Introduction

Background. The classical maximum modulus principles are fundamental properties of solutions to partial differential equations. They have numerous important applications to various problems in the theory of these equations, both linear and nonlinear.

Let Ω be a bounded domain in the Euclidean space $\mathbb{R}^n$. For the elliptic equation

$$(1) \qquad \sum_{j,k=1}^{n} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k} - \sum_{j=1}^{n} a_j(x)\frac{\partial u}{\partial x_j} - a_0(x)u = 0\ , \qquad x \in \Omega,$$

with bounded coefficients, positive-definite matrix $((a_{jk}(x)))$, and with $a_0(x) \geq 0$, two classical facts are basic: the weak and strong maximum modulus principles. By the weak one, a solution $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}(\overline{\Omega})$ to equation (1) satisfies

$$\max_{\overline{\Omega}}\ |u| = \max_{\partial\Omega}\ |u|\ .$$

According to the strong principle the maximum modulus of a non-constant solution u is never attained inside Ω. Note that the last property is not dealt with in the present book.

The weak and strong maximum modulus principles hold also for the parabolic equation

$$(2) \qquad \frac{\partial u}{\partial t} - \sum_{j,k=1}^{n} a_{jk}(x,t)\frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} a_j(x,t)\frac{\partial u}{\partial x_j} + a_0(x,t)u = 0$$

with bounded coefficients, positive-definite matrix $((a_{jk}(x,t)))$, and with $a_0(x,t) \geq 0$ in a cylinder $Q_T = \Omega \times (0,T]$. By the weak maximum modulus principle, if $u \in \mathrm{C}^{(2,1)}(Q_T) \cap \mathrm{C}(\overline{Q}_T)$ is a solution of (2), then

$$\max_{\overline{Q}_T}\ |u| = \max_{\overline{\Gamma}_T}\ |u|\ ,$$

where $\Gamma_T = \{(x,t) \in \partial Q_T :\ 0 \leq t < T\}$.

The maximum principles just mentioned were obtained for different classes of solutions and under various assumptions about the coefficients, but are valid almost exclusively for scalar equations of the second order (with weakly coupled systems as the only exception).

The not so sharp but incomparably more general property of the same nature, which holds for equations and systems of arbitrary order in smooth domains, is the so-called Miranda-Agmon maximum principle. In particular, for a homogeneous elliptic equation of order 2ℓ, this principle is the estimate

$$(3) \qquad \max_{\overline{\Omega}}\ |\nabla_{\ell-1}u| \leq c(\Omega)\max_{\partial\Omega}\ |\nabla_{\ell-1}u|.$$

A weaker variant of the Miranda-Agmon maximum principle runs as follows:

$$\max_{\overline{\Omega}} |\nabla_{\ell-1} u| \leq K \max_{\partial\Omega} |\nabla_{\ell-1} u| + C\|u\|_{L^1(\Omega)}. \tag{4}$$

Motivation, subject, and method. Various maximum principles for scalar equations of the second order are treated in numerous papers and in the monographs by Miranda [**Mir**], Protter and Weinberger [**PW**], Sperb [**Sp**], Fraenkel [**Fr**] , Pucci and Serrin[**PS**], López-Gómez [**LG**]. Less is written about systems and higher order equations and, in particular, there are no books dealing with this subject whatsoever.

The main goal of the present monograph, which has almost nothing in common both in results and methods with the books just mentioned, is to partially fill in this gap. Here we deal with different maximum modulus principles for elliptic and parabolic systems of arbitrary order. In particular, we present necessary and sufficient conditions for validity of the classical maximum modulus principles for systems of the second order. We obtain sharp constants in inequalities (3) and (4) and in many other inequalities of a similar nature. Somewhat related to this topic are explicit formulae for the norms and the essential norms of integral operators of potential type.

Our proofs are based on a unified approach, using on one hand, representations of the norms of matrix integral operators whose target spaces are linear and finite dimensional, and, on the other hand, on solving certain finite dimensional optimization problems.

The book reflects results obtained by the authors during the last three decades and published in [**Kr1**]-[**KM10**], [**KM12**], [**KM13**], [**MK**]. Some new material is added as well.

Structure of the book. Let us sketch briefly the main contents of the present volume which consists of two parts. Part I is mostly devoted to linear elliptic equations and systems.

Chapter 1 is auxiliary. It also contains general facts used throughout the book. We establish representations for the norms of linear integral operators acting from the space of n-component continuous or L^p-functions into the Euclidean space $\mathbb{R}^m$ or into the unitary space $\mathbb{C}^m$. Continuous vector-valued functions are defined on a locally compact Hausdorff space. Vector-valued functions in L^p are considered on spaces with a measure.

Chapter 2 concerns necessary and sufficient conditions for validity of the classical maximum modulus principle for strongly elliptic systems of the second order. First we consider the systems with constant coefficients and the second order derivatives only. It is shown that the classical maximum modulus principle holds for such a model system if and only if the differential operator of the system is the product of a positive-definite matrix and a scalar elliptic operator, which essentially means that the operator of the system is scalar.

