}.$$

Moreover, if $A \supseteq L$ and $B \supseteq M$, then Theorem 7.27(b) and the inequality just proved give us

$$R_v^{L\cup M} + R_v^{L\cap M} \leq R_v^{A\cup B} + R_v^{A\cap B} \leq R_v^A + R_v^B.$$

Therefore

$$R_v^{L\cup M} + R_v^{L\cap M} \leq R_v^A + \inf\{R_v^C : C \text{ is an open superset of } M\} = R_v^A + R_v^M,$$

and hence

$$R_v^{L\cup M} + R_v^{L\cap M} \leq \inf\{R_v^C : C \text{ is an open superset of } L\} + R_v^M = R_v^L + R_v^M,$$

so that (7.5) holds. The inequality (7.6) therefore holds almost everywhere on E, and hence everywhere on E because all the functions therein are supertemperatures.

For the last part, it follows from Theorem 11(b) and the strong subadditivity property just proved that

$$R_v^{L_+} \leq R_v^L \leq R_v^{L_+} + R_v^{L\backslash L_+} = R_v^{L_+} + R_0^{L\backslash L_+} = R_v^{L_+}.$$

□

7.5. The Thermal Capacity of Compact Sets

Throughout the remainder of this chapter, E denotes a fixed open set and all reductions are relative to E. The function 1 is defined by $1(p) = 1$ for all $p \in E$.

If K is a compact subset of E, then $\widehat{R}_1^K$ is a heat potential on E by Theorem 7.28, and is a temperature on $E\backslash\partial K$ by Theorem 7.27(d). Therefore the support of its associated Riesz measure is contained in ∂K, by Theorem 6.25.

Definition 7.33. We call $\widehat{R}_1^K$ the *thermal capacitary potential of* K, and its associated Riesz measure ω_K the *thermal capacitary distribution of* K. The *thermal capacity* $\mathcal{C}(K)$ of K is defined by

$$\mathcal{C}(K) = \omega_K(E).$$

Theorem 7.24 shows that $\mathcal{C}(K) = 0$ if and only if K is polar.

We also introduce the corresponding concepts relative to the adjoint equation. In Section 7.7, we see some nontrivial interaction between the two theories.

For a compact subset K of E, we denote by $\widehat{R}_1^{K*}$ the smoothed reduction of 1 over K relative to the adjoint equation. It is a coheat potential on E and a cotemperature on $E\backslash\partial K$. Its associated Riesz measure has its support in ∂K.

DEFINITION 7.34. We call $\widehat{R}_1^{K*}$ the *cothermal capacitary potential of* K, and its associated Riesz measure ω_K^* the *cothermal capacitary distribution of* K. The *cothermal capacity* $\mathcal{C}^*(K)$ of K is defined by

$$\mathcal{C}^*(K) = \omega_K^*(E).$$

EXAMPLE 7.35. Relative to $E = \mathbb{R}^{n+1}$, the cothermal capacity of the closed heat ball $K = \overline{\Omega}(0; c)$, and the heat sphere $\partial\Omega(0; c)$, is $(4\pi c)^{n/2}$. To see this, we let v be a nonnegative cosupertemperature on $\mathbb{R}^{n+1}$ such that $v \geq 1$ on $\overline{\Omega}(0; c)$ (equivalently $v \geq 1$ on $\partial\Omega(0; c)$). Since $G(0; \cdot)$ is a cotemperature on $\mathbb{R}^{n+1}\backslash\overline{\Omega}(0; c)$, and $G(0; q) \to 0$ as q tends to the point at infinity, the minimum principle shows that $v \geq (4\pi c)^{n/2} G(0; \cdot)$ on $\mathbb{R}^{n+1}\backslash\overline{\Omega}(0; c)$. Hence

$$v \geq (4\pi c)^{n/2}\big(G(0; \cdot) \wedge (4\pi c)^{-n/2}\big)$$

on $\mathbb{R}^{n+1}$. The function $(4\pi c)^{n/2}\big(G(0; \cdot) \wedge (4\pi c)^{-n/2}\big)$ is a cosupertemperature on $\mathbb{R}^{n+1}$, and is therefore $\widehat{R}_1^{K*}$. By Example 6.15, if

$$d\mu(x, t) = (4\pi c)^{-n/2} Q(x, -t)\, d\sigma(x, t)$$

on $\partial\Omega(0; c)$, where σ denotes the surface area measure, then the coheat potential of μ is

$$G^*\mu = G(0; \cdot) \wedge (4\pi c)^{-n/2}$$

on $\mathbb{R}^{n+1}$. Hence

$$\widehat{R}_1^{K*} = (4\pi c)^{n/2} G^*\mu,$$

and so

$$\mathcal{C}^*(K) = (4\pi c)^{n/2}\mu(K) = (4\pi c)^{n/2}.$$

The next three results are used to prove some important properties of thermal capacity.

THEOREM 7.36. *If* μ *and* ν *are nonnegative measures on* E, *then*

$$\int_E G_E\mu\, d\nu = \int_E G_E^*\nu\, d\mu.$$

PROOF. Using Theorem 6.10, and interchanging the order of the integrals, we obtain

$$\begin{aligned}\int_E G_E\mu\, d\nu &= \int_E \left(\int_E G_E(p; q)\, d\mu(q)\right) d\nu(p)\\ &= \int_E \left(\int_E G_E^*(q; p)\, d\nu(p)\right) d\mu(q)\\ &= \int_E G_E^*\nu\, d\mu.\end{aligned}$$

□

LEMMA 7.37. *Let* K *be a compact subset of* E.

(a) If μ *and* ν *are nonnegative measures on* E *such that* $G_E\mu \leq G_E\nu$ *on* E, *and* μ *has its support in* K, *then* $\mu(E) \leq \nu(E)$.

(b) If $\{G_E\mu_j\}$ *is a monotone sequence of heat potentials which converges on*

$E\backslash K$ to a heat potential $G_E\mu$, and each measure μ_j has its support in K, then $\mu_j(E) \to \mu(E)$ as $j \to \infty$.

PROOF. Let M be a compact subset of E whose interior contains K. Then the cothermal capacitary potential of M satisfies $G_E^*\omega_M^*(p) = 1$ for all $p \in K$.

(a) Because μ has its support in K, we have

$$\mu(E) = \int G_E^*\omega_M^* \, d\mu = \int G_E\mu \, d\omega_M^*,$$

by Theorem 7.36. Furthermore, because $G_E^*\omega_M^* \leq 1$ on E, we have

$$\nu(E) \geq \int G_E^*\omega_M^* \, d\nu = \int G_E\nu \, d\omega_M^*,$$

again by Theorem 7.36. Now the inequality $G_E\mu \leq G_E\nu$ implies that $\mu(E) \leq \nu(E)$.

(b) Because each measure μ_j has its support in K, each heat potential $G_E\mu_j$ is a temperature on $E\backslash K$, by Theorem 6.25. The sequence $\{G_E\mu_j\}$ converges monotonically to $G_E\mu$ on $E\backslash K$, and so $G_E\mu$ is also a temperature on $E\backslash K$, by the Harnack monotone convergence theorem. Therefore μ has its support in K, by Theorem 6.25. Hence, by Theorem 7.36,

$$\mu_j(E) = \int G_E^*\omega_M^* \, d\mu_j = \int G_E\mu_j \, d\omega_M^*$$

and

$$\mu(E) = \int G_E^*\omega_M^* \, d\mu = \int G_E\mu \, d\omega_M^*.$$

Since the support of ω_M^* is contained in $\partial M \subseteq E\backslash K$, the Lebesgue monotone convergence theorem now shows that $\mu_j(E) \to \mu(E)$ as $j \to \infty$. □

LEMMA 7.38. *Let u be a nonnegative supertemperature on an open set E.*

(a) If $\{L_i\}$ is an expanding sequence of subsets of E whose union D is open, then

$$\lim_{i\to\infty} \widehat{R}_u^{L_i} = \widehat{R}_u^D$$

on E.

(b) If $\{K_i\}$ is a contracting sequence of compact subsets of E with intersection K, and there is an open superset D of K such that $u \in C(D)$, then

$$\lim_{i\to\infty} R_u^{K_i} = R_u^K$$

on E.

PROOF. If $\{L_i\}$ is expanding with union D, then Theorem 7.27(b) shows that the sequence $\{\widehat{R}_u^{L_i}\}$ is increasing and majorized by $\widehat{R}_u^D$. Therefore the function $v = \lim_{i\to\infty} \widehat{R}_u^{L_i}$ is also majorized by $\widehat{R}_u^D$, and is thus finite on a dense subset of E. Theorem 3.60 now shows that v is a supertemperature on E. Moreover, for each i we have $\widehat{R}_u^{L_i} = R_u^{L_i} = u$ almost everywhere on L_i, by Theorem 7.13. It follows that $v = u$ almost everywhere on D, and hence everywhere on D because both are supertemperatures. Thus $v \geq R_u^D = \widehat{R}_u^D \geq v$ on E, which proves (a).

If $\{K_i\}$ is a contracting sequence of compact subsets of E with intersection K, then given any open superset C of K we can find a number i_0 such that $K_i \subseteq C$ whenever $i > i_0$. Therefore Theorem 7.27(b) shows that $R_u^K \leq R_u^{K_i} \leq R_u^C$ on E whenever $i > i_0$, and that the sequence $\{R_u^{K_i}\}$ is decreasing. It follows that

$R_u^K \le \lim_{i\to\infty} R_u^{K_i} \le R_u^C$ on E. Since C is arbitrary, Theorem 7.31(d) now shows that

$$R_u^K \le \lim_{i\to\infty} R_u^{K_i} \le \inf\{R_u^C : C \text{ is an open superset of } K\} = R_u^K,$$

which proves (b). □

The following theorem gives the basic properties of the thermal capacity of compact sets.

THEOREM 7.39. *The nonnegative, finite-valued set function $\mathcal{C}$, on the class of compact subsets of E, has the following properties.*

(a) $\mathcal{C}(\emptyset) = 0$, and if $K \subseteq L$ then $\mathcal{C}(K) \le \mathcal{C}(L)$.

(b) If $\{K_j\}$ is a contracting sequence with intersection K, then

$$\mathcal{C}(K) = \lim_{j\to\infty} \mathcal{C}(K_j).$$

(c) $\mathcal{C}(K \cup L) + \mathcal{C}(K \cap L) \le \mathcal{C}(K) + \mathcal{C}(L)$.

PROOF. (a) Since $R_1^\emptyset = 0$ on E, we have $G_E\omega_\emptyset = \widehat{R}_1^\emptyset = 0$, and so $\omega_\emptyset$ is null.

If $K \subseteq L$, then $\widehat{R}_1^K \le \widehat{R}_1^L$; that is, $G_E\omega_K \le G_E\omega_L$. Therefore, by Lemma 7.37(a), $\omega_K(E) \le \omega_L(E)$; that is, $\mathcal{C}(K) \le \mathcal{C}(L)$.

(b) Since $\{K_j\}$ is a contracting sequence with intersection K, the sequence $\{\widehat{R}_1^{K_j}\} = \{G_E\omega_{K_j}\}$ is decreasing and Lemma 7.38(b) shows that $\lim_{j\to\infty} R_1^{K_j} = R_1^K$ on E. Furthermore, the equalities $R_1^{K_j} = G_E\omega_{K_j}$ and $R_1^K = G_E\omega_K$ hold on $E\backslash K_1$, by Theorem 7.27(d), and hence $\lim_{j\to\infty} G_E\omega_{K_j} = G_E\omega_K$ on $E\backslash K_1$. The supports of the measures ω_{K_j} are all contained in K_1, and so we can apply Lemma 7.37(b) to obtain

$$\mathcal{C}(K) = \omega_K(E) = \lim_{j\to\infty} \omega_{K_j}(E) = \lim_{j\to\infty} \mathcal{C}(K_j).$$

(c) By Theorem 7.32, $\widehat{R}_1^{K\cup L} + \widehat{R}_1^{K\cap L} \le \widehat{R}_1^K + \widehat{R}_1^L$; that is,

$$G_E\omega_{K\cup L} + G_E\omega_{K\cap L} \le G_E\omega_K + G_E\omega_L.$$

The support of the measure $\omega_{K\cup L} + \omega_{K\cap L}$ is contained in $K \cup L$, and so it follows from Lemma 7.37(a) that $(\omega_{K\cup L} + \omega_{K\cap L})(E) \le (\omega_K + \omega_L)(E)$; that is,

$$\mathcal{C}(K \cup L) + \mathcal{C}(K \cap L) \le \mathcal{C}(K) + \mathcal{C}(L).$$

□

7.6. The Thermal Capacity of More General Sets

We now consider a definition of thermal capacity for sets that are perhaps not compact. The definition involves the notions of inner and outer thermal capacity.

DEFINITION 7.40. If S is an arbitrary subset of E, the *inner thermal capacity* of S is defined by

$$\mathcal{C}_-(S) = \sup\{\mathcal{C}(K) : K \text{ is a compact subset of } S\},$$

and the *outer thermal capacity* of S by

$$\mathcal{C}_+(S) = \inf\{\mathcal{C}_-(D) : D \text{ is an open superset of } S\}.$$

These two set functions take nonnegative, extended real values. If $S \subseteq T \subseteq E$, then $\mathcal{C}_-(S) \leq \mathcal{C}_-(T)$ and $\mathcal{C}_+(S) \leq \mathcal{C}_+(T)$. Furthermore, if K is a compact subset of S and D is an open superset of S, then $K \subseteq D$ so that $\mathcal{C}(K) \leq \mathcal{C}_-(D)$. Taking the supremum over all choices of K, we get $\mathcal{C}_-(S) \leq \mathcal{C}_-(D)$. Now taking the infimum over all choices of D, we obtain $\mathcal{C}_-(S) \leq \mathcal{C}_+(S)$.

DEFINITION 7.41. If $S \subseteq E$ and $\mathcal{C}_-(S) = \mathcal{C}_+(S)$, then S is called *(thermal) capacitable.*

Note that, if S is open and D is an open superset of S, then $\mathcal{C}_-(S) \leq \mathcal{C}_-(D)$, so that $\mathcal{C}_+(S) = \mathcal{C}_-(S)$ and S is capacitable.

LEMMA 7.42. *If K is a compact subset of E, then K is capacitable and*

$$\mathcal{C}_-(K) = \mathcal{C}_+(K) = \mathcal{C}(K).$$

PROOF. Let $\{K_i\}$ be a contracting sequence of compact subsets of E, such that $K \subseteq K_i^\circ$ for all i and $\bigcap_{i=1}^\infty K_i = K$. Then, by Theorem 7.39(b),

$$\mathcal{C}_-(K) \leq \mathcal{C}_+(K) \leq \mathcal{C}_-(K_i^\circ) \leq \mathcal{C}_-(K_i) = \mathcal{C}(K_i) \to \mathcal{C}(K) = \mathcal{C}_-(K)$$

as $i \to \infty$. Therefore $\mathcal{C}_+(K) = \mathcal{C}_-(K) = \mathcal{C}(K)$. □

DEFINITION 7.43. If a subset S of E is capacitable, we write $\mathcal{C}(S)$ for the common value of $\mathcal{C}_+(S)$ and $\mathcal{C}_-(S)$, and call it the *thermal capacity* of S.

Lemma 7.42 shows that this definition of thermal capacity is consistent with the one given earlier for compact sets.

The corresponding notions related to the adjoint equation are called the *inner cothermal capacity*, the *outer cothermal capacity*, the *(cothermal) capacitable* and the *cothermal capacity.* They are denoted with a $\mathcal{C}^*$ rather than a $\mathcal{C}$, and with subscripts if appropriate.

We now consider the properties of thermal capacity for the class of open subsets of E.

THEOREM 7.44. *(a) If $\{U_i\}$ is an expanding sequence of open sets, then*

$$\lim_{i\to\infty} \mathcal{C}(U_i) = \mathcal{C}\left(\bigcup_{i=1}^\infty U_i\right).$$

(b) If U and V are open sets, then

$$\mathcal{C}(U \cup V) + \mathcal{C}(U \cap V) \leq \mathcal{C}(U) + \mathcal{C}(V).$$

(c) If $\{V_i\}$ is an arbitrary sequence of open sets, then

$$\mathcal{C}\left(\bigcup_{i=1}^\infty V_i\right) \leq \sum_{i=1}^\infty \mathcal{C}(V_i).$$

(d) If U is a bounded open set such that $\overline{U} \subseteq E$, then R_1^U is a heat potential whose associated Riesz measure ω_U satisfies $\omega_U(E) = \mathcal{C}(U)$.

Proof. (a) We first note that the sequence $\{\mathcal{C}(U_i)\}$ is increasing, and that $\lim_{i\to\infty}\mathcal{C}(U_i) \le \mathcal{C}(\cup_{i=1}^{\infty}U_i)$. To prove the reverse inequality, we take an arbitrary compact subset K of $\cup_{i=1}^{\infty}U_i$, and note that $K \subseteq U_m$ for some integer m, so that $\mathcal{C}(K) \le \mathcal{C}(U_m) \le \lim_{i\to\infty}\mathcal{C}(U_i)$. Taking the supremum over all choices of K, we obtain $\mathcal{C}(\bigcup_{i=1}^{\infty} U_i) \le \lim_{i\to\infty}\mathcal{C}(U_i)$, as required.