Further we turn to strongly elliptic systems of the second order with variable coefficients and obtain, in particular, algebraic sufficient conditions for validity of the classical modulus principle. These conditions prove to be also necessary if the maximum modulus principle holds for every subdomain of Ω. We give an example showing that the scalarity of the principal part of the system with variable coefficients may not be necessary for validity of the classical maximum modulus principle.

In Chapter 3 we find representations for the best constants in the Miranda-Agmon maximum principle for solutions of homogeneous strongly elliptic systems of the second order with constant coefficients in a half-space. We also obtain explicit formulas for sharp constants in inequalities of the Miranda-Agmon type for solutions of the Stokes and Lamé systems as well as a system of viscoelasticity in a half-space. For instance, we prove that the velocity vector $\boldsymbol{u}$ satisfying the homogeneous Stokes system in $\mathbb{R}^n_+ = \big\{x = (x_1, \dots, x_n) : x_n > 0\big\}$ admits the estimate

$$|\boldsymbol{u}(x)| \le K_n \sup\{|\boldsymbol{u}(x', 0)| : x' \in \partial\mathbb{R}^n_+\}$$

with the sharp constant

$$K_n = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n+1}{2})}.$$

In particular, $K_2 = 4/\pi$ and $K_3 = 3/2$.

It is proved that under certain algebraic conditions imposed on the Dirichlet boundary data the constant in an inequality of the Miranda-Agmon type is equal to one, i.e., the classical maximum modulus principle holds.

Chapter 4 concerns linear elliptic systems of the second order with constant coefficients without lower order terms. We find a representation of the sharp coefficient in a pointwise estimate for the modulus of a solution of the system in a half-space by the norm of the Dirichlet data in L^p. Solving a certain optimization problem on the unit sphere in a finite dimensional space, we derive explicitly the coefficients just mentioned for the Lamé and Stokes systems.

Unlike previous chapters, in Chapters 5 and 6 we turn to pointwise estimates for derivatives of solutions to elliptic equations. In Chapter 5 we are concerned with the best coefficients in the Miranda-Agmon type estimates (3) and (4). For example, we obtain the C. Miranda inequality for biharmonic functions with the sharp constant

$$\sup_{\mathbb{R}^n_+} |\nabla u| \le K \sup_{\partial\mathbb{R}^n_+} |\nabla u|.$$

In particular, $K = 4/\pi$ for $n = 2, K = 1/2 + 2\pi\sqrt{3}/9$ for $n = 3$ and $K = 2$ for $n = 4$.

Chapter 6 contains inequalities with sharp coefficients for the modulus of directional derivatives of harmonic functions in a half-space and in a ball. The boundary values of harmonic functions are assumed to be bounded, semibounded or L^p-integrable. Simultaneously with deriving explicit formulae for the best coefficients in pointwise estimates of the modulus of the gradient of a harmonic function, we find extremal directions for harmonic fields in a half-space or in a ball. This leads to a solution of the Khavinson type extremal problems for harmonic functions. In particular, we obtain the inequality with the sharp constant

$$|\nabla u(x)| \le \frac{4(n-1)^{(n-1)/2}\, \omega_{n-1}}{n^{n/2}\, \omega_n x_n} \sup_{y \in \mathbb{R}^n_+} |u(y)|\,,$$

where ω_n is the area of the unit sphere in $\mathbb{R}^n$. The estimates obtained in this chapter can be interpreted as multidimensional analogues of certain "sharp real-part theorems" going back to the famous Hadamard theorem in complex function theory.

In Chapter 7, which is the last chapter of Part I, we consider vector-valued potentials of the double layer type and their corresponding matrix-valued integral operators in spaces of continuous vector-valued functions on the boundaries of irregular domains. The potentials and operators in question arise when the Dirichlet problem for elliptic systems is reduced to regular boundary integral equations. Unlike the case of smooth boundaries, the boundary integral operators under consideration are not compact but only bounded. As a measure of non-compactness of a bounded operator L one uses the essential norm $\mathrm{ess}\|L\|$ which appears in the estimate $R(L) \geq \big(\mathrm{ess}\|L\|\big)^{-1}$ for the Fredholm radius $R(L)$ of L. The notions of the essential norm and of the Fredholm radius were introduced and applied by Radon who developed a logarithmic potential theory on curves with bounded rotation. We find representations for the norms and for the essential norms of matrix-valued integral operators of double layer potential type in spaces of continuous vector-valued functions. Representations for the essential norm are concretized for boundary integral operators of the elasticity theory (Lamé system) and hydrodynamics (Stokes system) both in the planar case for a domain with angular points, and in the three-dimensional case for a domain with conic points or edges.

The second part of the book concerns equations and systems of parabolic type.