(b) We take any compact subset K of $U \cap V$, any compact subset L of $U \cup V$, choose disjoint open sets C and D such that $L\backslash V \subseteq C \subseteq U$ and $L\backslash U \subseteq D \subseteq V$, and put $L_1 = L\backslash D$ and $L_2 = L\backslash C$. Then $L_1 \subseteq U$, $L_2 \subseteq V$, and $L_1 \cup L_2 = L$. It therefore follows from Theorem 7.39(c) that

$$\begin{aligned}\mathcal{C}(L) + \mathcal{C}(K) &\le \mathcal{C}(K \cup L) + \mathcal{C}(K \cup (L_1 \cap L_2))\\ &= \mathcal{C}((K \cup L_1) \cup (K \cup L_2)) + \mathcal{C}((K \cup L_1) \cap (K \cup L_2))\\ &\le \mathcal{C}(K \cup L_1) + \mathcal{C}(K \cup L_2)\\ &\le \mathcal{C}(U) + \mathcal{C}(V).\end{aligned}$$

Taking the suprema over all choices of K and L, we obtain the result.

(c) It follows from (b) that $\mathcal{C}(V_1 \cup V_2) \le \mathcal{C}(V_1) + \mathcal{C}(V_2)$, and hence, by induction, that

$$\mathcal{C}\left(\bigcup_{i=1}^{m} V_i\right) \le \sum_{i=1}^{m}\mathcal{C}(V_i) \le \sum_{i=1}^{\infty}\mathcal{C}(V_i)$$

for every integer m. Putting $D_m = \bigcup_{i=1}^{m} V_i$ for all m, and using (a), we obtain

$$\mathcal{C}\left(\bigcup_{i=1}^{\infty} V_i\right) = \lim_{i\to\infty}\mathcal{C}(D_m) \le \sum_{i=1}^{\infty}\mathcal{C}(V_i).$$

(d) By Theorem 7.31(a), we have $R_1^U = \widehat{R}_1^U$ on E, so that Theorem 7.28 shows that R_1^U is a heat potential. Now let $\{U_i\}$ be an expanding sequence of bounded open sets such that $\overline{U}_i \subseteq U_{i+1}$ for all i and $\bigcup_{i=1}^{\infty} U_i = U$. Put $K_i = \overline{U}_i$ for all i. Then $\{K_i\}$ is an expanding sequence of sets whose union U is open, so that $\lim_{i\to\infty}\widehat{R}_1^{K_i} = \widehat{R}_1^U$ on E, by Lemma 7.38(a); that is, $\lim_{i\to\infty} G_E\omega_{K_i} = G_E\omega_U$. Therefore, since all these measures have their supports in the compact set $\overline{U}$, Lemma 7.37(b) shows that $\lim_{i\to\infty}\omega_{K_i}(E) = \omega_U(E)$; that is, $\lim_{i\to\infty}\mathcal{C}(K_i) = \omega_U(E)$. If K is any compact subset of U, then there is an integer m such that $K \subseteq K_m$, and hence $\mathcal{C}(K) \le \mathcal{C}(K_m) \le \mathcal{C}(U)$. Taking the supremum over all choices of K, we obtain $\mathcal{C}(U) = \lim_{m\to\infty}\mathcal{C}(K_m) = \omega_U(E)$. □

We now prove an extension of Theorem 7.44 to arbitrary subsets of E. Such sets may not be capacitable, and so the theorem is about outer thermal capacity rather than thermal capacity itself.

Theorem 7.45. *(a) If $\{S_i\}$ is an expanding sequence of sets, then*

$$\lim_{i\to\infty}\mathcal{C}_+(S_i) = \mathcal{C}_+\left(\bigcup_{i=1}^{\infty} S_i\right).$$

(b) For any sets S and T, we have

$$\mathcal{C}_+(S \cup T) + \mathcal{C}_+(S \cap T) \le \mathcal{C}_+(S) + \mathcal{C}_+(T).$$

(c) If $\{T_i\}$ is an arbitrary sequence of sets, then

$$\mathcal{C}_+\left(\bigcup_{i=1}^{\infty} T_i\right) \leq \sum_{i=1}^{\infty} \mathcal{C}_+(T_i).$$

(d) If S is a bounded set such that $\overline{S} \subseteq E$, then $\widehat{R}_1^S$ is a heat potential whose associated Riesz measure ω_S satisfies $\omega_S(E) = \mathcal{C}_+(S)$.

Proof. (a) The conclusion is trivial if $\mathcal{C}_+(S_i) = +\infty$ for some i, so we suppose otherwise. Furthermore, it is obvious that

$$\lim_{i\to\infty} \mathcal{C}_+(S_i) \leq \mathcal{C}_+\left(\bigcup_{i=1}^{\infty} S_i\right),$$

and so we have only to prove the reverse inequality. We begin this by taking any $\epsilon > 0$, and for each positive integer i choosing an open set U_i such that $S_i \subseteq U_i$ and $\mathcal{C}(U_i) < \mathcal{C}_+(S_i) + 2^{-i}\epsilon$.

The proof depends on the inequality

$$\mathcal{C}\left(\bigcup_{i=1}^{m} U_i\right) < \mathcal{C}_+(S_m) + (1 - 2^{-m})\epsilon,$$

for every positive integer m, which we prove by induction. It is obviously true when $m = 1$. Suppose that it holds when $m = k$. Since

$$S_k \subseteq U_k \cap S_{k+1} \subseteq \left(\bigcup_{i=1}^{k} U_i\right) \cap U_{k+1},$$

it follows from Theorem 7.44(b) that

$$\begin{aligned}\mathcal{C}\left(\bigcup_{i=1}^{k+1} U_i\right) + \mathcal{C}_+(S_k) &\leq \mathcal{C}\left(\bigcup_{i=1}^{k+1} U_i\right) + \mathcal{C}\left(\left(\bigcup_{i=1}^{k} U_i\right) \cap U_{k+1}\right) \\ &\leq \mathcal{C}\left(\bigcup_{i=1}^{k} U_i\right) + \mathcal{C}(U_{k+1}) \\ &< \mathcal{C}_+(S_k) + \mathcal{C}_+(S_{k+1}) + (1 - 2^{-k-1})\epsilon.\end{aligned}$$

Since $\mathcal{C}_+(S_k) < +\infty$, we can cancel it to obtain the required inequality when $m = k + 1$. By induction, the inequality holds for all m.

We now make $m \to \infty$, and use Theorem 7.44(a), to obtain

$$\mathcal{C}_+\left(\bigcup_{i=1}^{\infty} S_i\right) \leq \mathcal{C}\left(\bigcup_{i=1}^{\infty} U_i\right) = \lim_{m\to\infty} \mathcal{C}\left(\bigcup_{i=1}^{m} U_i\right) \leq \lim_{m\to\infty} \mathcal{C}_+(S_m) + \epsilon.$$

Since ϵ is arbitrary, we deduce the required reverse inequality, and so the proof of (a) is complete.

(b) Let U and V be open sets such that $S \subseteq U$ and $T \subseteq V$. Then

$$\mathcal{C}_+(S \cup T) + \mathcal{C}_+(S \cap T) \leq \mathcal{C}(U \cup V) + \mathcal{C}(U \cap V) \leq \mathcal{C}(U) + \mathcal{C}(V),$$

by Theorem 7.44(b). Taking the infima over all choices of U and V, we obtain the result.

(c) It follows from (b) that $\mathcal{C}_+(T_1 \cup T_2) \leq \mathcal{C}_+(T_1) + \mathcal{C}_+(T_2)$, and hence, by induction, that

$$\mathcal{C}_+\left(\bigcup_{i=1}^{m} T_i\right) \leq \sum_{i=1}^{m} \mathcal{C}_+(T_i) \leq \sum_{i=1}^{\infty} \mathcal{C}_+(T_i)$$

for every integer m. Using (a), we deduce that

$$\mathcal{C}_+\left(\bigcup_{i=1}^{\infty} T_i\right) = \lim_{i\to\infty} \mathcal{C}_+\left(\bigcup_{i=1}^{m} T_i\right) \leq \sum_{i=1}^{\infty} \mathcal{C}_+(T_i).$$

(d) Theorem 7.28 shows that $\widehat{R}_1^S$ is a heat potential. Let $0 < \epsilon < 1$, and let U be a bounded open set such that $S \subseteq U$, $\overline{U} \subseteq E$, and $\mathcal{C}(U) < \mathcal{C}_+(S) + \epsilon$. Let $\mathcal{F}$ denote the family of nonnegative supertemperatures v on E that satisfy the condition $v \geq 1$ on S, so that $R_1^S = \inf \mathcal{F}$. By Lemma 7.12, there is a sequence $\{u_k\}$ of functions in $\mathcal{F}$ such that, if $u = \inf\{u_k : k \in \mathbb{N}\}$ then $\widehat{u} = \widehat{R}_1^S$. We can take $u_1 = 1$. We now put

$$v_k = u_1 \wedge u_2 \wedge \ldots \wedge u_k, \qquad w_k = R_{v_k}^U, \qquad U_k = \{p \in U : w_k(p) > 1 - \epsilon\},$$

and note that $v_k, w_k \in \mathcal{F}$ and $U_k \supseteq S$, for all k. Moreover, the sequences $\{v_k\}$ and $\{w_k\}$ are decreasing, with $\lim_{k\to\infty} v_k = u$ on E. Each function w_k is a heat potential, by Theorems 7.31(a) and 7.28, and is a temperature on $E\backslash\overline{U}$ by Theorem 7.27(d). Since $v_k \geq 1$ on S and $S \subseteq U$, we have $R_1^S \leq R_{v_k}^U = w_k \leq v_k$ on E, and hence

$$\widehat{R}_1^S \leq \lim_{k\to\infty} w_k \leq \lim_{k\to\infty} v_k = u$$

on E. By the Harnack monotone convergence theorem, the function $w = \lim_{k\to\infty} w_k$ is a temperature on $E\backslash\overline{U}$, and therefore $w = \widehat{w}$ there. Hence

$$\widehat{R}_1^S \leq w = \widehat{w} \leq \widehat{u} = \widehat{R}_1^S,$$

and so $w = \widehat{R}_1^S$, on $E\backslash\overline{U}$.

Furthermore, since the sequence $\{w_k\}$ is decreasing, the sequence of sets $\{U_k\}$ is contracting, and so the sequence of reductions $\{R_1^{U_k}\}$ is decreasing. Since each w_k is lower semicontinuous on E, each set U_k is open. Hence, using Theorems 7.31(a), 7.28 and 6.25, the sequence $\{R_1^{U_k}\} = \{G_E\omega_{U_k}\}$ is a decreasing sequence of heat potentials of measures supported in the compact set $\overline{U}$. Therefore Lemma 7.37(a) shows that the sequence $\{\omega_{U_k}(E)\}$ is decreasing. Since $w_k = R_{v_k}^U$ is a heat potential $G_E\nu_k$ and a temperature on $E\backslash\overline{U}$, the measure ν_k has its support in $\overline{U}$. Moreover, since w is a temperature on $E\backslash\overline{U}$, and $w = \widehat{R}_1^S = G_E\omega_S$ there, the measure ω_S also has its support in $\overline{U}$. It therefore follows from Lemma 7.37(b) that $\nu_k(E) \to \omega_S(E)$ as $k \to \infty$.

Because $R_1^U \geq \widehat{R}_1^S$, Lemma 7.37(a) implies that $\omega_U(E) \geq \omega_S(E)$. Furthermore, the definition of U_k implies that $w_k \geq (1-\epsilon)R_1^{U_k}$ on E; that is,

$$G_E\nu_k \geq (1-\epsilon)G_E\omega_{U_k}.$$

So it follows from Lemma 7.37(a) that $\nu_k(E) \geq (1-\epsilon)\omega_{U_k}(E)$, and hence that

$$\omega_S(E) = \lim_{k\to\infty} \nu_k(E) \geq (1-\epsilon) \lim_{k\to\infty} \omega_{U_k}(E) = (1-\epsilon) \lim_{k\to\infty} \mathcal{C}(U_k),$$

by Theorem 7.44(d). It now follows, using Theorem 7.44(d) again, that

$$\mathcal{C}_+(S) + \epsilon > \mathcal{C}(U) = \omega_U(E) \geq \omega_S(E) \geq (1-\epsilon) \lim_{k\to\infty} \mathcal{C}(U_k) \geq (1-\epsilon)\mathcal{C}_+(S).$$

Making $\epsilon \to 0+$, we obtain $\mathcal{C}_+(S) = \omega_S(E)$, as required. $\square$

7.7. Thermal and Cothermal Capacities

In this section, we use Theorem 7.36 to show that the thermal and cothermal capacities coincide, and that the polar sets are the same as the copolar sets.

THEOREM 7.46. *If S is an arbitrary subset of E, then:*
(a) $\mathcal{C}_-(S) = \mathcal{C}^*_-(S)$ and $\mathcal{C}_+(S) = \mathcal{C}^*_+(S)$,
(b) S is polar if and only if $\mathcal{C}(S) = 0$,
and
(c) S is polar if and only if S is copolar.

PROOF. (a) Let K be a compact subset of E, and let U be any bounded open superset of K such that $\overline{U} \subseteq E$. By Theorem 7.44(d), R_1^U is a heat potential whose associated Riesz measure ω_U satisfies $\omega_U(E) = \mathcal{C}(U)$. Therefore, because $\widehat{R}_1^{K*} = G_E^*\omega_K^* \leq 1$ on E, we have

$$\mathcal{C}(U) = \omega_U(E) \geq \int_E G_E^*\omega_K^* \, d\omega_U = \int_K G_E\omega_U \, d\omega_K^* = \omega_K^*(K) = \mathcal{C}^*(K)$$

by Theorem 7.36, since ω_K^* is supported in K and $G_E\omega_U = R_1^U = 1$ on $U \supseteq K$. Taking the infimum over all choices of U and using Lemma 7.42, we obtain the inequality $\mathcal{C}(K) \geq \mathcal{C}^*(K)$. The dual of this result is the reverse inequality, and so equality holds. It now follows that $\mathcal{C}_-(S) = \mathcal{C}^*_-(S)$ for any S, and then that $\mathcal{C}_+(S) = \mathcal{C}^*_+(S)$.

(b) Let $\{U_i\}$ be a sequence of bounded open sets, such that $\overline{U}_i \subseteq E$ for all i and $\bigcup_{i=1}^{\infty} U_i = E$. By Theorem 7.24, each set $S \cap U_i$ is polar if and only if $\widehat{R}_1^{S\cap U_i} = 0$, which occurs if and only if the associated Riesz measure $\omega_{S\cap U_i}$ satisfies $\omega_{S\cap U_i}(E) = 0$. By Theorem 7.45(d), we have $\omega_{S\cap U_i}(E) = \mathcal{C}_+(S\cap U_i)$, and so $S\cap U_i$ is polar if and only if $\mathcal{C}_+(S \cap U_i) = 0$.

It follows that, if S is polar then

$$\mathcal{C}_+(S) = \mathcal{C}_+\left(\bigcup_{i=1}^{\infty}(S \cap U_i)\right) \leq \sum_{i=1}^{\infty} \mathcal{C}_+(S \cap U_i) = 0,$$

by Theorem 7.45(c). Conversely, if $\mathcal{C}_+(S) = 0$ then $\mathcal{C}_+(S \cap U_i) = 0$ for all i, so that $S \cap U_i$ is polar for all i, and so S is polar by Theorem 7.4.

(c) This follows from (b), (a), and the dual of (b). $\square$

REMARK 7.47. It follows from Theorem 7.46(a) and Example 7.35 that, relative to $E = \mathbb{R}^{n+1}$, the *thermal* capacity of the closed heat ball $\overline{\Omega}(0; c)$, and the heat sphere $\partial\Omega(0; c)$, is $(4\pi c)^{n/2}$.

7.8. Capacitable Sets

We already know that the compact sets, the open sets, and the polar sets are capacitable. In this section, we prove that the collection of capacitable sets is very large, and includes all the Borel sets. This involves the concept of an analytic set, which we define after introducing some notation.

We use the standard notation Y^X for the set of all functions from X into Y. Thus $\mathbb{N}^{\mathbb{N}}$ denotes the collection of all sequences of positive integers. This should not be confused with $\bigcup_{k=1}^{\infty} \mathbb{N}^k$, which is the collection of all *finite* sequences of positive integers. We also denote by $\mathcal{K}$ the collection of all compact subsets of $\mathbb{R}^{n+1}$.

DEFINITION 7.48. A subset A of $\mathbb{R}^{n+1}$ is called *analytic* if there exists a mapping $\phi : \bigcup_{k=1}^{\infty} \mathbb{N}^k \to \mathcal{K}$ such that

$$A = \bigcup_{\{m_i\}\in\mathbb{N}^{\mathbb{N}}} \big(\phi(m_1) \cap \phi(m_1, m_2) \cap \phi(m_1, m_2, m_3) \cap ...\big). \tag{7.8}$$

We denote by $\mathcal{A}$ the collection of all analytic subsets of $\mathbb{R}^{n+1}$.

LEMMA 7.49. *(a) If $\{A_j\}$ is a sequence of analytic sets, then $\bigcup_{j=1}^{\infty} A_j$ and $\bigcap_{j=1}^{\infty} A_j$ are also analytic.*

(b) Every Borel set is analytic.

(c) If A is an analytic subset of E, then the compact sets $\phi(m_1, ..., m_k)$ in (7.8) *can be chosen to be subsets of E.*

PROOF. (a) For each j, there is a mapping $\phi_j : \bigcup_{k=1}^{\infty} \mathbb{N}^k \to \mathcal{K}$ such that

$$A_j = \bigcup_{\{m_i\}\in\mathbb{N}^{\mathbb{N}}} \big(\phi_j(m_1) \cap \phi_j(m_1, m_2) \cap \phi_j(m_1, m_2, m_3) \cap ...\big).$$

Let the mapping $k \mapsto (\alpha(k), \beta(k))$ be a bijection from $\mathbb{N}$ to $\mathbb{N}^2$. If we define the map $\phi : \bigcup_{k=1}^{\infty} \mathbb{N}^k \to \mathcal{K}$ by putting

$$\phi(m_1, ..., m_k) = \phi_{\alpha(m_1)}(\beta(m_1), m_2, m_3, ..., m_k),$$

then we can write A_j as

$$A_j = \bigcup_{\{\{m_i\}:\alpha(m_1)=j\}} \big(\phi(m_1) \cap \phi(m_1, m_2) \cap \phi(m_1, m_2, m_3) \cap ...\big).$$

We can now write $\bigcup_{j=1}^{\infty} A_j$ as

$$\bigcup_{j=1}^{\infty} A_j = \bigcup_{\{m_i\}\in\mathbb{N}^{\mathbb{N}}} \big(\phi(m_1) \cap \phi(m_1, m_2) \cap \phi(m_1, m_2, m_3) \cap ...\big),$$

which shows that the union is analytic.