In Chapter 8 we consider systems of partial differential equations of the first order in t and of order 2ℓ in x variables, which are uniformly parabolic in the sense of Petrovskiĭ. We show that the classical maximum modulus principle is not valid in $\mathbb{R}^{n+1}_T = \mathbb{R}^n \times (0, T]$ for $\ell \geq 2$. For the system

$$\frac{\partial \boldsymbol{u}}{\partial t} - \sum_{j,k=1}^{n} \mathcal{A}_{jk}(x,t)\frac{\partial^2 \boldsymbol{u}}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \mathcal{A}_j(x,t)\frac{\partial \boldsymbol{u}}{\partial x_j} + \mathcal{A}_0(x,t)\boldsymbol{u} = \boldsymbol{0} \tag{5}$$

we obtain necessary and, separately, sufficient conditions for the classical maximum modulus principle to hold in the layer $\mathbb{R}^{n+1}_T$ and in the cylinder $Q_T = \Omega \times (0, T]$, where Ω is a bounded subdomain of $\mathbb{R}^n$. If the coefficients of the system do not depend on t, these conditions coincide. A necessary and sufficient condition in this case is that the principal part of the system is scalar, i.e., $\mathcal{A}_{jk}(x) = a_{jk}(x)I,\ 1 \leq j,\ k \leq n$, and that the coefficients of the system satisfy the algebraic inequality

$$\sum_{j,k=1}^{n} a_{jk}(x)(\boldsymbol{\xi}_j, \boldsymbol{\xi}_k) + \sum_{j=1}^{n} (\mathcal{A}_j(x)\boldsymbol{\xi}_j, \boldsymbol{\zeta}) + (\mathcal{A}_0(x)\boldsymbol{\zeta}, \boldsymbol{\zeta}) \geq 0$$

for all $x \in \Omega$ ($x \in \mathbb{R}^n$) and for any $\boldsymbol{\xi}_j,\ \boldsymbol{\zeta} \in \mathbb{R}^m,\ j = 1, \dots, n$, with $(\boldsymbol{\xi}_j, \boldsymbol{\zeta}) = 0$. Here $((a_{jk}))$ is a positive-definite $(n \times n)$-matrix-valued function, and I is the identity matrix of order m.

The topic of Chapter 9 is related to one of the main assertions in the preceding chapter. We consider solutions to the initial boundary value problem in the cylinder $\Omega \times (0, T]$ with zero Dirichlet data for a linear second order system, which is strongly parabolic, with smooth $(m \times m)$-matrix-valued coefficients depending only on x. It is assumed that Ω is a bounded subdomain of $\mathbb{R}^n$ with smooth boundary. It is shown that the criterion for validity of the classical maximum modulus principle, obtained in the previous chapter for the case of the parabolic second order system with coefficients independent of t remains necessary if one assumes a priori that the boundary data are zero. We note that this result can be interpreted as a criterion for the $[\mathrm{C}(\overline{\Omega})]^m$-contractivity of a semigroup generated by the operator $\mathfrak{A}(x, D_x)$ of the strongly parabolic system

$$\frac{\partial \boldsymbol{u}}{\partial t} - \mathfrak{A}(x, D_x)\boldsymbol{u} = \boldsymbol{0}.$$

Unlike the rest of Part II, in Chapters 10 and 11 we deal with a *maximum norm principle*, understanding the norm in a generalized sense, i.e., as the Minkowski functional of a compact convex body in $\mathbb{R}^m$ containing the origin in its interior.

In Chapter 10 we consider systems of partial differential equations, which contain only second derivatives in the x variables and which are uniformly parabolic in the sense of Petrovskiĭ. For such systems we obtain necessary and, separately, sufficient conditions for the maximum norm principle to hold in the layer $\mathbb{R}^n \times (0, T]$ and in the cylinder $\Omega \times (0, T]$, where Ω is a bounded subdomain of $\mathbb{R}^n$. The necessary and sufficient conditions coincide if the coefficients of the system do not depend on t. The above mentioned criterion for validity of the maximum norm principle is formulated as a number of equivalent algebraic conditions describing the relation between the geometry of the unit sphere of the given norm and coefficients of the system under consideration. Simpler formulated criteria ensuring the maximum norm principle are given for certain classes of norms: for differentiable norms, p-norms $(1 \leq p \leq \infty)$ in $\mathbb{R}^m$, as well as for norms whose unit balls are m-pyramids, m-bipyramids, cylindrical bodies, m-parallelepipeds.

In Chapter 11, the question of validity of the maximum norm principle is considered for linear strongly coupled systems of partial differential equations of the first order in t and of the second order in the x variables, which are uniformly parabolic in the sense of Petrovskiĭ. It is supposed that the norm is twice continuously differentiable. For such systems necessary and, separately, sufficient conditions for the maximum norm principle are obtained. These necessary and sufficient conditions coincide if the coefficients of the system do not depend on t. The criterion for validity of the maximum norm principle in this case is that the principal part of the system is scalar and that the coefficients satisfy a certain algebraic inequality, describing a relation between the system and the geometry of the sphere of the given norm. Simpler formulated criteria are given for certain classes of norms, in particular, for p-norms $(2 < p < \infty)$ in $\mathbb{R}^m$.

More detailed information on the material in this book can be found in the introductions to the chapters. We collect historical and bibliographical notes in comments to each chapter.

Readership. The volume is addressed to mathematicians working in partial differential equations, in particular, in elasticity theory and hydrodynamics.

Prerequisites for reading this book are undergraduate courses in linear algebra, theory of partial differential equations, and functional analysis.

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