We now show that the intersection is also analytic. A point p belongs to $\bigcap_{j=1}^{\infty} A_j$ if and only if, for each $j \in \mathbb{N}$ there is an element $\{m_i^{(j)}\}$ of $\mathbb{N}^{\mathbb{N}}$ such that

$$p \in \phi_j(m_1^{(j)}) \cap \phi_j(m_1^{(j)}, m_2^{(j)}) \cap \phi_j(m_1^{(j)}, m_2^{(j)}, m_3^{(j)}) \cap \dots .$$

That is, if and only if there is a function $f : \mathbb{N}^2 \to \mathbb{N}$ such that

$$p \in \bigcap_{j=1}^{\infty} \big(\phi_j(f(j,1)) \cap \phi_j(f(j,1), f(j,2)) \cap \phi_j(f(j,1), f(j,2), f(j,3)) \cap \dots\big).$$

Given any element $\{m_k\}$ of $\mathbb{N}^{\mathbb{N}}$, we let $E(j,i)$ denote the (j,i) entry of the infinite matrix

$$\begin{matrix} m_1 & m_2 & m_4 & m_7 & \dots \\ m_3 & m_5 & m_8 & \dots & \dots \\ m_6 & m_9 & \dots & \dots & \dots. \\ m_{10} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{matrix}$$

We then define $\phi(m_1, ..., m_k)$ by looking at the position of m_k in the above matrix. If $m_k = E(j,i)$, we define

$$\phi(m_1, ..., m_k) = \phi_j(f(j,1), \ldots, f(j,i)).$$

Then

$$\bigcap_{j=1}^{\infty} \big(\phi_j(f(j,1)) \cap \phi_j(f(j,1), f(j,2)) \cap \phi_j(f(j,1), f(j,2), f(j,3)) \cap \ldots\big)$$
$$= \bigcap_{k=1}^{\infty} \phi(m_1, ..., m_k).$$

Thus

$$\bigcap_{j=1}^{\infty} A_j = \bigcup_{\{m_k\}\in\mathbb{N}^{\mathbb{N}}} \big(\phi(m_1) \cap \phi(m_1, m_2) \cap \phi(m_1, m_2, m_3) \cap \ldots\big),$$

as required.

(b) Any compact set K is analytic, because it can be written in the form (7.8) by taking $\phi(m_1, ..., m_k) = K$ for any choice of $(m_1, ..., m_k)$. Since any open or closed set can be written as the union of a sequence of compact sets, it follows from part (a) that such sets are also analytic. We now consider $\mathcal{C}$, the collection of all analytic sets A such that $\mathbb{R}^{n+1}\backslash A$ is also analytic. If $\{A_k\}$ is a sequence of sets in $\mathcal{C}$, then part (a) shows that $\bigcup_{k=1}^{\infty} A_k$ is analytic, and that

$$\mathbb{R}^{n+1}\backslash \bigcup_{k=1}^{\infty} A_k = \bigcap_{k=1}^{\infty} \big(\mathbb{R}^{n+1}\backslash A_k\big)$$

is too. Hence $\bigcup_{k=1}^{\infty} A_k \in \mathcal{C}$, and so $\mathcal{C}$ is a σ-algebra that contains the open sets. Therefore $\mathcal{C}$, and hence $\mathcal{A}$, contains the Borel sets.

(c) Let A be an analytic subset of E. Since A is analytic, there is a mapping $\psi : \bigcup_{k=1}^{\infty} \mathbb{N}_k \to \mathcal{K}$ such that

$$A = \bigcup_{\{m_i\}\in\mathbb{N}^{\mathbb{N}}} \big(\psi(m_1) \cap \psi(m_1, m_2) \cap \psi(m_1, m_2, m_3) \cap ...\big).$$

Let $\{K_j\}$ be a sequence of compact sets with union E. Given any $\{m_i\} \in \mathbb{N}^{\mathbb{N}}$, we put $\phi(m_1) = K_{m_1}$ and $\phi(m_1, \ldots, m_k) = K_{m_1} \cap \psi(m_2, \ldots, m_k)$ whenever $k \geq 2$. Then each $\phi(m_1, \ldots, m_k)$ is a compact subset of E, and

$$\phi(m_1) \cap \phi(m_1, m_2) \cap \phi(m_1, m_2, m_3) \cap \ldots$$
$$= K_{m_1} \cap \psi(m_2) \cap \psi(m_2, m_3) \cap \psi(m_2, m_3, m_4) \cap \ldots,$$

so that

$$\bigcup_{\{m_i\}\in\mathbb{N}^{\mathbb{N}}} \big(\phi(m_1) \cap \phi(m_1, m_2) \cap \phi(m_1, m_2, m_3) \cap \ldots\big)$$
$$= \bigcup_{j=1}^{\infty} K_j \cap \bigcup_{\{m_i\}\in\mathbb{N}^{\mathbb{N}}} \big(\psi(m_2) \cap \psi(m_2, m_3) \cap \psi(m_2, m_3, m_4) \cap \ldots\big)$$
$$= E \cap A$$
$$= A.$$

□

LEMMA 7.50. *Let A be given by (7.8), and let $\{n_i\} \in \mathbb{N}^{\mathbb{N}}$. For all $k \in \mathbb{N}$, we define*

$$S_k = \bigcup_{\{\{m_i\} \in \mathbb{N}^{\mathbb{N}} : m_i \leq n_i \text{ } if \, i \leq k\}} \big(\phi(m_1) \cap \phi(m_1, m_2) \cap \phi(m_1, m_2, m_3) \cap \dots\big)$$

and

$$T_k = \bigcup_{\{\{m_i\} \in \mathbb{N}^{k} : m_i \leq n_i \text{ } if \, i \leq k\}} \big(\phi(m_1) \cap \dots \cap \phi(m_1, \dots, m_k)\big).$$

Then

(a) $S_k \subseteq A \cap T_k$ for all k, and the sequence $\{S_k\}$ is contracting.

(b) $\{T_k\}$ is a contracting sequence of compact sets whose intersection is a subset of A.

PROOF. It is clear that (a) holds. Furthermore, because each set T_k is a union of finitely many compact sets, each T_k is compact. It is also clear that $\{T_k\}$ is contracting, and so it remains only to prove that its intersection T is contained in A.

Let $p \in T$. Then for each $k \in \mathbb{N}$, there is an element $(m_1^{(k)}, \dots, m_k^{(k)})$ of $\mathbb{N}^k$ such that $m_i^{(k)} \leq n_i$ whenever $i \leq k$ and such that

$$p \in \phi(m_1^{(k)}) \cap \dots \cap \phi(m_1^{(k)}, \dots, \phi(m_k^{(k)}).$$

Since $1 \leq m_1^{(k)} \leq n_1$ for all $k \in \mathbb{N}$, there is an integer $m_1' \leq n_1$ such that $m_1^{(k)} = m_1'$ for infinitely many values of k. Since $1 \leq m_2^{(k)} \leq n_2$ for all those values of k for which $m_1^{(k)} = m_1'$, we similarly deduce that there is an integer $m_2' \leq n_2$ such that $(m_1^{(k)}, m_2^{(k)}) = (m_1', m_2')$ for infinitely many values of k. Continuing in this manner indefinitely, we obtain a sequence $\{m_i'\}$ such that

$$p \in \phi(m_1') \cap \phi(m_1', m_2') \cap \phi(m_1', m_2', m_3') \cap \dots \subseteq A.$$

Hence $T \subseteq A$, and the proof is complete. $\square$

THEOREM 7.51. *Every analytic subset of E is (thermal) capacitable.*

PROOF. Let A be an analytic subset of E. Then, by Lemma 7.49(c), we can write A in the form (7.8) with the compact sets $\phi(m_1, \dots, m_k)$ contained in E. We choose any number $\alpha < \mathcal{C}_+(A)$, and define a sequence $\{n_i\} \in \mathbb{N}^{\mathbb{N}}$ inductively, as follows.

At stage 1 of the process, we define the sequence of sets $\{I_{1,j}\}$ by putting

$$I_{1,j} = \{\{m_i\} \in \mathbb{N}^{\mathbb{N}} : m_1 \leq j\}$$

for all j. The sequence is expanding with union $\mathbb{N}^{\mathbb{N}}$. Hence the sequence of sets $\{A_{1,j}\}$, where

$$A_{1,j} = \bigcup_{\{m_i\} \in I_{1,j}} \big(\phi(m_1) \cap \phi(m_1, m_2) \cap \phi(m_1, m_2, m_3) \cap \dots\big)$$

for all j, is also expanding, and its union is A. It therefore follows from Theorem 7.45(a) that we can choose an integer n_1 such that $\mathcal{C}_+(A_{1,n_1}) > \alpha$.

At stage k+1 of the process, we have integers $n_1, ..., n_k$ and sets

$$I_{k,n_k} = \{\{m_i\} \in \mathbb{N}^{\mathbb{N}} : m_1 \leq n_1, ..., m_k \leq n_k\}$$

such that the set

$$A_{k,n_k} = \bigcup_{\{m_i\}\in I_{k,n_k}} \big(\phi(m_1)\cap\phi(m_1,m_2)\cap\phi(m_1,m_2,m_3)\cap\dots\big)$$

satisfies $\mathcal{C}_+(A_{k,n_k}) > \alpha$. The sequence of sets $\{I_{k+1,j}\}$, defined by

$$I_{k+1,j} = \{\{m_i\}\in\mathbb{N}^{\mathbb{N}} : m_1\le n_1,\dots,m_k\le n_k, m_{k+1}\le j\}$$

for all j, is expanding with union I_{k,n_k}. Therefore the sequence of sets $\{A_{k+1,j}\}$, defined by

$$A_{k+1,j} = \bigcup_{\{m_i\}\in I_{k+1,j}} \big(\phi(m_1)\cap\phi(m_1,m_2)\cap\phi(m_1,m_2,m_3)\cap\dots\big)$$

for all j, is also expanding, and its union is A_{k,n_k}. It follows that we can find an integer n_{k+1} such that $\mathcal{C}_+(A_{k+1,n_{k+1}}) > \alpha$.

The sets A_{k,n_k} are the sets S_k of Lemma 7.49, so that $\mathcal{C}_+(S_k) > \alpha$ for all k. Having chosen the sequence $\{n_k\}$ in this way, we use it to define the sets T_k as in Lemma 7.49. By that result, $\{T_k\}$ is a contracting sequence of compact sets whose intersection T is contained in A, and $S_k\subseteq T_k$ for all k. Therefore, by Theorem 7.39(b),

$$\mathcal{C}_-(A)\ge\mathcal{C}(T)=\lim_{k\to\infty}\mathcal{C}(T_k)\ge\lim_{k\to\infty}\mathcal{C}_+(S_k)\ge\alpha.$$

Thus $\mathcal{C}_-(A)\ge\alpha$ for any number $\alpha<\mathcal{C}_+(A)$, so that $\mathcal{C}_+(A)\le\mathcal{C}_-(A)$. The reverse inequality is always true, and so A is capacitable. □

7.9. Polar Sets and Heat Potentials

The first theorem in this section characterizes the polar Borel sets as those that cannot support a nontrivial, bounded heat potential. Part of this result generalizes Theorem 7.29 from relatively closed sets to Borel sets.

THEOREM 7.52. *If S is a Borel subset of E, then the following statements are equivalent.*

(a) S is polar.

(b) If μ is a nonnegative measure on E such that the heat potential $G_E\mu$ is bounded, then $\mu(S)=0$.

PROOF. Suppose first that S is polar, and let μ be a nonnegative measure on E such that $G_E\mu$ is bounded. By Theorem 7.46, S is also copolar. Therefore, by the dual of Theorem 7.3, there is a coheat potential $G_E^*\nu^*$ such that $G_E^*\nu^*(p)=+\infty$ for all $p\in S$, and $\nu^*(E)<+\infty$. By Theorem 7.36,

$$\int_E G_E^*\nu^*\,d\mu=\int_E G_E\mu\,d\nu^*<+\infty,$$

because $G_E\mu$ is bounded and $\nu^*(E)<+\infty$. Since $G_E^*\nu^*(p)=+\infty$ for all $p\in S$, this implies that $\mu(S)=0$.

Now suppose, conversely, that S is not polar. By Lemma 7.49(b) and Theorem 7.51, the Borel sets are capacitable, so that $\mathcal{C}_-(S)=\mathcal{C}(S)>0$ by Theorem 7.46. Therefore there exists a compact subset K of S such that $\mathcal{C}(K)>0$. The thermal capacitary potential $\widehat{R}_1^K=G_E\omega_K$ of K is bounded on E. Moreover, the thermal capacitary distribution ω_K has its support in $K\subseteq S$ and $\omega_K(E)=\mathcal{C}(K)>0$. Thus ω_K is a nonnegative measure on E such that $G_E\omega_K$ is bounded and $\omega_K(S)>0$. □

If Z is a polar subset of E and $p_0 \in E\backslash Z$, then we know from Theorem 7.3 that there is a heat potential $G_E\mu$ such that $G_E\mu(p) = +\infty$ for all $p \in Z$ and $G_E\mu(p_0) < +\infty$. Our next result shows that, if Z is a *compact* polar subset of E, then μ can be chosen such that $G_E\mu(p) < +\infty$ for *all* $p \in E\backslash Z$.

THEOREM 7.53. *If Z is a compact polar subset of E, then there is a nonnegative measure μ on E such that $G_E\mu(p) = +\infty$ for all $p \in Z$ and $G_E\mu(p) < +\infty$ for all $p \in E\backslash Z$.*

PROOF. We take a contracting sequence of bounded open sets $\{V_j\}$ such that $\overline{V}_{j+1} \subseteq V_j$ and $\overline{V}_j \subseteq E$ for all j, and $\bigcap_{j=1}^{\infty} V_j = Z$. We put $K_j = \overline{V}_j$ and $v_j = \widehat{R}_1^{K_j}$ for all j. Then $\{v_j\}$ is a decreasing sequence of heat potentials, by Theorem 7.28, such that $v_j(p) = 1$ for all $p \in Z$ and v_j is a temperature on $E\backslash K_j$, by Theorem 7.27(d). We put $v = \lim_{j\to\infty} v_j$ on E.

Given any integer k, the functions v_j for $j \geq k$ are all temperatures on $E\backslash K_k$, and so v is a temperature on $E\backslash K_k$ by the Harnack monotone convergence theorem. Since k is arbitrary, v is a temperature on $E\backslash Z$. Because Z is a closed polar set and $0 \leq v \leq 1$ on E, the restriction of v to $E\backslash Z$ has a unique extension to a temperature $\bar{v}$ on E, by Corollary 7.15. Moreover, $0 \leq \bar{v} \leq v_1$ on E, and v_1 is a heat potential, so it follows from Theorem 6.19 that $\bar{v} = 0$ on E. Hence $v = 0$ on $E\backslash Z$.

We now take an expanding sequence of bounded open sets $\{D_i\}$ such that $V_1 \subseteq D_1$, $\overline{D}_i \subseteq E$ for all i, and $\bigcup_{i=1}^{\infty} D_i = E$. Given any integer l, the heat potentials v_j for $j > l$ are temperatures on $E\backslash\overline{V}_j$, and hence are continuous on the compact set $\overline{D}_l\backslash V_l$. The sequence $\{v_j\}$ decreases to the constant 0, and so Dini's theorem implies that the convergence is uniform on $\overline{D}_l\backslash V_l$. Hence there is an integer $j_l \geq l$ such that $v_{j_l} < 2^{-l}$ on $\overline{D}_l\backslash V_l$. We now put $w = \sum_{l=1}^{\infty} v_{j_l}$, and note that if $p \in Z$ then $w(p) = +\infty$ because $v_{j_l}(p) = 1$ for all l. If $q \in E\backslash Z$, then there is an integer m such that $q \in \overline{D}_l\backslash V_l$ for all $l \geq m$. Hence

$$w(q) = \sum_{l=1}^{m-1} v_{j_l}(q) + \sum_{l=m}^{\infty} v_{j_l}(q) \leq (m-1) + \sum_{l=m}^{\infty} 2^{-l} < +\infty.$$

Therefore w is a nonnegative supertemperature on E, by Theorem 3.60, and is finite on $E\backslash Z$. By the Riesz decomposition theorem, $G_E\mu_w$ is a heat potential and $w = G_E\mu_w + h$ on E, where h is the greatest thermic minorant of w on E. Thus $G_E\mu_w(p) = +\infty$ if and only if $p \in Z$. □

7.10. Thermal Capacity and Lebesgue Measure

In this section, we replace the arbitrary open set E by a strip $E_s = \mathbb{R}^n \times]a, b[$, where $-\infty \leq a < b \leq +\infty$. All reductions, and hence the thermal capacity, are relative to E_s. By Example 6.2, the Green function for E_s is G.

The main result is that, for analytic subsets of a hyperplane of the form $\mathbb{R}^n \times \{c\}$, the thermal capacity is the same as the n-dimensional Lebesgue measure.

We denote by $M^+(K)$ the class of nonnegative measures on E_s which have their support in the compact set K. We know that $\widehat{R}_1^K = G\omega_K \leq 1$ on E_s, and that $\omega_K \in M^+(K)$. The next lemma shows that $G\omega_K$ is the largest such heat potential.

LEMMA 7.54. *For any compact subset K of E_s, we have*

$$G\omega_K = \sup\{G\mu : G\mu \le 1 \text{ on } E_s,\ \mu \in M^+(K)\}.$$

PROOF. Let K be a compact subset of E_s, and let $\mu \in M^+(K)$ and satisfy $G\mu \le 1$ on E_s.

We claim that $G\mu(x,t) \to 0$ as (x,t) tends to the point at infinity in such a way that t remains upper bounded. Clearly $G\mu(x,t) = 0$ if $t < s$ for every point $(y,s) \in K$. Let $D = \{(y,s) \in E_s : |y| < \rho,\ c < s < d\}$ be a circular cylinder that contains K. There is a constant C such that

$$G\mu(x,t) = \int_D W(x-y,t-s)\,d\mu(y,s) \le C\int_D |x-y|^{-n}\,d\mu(y,s).$$

If $|x| > \rho + r$, then $|x-y| > r$ whenever $(y,s) \in D$, and so it follows that

$$G\mu(x,t) \le Cr^{-n}\mu(D) \to 0 \text{ as } r \to +\infty.$$

We now take any nonnegative supertemperature v on E_s such that $v \ge 1$ on K. Then $v - G\mu \ge 0$ on K, and we need to show that this inequality is also true on $E_s\backslash K$. For all points $q \in \partial K$, we have

$$\liminf_{p\to q}\big(v(p) - G\mu(p)\big) \ge \liminf_{p\to q} v(p) - 1 \ge v(q) - 1 \ge 0.$$

Furthermore, as $p = (x,t)$ tends to the point at infinity in such a way that t remains upper bounded, we have

$$\liminf\big(v(p) - G\mu(p)\big) \ge \liminf v(p) - \lim G\mu(p) \ge 0.$$

The function $v - G\mu$ is a supertemperature on $E_s\backslash K$, because $G\mu$ is a temperature there (by Corollary 6.22). It therefore follows from the minimum principle that $v - G\mu \ge 0$ on $E_s\backslash K$. Hence $v \ge G\mu$ on E_s, and it follows that $R_1^K \ge G\mu$. Since $G\mu$ is lower semicontinuous, we obtain $G\omega_K = \widehat{R}_1^K \ge G\mu$, as required. $\square$

THEOREM 7.55. *Let A be an analytic subset of $\mathbb{R}^n$, let $c \in]a,b[$, and let m_n denote n-dimensional Lebesgue measure on $\mathbb{R}^n$. Then $\mathcal{C}(A\times\{c\}) = m_n(A)$, and if A is compact then the thermal capacitary distribution $\omega_{A\times\{c\}}$ is the product of the restriction of m_n to A with the unit mass at c.*

PROOF. Let K be a compact subset of $\mathbb{R}^n$, and let $G\omega_{K\times\{c\}}$ denote the thermal capacitary potential of $K\times\{c\}$. Then $\omega_{K\times\{c\}}$ is supported by $K\times\{c\}$, so that we can write $\omega_{K\times\{c\}} = \nu_K \times \delta_c$, where ν_K is supported by K and δ_c is the unit mass at c. Hence

$$G\omega_{K\times\{c\}}(x,t) = \int_{K\times\{c\}} W(x-y,t-s)\,d\omega_{K\times\{c\}}(y,s) = \int_K W(x-y,t-c)\,d\nu_K(y)$$

if $c < t < b$, and $G\omega_{K\times\{c\}}(x,t) = 0$ if $a < t \le c$. In particular, $G\omega_{K\times\{c\}}$ is the Gauss-Weierstrass integral of ν_K on $\mathbb{R}^n\times]c,b[$. Therefore, since $0 \le G\omega_{K\times\{c\}} \le 1$ on $\mathbb{R}^n\times]c,b[$, it follows from Corollary 5.35 that there is a function f on $\mathbb{R}^n\times\{c\}$ such that $|f| \le 1$ and

$$G\omega_{K\times\{c\}}(x,t) = \int_{\mathbb{R}^n} W(x-y,t-c)f(y,c)\,dy$$

for all $(x,t) \in \mathbb{R}^n\times]c,b[$. Gauss-Weierstrass integrals have unique representing measures, by Theorem 4.11, and hence $f(y,c)\,dy = d\nu_K(y)$. In particular, f is supported by K, so that

$$G\omega_{K\times\{c\}}(x,t) \le \int_K W(x-y,t-c)\,dy \tag{7.9}$$

on $\mathbb{R}^n\times\,]c,b[$. By Lemma 7.54,

$$G\omega_{K\times\{c\}} = \sup\{G\mu : G\mu \le 1 \text{ on } E_s,\ \mu\in M^+(K\times\{c\})\},$$

so that equality holds in (7.9). Therefore the product of the restriction of m_n to K with δ_c is the thermal capacitary distribution of $K\times\{c\}$, and $\mathcal{C}(K\times\{c\}) = m_n(K)$. Now the inner regularity of Lebesgue measure implies that, for any analytic subset A of $\mathbb{R}^n$, we have

$$m_n(A) = \sup\{m_n(K) : K \text{ is a compact subset of } A\} = \mathcal{C}_-(A\times\{c\}),$$

and the result follows because $A\times\{c\}$ is thermal capacitable (Theorem 7.51). □

It is important to realize that the result of Theorem 7.55 does *not* extend to subsets of more than one hyperplane of the form $\mathbb{R}^n\times\{c\}$, and that although Lebesgue measure is additive, thermal capacity is only strongly subbadditive. The following example illustrates these points.

EXAMPLE 7.56. Let A and B be compact subsets of $\mathbb{R}^n$ with positive Lebesgue measure, let $-\infty < a < b < +\infty$, and let $K = (A\times\{a\})\cup(B\times\{b\})$. Then K is compact, and its thermal capacitary distribution ω_K (relative to $\mathbb{R}^{n+1}$) is supported by K, so that we can write $\omega_K = (\mu_A\times\delta_a)+(\mu_B\times\delta_b)$, where δ_c denotes the unit mass at $c\in\{a,b\}$. Therefore $G\omega_K(x,t)=0$ if $t\le a$,

$$G\omega_K(x,t) = \int_A W(x-y,t-a)\,d\mu_A(y)$$

if $a<t\le b$, and

$$G\omega_K(x,t) = \int_A W(x-y,t-a)\,d\mu_A(y) + \int_B W(x-y,t-b)\,d\mu_B(y)$$

if $t>b$.

We consider first $G\omega_K$ on the strip $\mathbb{R}^n\times\,]a,b]$. Since $0\le G\omega_K\le 1$, it follows from Corollary 5.35 that there is a function f on $\mathbb{R}^n\times\{a\}$ such that $|f|\le 1$ and

$$G\omega_K(x,t) = \int_{\mathbb{R}^n} W(x-y,t-a)f(y,a)\,dy$$

for all $(x,t)\in\mathbb{R}^n\times\,]a,b[$. The uniqueness of the Gauss-Weierstrass representation (Theorem 4.11) shows that $f(y,a)\,dy = d\mu_A(y)$, so that f is supported by A and

$$G\omega_K(x,t) \le \int_A W(x-y,t-a)\,dy$$

on $\mathbb{R}^n\times\,]a,b[$. By Theorem 2.2, this integral represents a continuous function on $\mathbb{R}^n\times\,]a,+\infty[$, and it therefore follows from Lemma 3.16 that this inequality holds on $\mathbb{R}^n\times\,]a,b]$. Moreover Lemma 7.37(a), with $E=\mathbb{R}^n\times\,]-\infty,b[$, shows that $\mu_A(A)=\omega_K(A\times\{a\})\le m_n(A)$.

We now put

$$g(x,b) = \int_A W(x-y,b-a)\,d\mu_A(y)$$

for all $x \in \mathbb{R}^n$, so that

$$G\omega_K(x,t) = \int_{\mathbb{R}^n} W(x-y,t-b)g(y,b)\,dy + \int_B W(x-y,t-b)\,d\mu_B(y)$$

if $t > b$, in view of Theorem 4.10. Since $0 \le G\omega_K \le 1$, Corollary 5.35 shows that there is a function h on $\mathbb{R}^n \times \{b\}$ such that $|h| \le 1$ and

$$G\omega_K(x,t) = \int_{\mathbb{R}^n} W(x-y,t-b)h(y,b)\,dy$$

whenever $t > b$. By Theorem 4.11 the representing measure is unique, so that $d\mu_B(y) = (h(y,b) - g(y,b))\,dy$. Therefore $h - g$ is supported by $B \times \{b\}$, and in particular $h(\cdot,b) = g(\cdot,b)$ on $\mathbb{R}^n \backslash B$. Moreover, since $h - g \le 1 - g$ on $B \times \{b\}$, we have $d\mu_B(y) \le (1 - g(y,b))\chi_B(y)\,dy$. Since $m_n(A) > 0$ we have $g > 0$, so that since $m_n(B) > 0$ it follows that $\mu_B(B) < m_n(B)$. Hence

$$\begin{aligned}\mathcal{C}((A \times \{a\}) \cup (B \times \{b\})) &= \mu_A(A) + \mu_B(B) \\ &< m_n(A) + m_n(B) = \mathcal{C}(A \times \{a\}) + \mathcal{C}(B \times \{b\}).\end{aligned}$$

Using Theorem 7.55 and the equivalence of polarity and zero thermal capacity, we deduce the following result, which we then apply to improve upon Theorem 5.1.

THEOREM 7.57. *If $Z \subseteq \mathbb{R}^n$, $m_n(Z) = 0$ and $c \in \mathbb{R}$, then there is a positive temperature u on $\mathbb{R}^n \times]c, +\infty[$ such that*

$$\lim_{(x,t)\to(y,c+)} u(x,t) = +\infty$$

for all $y \in Z$.

PROOF. We choose a, b such that $-\infty \le a < c < b \le +\infty$, and consider the thermal capacity relative to $E_s = \mathbb{R}^n \times]a,b[$. By Theorem 7.55, $\mathcal{C}(Z \times \{c\}) = 0$. Therefore $Z \times \{c\}$ is polar, by Theorem 7.46, and so there is a heat potential $G\mu$ of a finite measure such that $G\mu(x,c) = +\infty$ for all $x \in Z$, by Theorem 7.3. Let ν denote the restriction of μ to the half-space $\mathbb{R}^n \times]-\infty, c]$. Then ν is finite, and so $G\nu$ is a heat potential by Theorem 6.18. Therefore $G\nu$ is a temperature on $\mathbb{R}^n \times]c, +\infty[$, by Corollary 6.22. Furthermore, for any $y \in Z$ we have

$$\begin{aligned} G\nu(y,c) &= \int_{\mathbb{R}^n \times]-\infty,c[} W(y-z,c-s)\,d\nu(y,s) \\ &= \int_{\mathbb{R}^n \times]-\infty,c[} W(y-z,c-s)\,d\mu(y,s) \\ &= G\mu(y,c) \\ &= +\infty, \end{aligned}$$

so that $G\nu = +\infty$ on $Z \times \{c\}$. The lower semicontinuity of $G\nu$ now implies that $G\nu(x,t) \to +\infty$ as $(x,t) \to (y,c)$ for each $y \in Z$. Hence the restriction of $G\nu$ to $\mathbb{R}^n \times]-\infty, c[$ is the required temperature. □

THEOREM 7.58. *Suppose that $0 \le s < b$, and that w is a subtemperature on $\mathbb{R}^n \times]s,b[$. If the hyperplane mean $M_b(w^+;\cdot)$ is a locally integrable function on the half-closed interval $[s,b[$, if*

$$\limsup_{(x,t)\to(\xi,s+)} w(x,t) < +\infty \tag{7.10}$$

for all $\xi \in \mathbb{R}^n$*, and if*

$$\limsup_{(x,t)\to(\xi,s+)} w(x,t) \leq A \tag{7.11}$$

for almost all $\xi \in \mathbb{R}^n$*, then* $w \leq A$ *on* $\mathbb{R}^n \times]s,b[$*.*

PROOF. Let Z denote the set of all ξ such that (7.11) does not hold. Then $m_n(Z) = 0$, and so there is a positive temperature u on $\mathbb{R}^n \times]s,b[$ such that

$$\lim_{(x,t)\to(\xi,s+)} u(x,t) = +\infty \tag{7.12}$$

for all $\xi \in Z$, by Theorem 7.57. Given any positive number ϵ, we put $w_\epsilon = w - \epsilon u$ on $\mathbb{R}^n \times]s,b[$. Then w_ϵ is a subtemperature on $\mathbb{R}^n \times]s,b[$, and since $w_\epsilon \leq w$ the hyperplane mean $M_b(w_\epsilon^+;\cdot)$ is locally integrable on $[s,b[$. Furthermore, for all $\xi \in Z$ we have

$$\limsup_{(x,t)\to(\xi,s+)} w_\epsilon(x,t) = \limsup_{(x,t)\to(\xi,s+)} w(x,t) - \epsilon \lim_{(x,t)\to(\xi,s+)} u(x,t) = -\infty,$$

in view of (7.10) and (7.12). Moreover, for all $\xi \in \mathbb{R}^n \backslash Z$ we have

$$\limsup_{(x,t)\to(\xi,s+)} w_\epsilon(x,t) \leq \limsup_{(x,t)\to(\xi,s+)} w(x,t) \leq A,$$

in view of (7.11). Thus

$$\limsup_{(x,t)\to(\xi,s+)} w_\epsilon(x,t) \leq A$$

for all $\xi \in \mathbb{R}^n$, and so it follows from Theorem 5.1 that $w_\epsilon \leq A$ on $\mathbb{R}^n \times]s,b[$. Making $\epsilon \to 0+$, we deduce that $w \leq A$ on $\mathbb{R}^n \times]s,b[$, as required. □

7.11. Notes and Comments

The main references for the results in this chapter are Watson [**72, 73**] and Doob [**14**]. Some results about the thermal capacity of compact sets were also proved by Landis [**49**] and Lanconelli [**46**] for $E = \mathbb{R}^{n+1}$, taking as the definition the characterization given in Lemma 7.54. However, the treatment here is heavily influenced by Armitage & Gardiner [**3**], both in the methods used and in the order many of the results are proved. This is especially true of sections, 7.5, 7.6 and 7.8.

Theorem 7.6 is new, as are Examples 7.35 and 7.56.

Corollary 7.15 can be referred to as a removable sets theorem for temperatures. Several authors have considered such results, including Král [**43**], Kaiser & Müller [**37**], Umanskiĭ [**68**], Watson [**86**] and Hui [**36**].

A slightly different version of Theorem 7.17, using the means $\mathcal{M}$ or $\mathcal{V}$ instead of $\mathcal{L}$, was given by Watson [**71**]. In [**72**], Watson showed that there are bounded supertemperatures u such that

$$\limsup_{(x,t)\to(x_0,t_0-)} u(x,t) > u(x_0,t_0),$$

which shows that supertemperatures do not generally satisfy the corresponding one-sided continuity condition.

In the case where $E = \mathbb{R}^{n+1}$ and S is analytic, Theorem 7.46(a) was proved in a different form by Gariepy & Ziemer [**25**]. Related work can be found in Maeda [**51**]. Theorem 7.46(b),(c) were proved by Watson [**73**] for subsets of a quasi-regular (called 'admissible' in [**73**]) open set E. Since $\mathbb{R}^{n+1}$ is regular, and a subset of E is polar if and only if it is polar as a subset of $\mathbb{R}^{n+1}$, the equivalence of polar and copolar follows for any E. That equivalence was later proved by Doob [**14**].

Theorem 7.55 was first proved by Lanconelli [**46**], for the case where $E_s = \mathbb{R}^{n+1}$ and A is compact. The general case was proved independently by Watson [**73**].

Kaufman & Wu [**39**] began the comparison of polarity with classical capacities by showing that, if $n = 1$ and $S \subseteq \{x_0\} \times \mathbb{R}$, then S is polar if and only if it has zero Riesz $\frac{1}{2}$-capacity. On the other hand, if $S \subseteq \mathbb{R}^n \times \{t_0\}$ for any n, then Theorem 7.55 shows that S is polar if and only if it has n-dimensional Lebesgue measure zero. Such a distinct difference between the coordinates means that any spherically symmetric measure or capacity will be of little use here, since for $n = 1$ the critical dimension is 1 in the x-coordinate and $\frac{1}{2}$ in the t-coordinate. To overcome this, Taylor & Watson [**67**] defined measures, of Hausdorff type, using a restricted class of covering sets. These measures effectively double the classical dimension in the t-direction whilst leaving it unchanged in the x-direction. They found a class of sets in $\mathbb{R}^{n+1}$, not all subsets of $\{x_0\} \times \mathbb{R}$ for any x_0, for which the thermal capacity is zero if and only if the Riesz $\frac{n}{2}$-capacity of the projection onto the t-axis is zero. They found another class of sets, not all subsets of $\mathbb{R}^n \times \{t_0\}$ for any t_0, for which the thermal capacity is zero if and only if the n-dimensional Lebesgue measure of the projection onto the hyperplane $\mathbb{R}^n \times \{0\}$ is zero. They obtained comparison theorems in both directions, and left some **open questions**. Mysovskikh [**54**] also obtained comparison theorems for polarity, using both anisotropic and classical Hausdorff measures. Wu [**93**] considered the product of compact subsets X and T of $\mathbb{R}$, and gave criteria for the thermal capacity of $X \times T$ to be positive in terms of the classical Hausdorff measures and Riesz capacities of X and T. She also left an **open question**.