CHAPTER 1

Preliminaries

We collect here some definitions and properties of plane quasiconformal mappings. Two basic references for this material are the books by Ahlfors [**7**] and Lehto and Virtanen [**117**], to which we refer the reader for further details. A more recent title is the book by Astala, Iwaniec, and Martin [**16**].

In what follows $\mathbf{R}^2$ denotes the Euclidean plane with its usual identification with the complex plane $\mathbf{C}$. The one-point compactification $\overline{\mathbf{R}}^2 = \mathbf{R}^2 \cup \{\infty\}$ is equipped with the chordal metric

$$\mathrm{ch}(z,w) = \frac{2|z-w|}{\sqrt{|z|^2+1}\,\sqrt{|w|^2+1}},$$

where we employ the usual conventions regarding ∞.

Let D and D' be subdomains of $\overline{\mathbf{R}}^2$. We will assume, unless stated otherwise, that $\mathrm{card}(\overline{\mathbf{R}}^2 \setminus D) \geq 2$. The exterior of D is denoted by $D^* = \overline{\mathbf{R}}^2 \setminus \overline{D}$. Let $\mathbf{B}(z,r)$ be the open Euclidean disk with center $z \in \mathbf{R}^2$ and radius r, and let $\mathbf{B}$ be the unit disk $\mathbf{B}(0,1)$. Finally, $\mathbf{H}$ will denote the upper or right half-planes

$$\{z = x + i\,y : y > 0\} \qquad \text{or} \qquad \{z = x + i\,y : x > 0\}.$$

1.1. Quasiconformal mappings

There are several different ways to view a quasiconformal mapping. Perhaps the most geometrically intuitive is in terms of the linear dilatation of a homeomorphism.

Suppose that $f : D \to D'$ is a homeomorphism. For $z \in D \setminus \{\infty, f^{-1}(\infty)\}$ and $0 < r < \mathrm{dist}(z, \partial D)$ we let

$$\begin{aligned} l_f(z,r) &= \min_{|z-w|=r} |f(z) - f(w)|, \\ L_f(z,r) &= \max_{|z-w|=r} |f(z) - f(w)| \end{aligned} \tag{1.1.1}$$

and call

$$H_f(z) = \limsup_{r\to 0} \frac{L_f(z,r)}{l_f(z,r)}$$

the *linear dilatation* of f at z. See Figure 1.1.

Recall that a homeomorphism in $\mathbf{R}^2$ is either sense-preserving or sense-reversing [**117**]. Menchoff showed in 1937 [**129**] that if $D, D' \subset \mathbf{R}^2$, a sense-preserving homeomorphism $f : D \to D'$ is analytic, and hence conformal, whenever

$$H_f(z) = 1 \tag{1.1.2}$$

for all but a countable set of $z \in D$.

The following definition for quasiconformality is a natural counterpart of Menchoff's theorem (Gehring [**47**]).

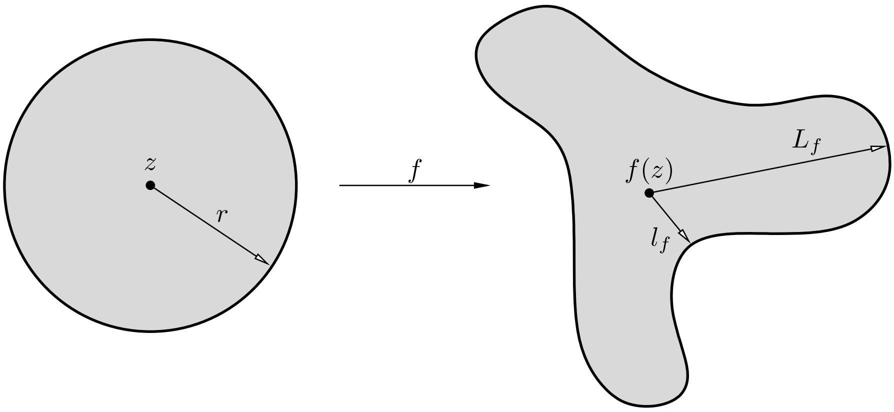

FIGURE 1.1

DEFINITION 1.1.3. A homeomorphism $f : D \to D'$ is K-*quasiconformal* where $1 \le K < \infty$ if $H_f(z) < \infty$ for every $z \in D \setminus \{\infty, f^{-1}(\infty)\}$ and

$$H_f(z) \le K$$

almost everywhere in D.

The inequality in the above definition can be weakened significantly to yield the same class of mappings. Letting

$$h_f(z) = \liminf_{r \to 0} \frac{L_f(z,r)}{l_f(z,r)},$$

where l_f and L_f are as in (1.1.1), Heinonen and Koskela [**81**] and Kallunki and Koskela [**96**] obtained the following surprising result.

THEOREM 1.1.4. *A homeomorphism $f : D \to D'$ is K-quasiconformal, where $1 \le K < \infty$, if $H_f(z) < \infty$ for every $z \in D \setminus \{\infty, f^{-1}(\infty)\}$ and*

$$h_f(z) \le K$$

almost everywhere in D.

The next results, which can be found in Lehto-Virtanen [**117**], identify mappings which are 1-quasiconformal or which are the composition and inverses of quasiconformal mappings.

THEOREM 1.1.5. *A homeomorphism $f : D \to D'$ is 1-quasiconformal if and only if f or its complex conjugate $\overline{f}$ is a conformal mapping, i.e., analytic in $D \setminus \{\infty, f^{-1}(\infty)\}$.*

THEOREM 1.1.6. *If $f : D \to D'$ is K_1-quasiconformal and $g : D' \to D''$ is K_2-quasiconformal, then $g \circ f : D \to D''$ is $K_1 K_2$-quasiconformal. The inverse of a K-quasiconformal mapping is K-quasiconformal.*

Menchoff's theorem asserts that a sense-preserving homeomorphism f of D is a conformal mapping if, except at a countable set of points $z \in D$, f maps infinitesimal circles about z onto infinitesimal circles about $f(z)$. Theorems 1.1.4 and 1.1.5 extend this result by first replacing the countable exceptional set where

(1.1.2) was not required to hold by a set of *measure zero* and then by requiring that f preserves only a *sequence* of infinitesimal circles about the remaining points $z \in D$.

DEFINITION 1.1.7. A real-valued function u is *absolutely continuous on lines*, or *ACL*, in a domain D if for each rectangle $[a, b] \times [c, d] \subset D$,

1° $u(x + i\,y)$ is absolutely continuous in x for almost all $y \in [c, d]$,
2° $u(x + i\,y)$ is absolutely continuous in y for almost all $x \in [a, b]$.

A complex-valued function f is ACL in D if its real and imaginary parts are ACL in D.

If a homeomorphism f is ACL in D, then a measure theoretic argument shows that f has finite partial derivatives a.e. in D and hence, in fact, a differential a.e. in D by Gehring-Lehto [**63**].

A quasiconformal mapping can then be described in terms of its analytic properties as follows. See e.g. Lehto-Virtanen [**117**].

THEOREM 1.1.8. *A homeomorphism $f : D \to D'$ is K-quasiconformal if and only if f is ACL in D and*

$$\max_{\alpha} |\partial_\alpha f(z)|^2 \le K\,|J_f(z)| \tag{1.1.9}$$

almost everywhere in D. Here $\partial_\alpha f(z)$ denotes the derivative of f at z in the direction α and $J_f(z)$ denotes the Jacobian of f at z. Moreover, if f is quasiconformal, we have that $J_f(z) \neq 0$ a.e. in D and that it satisfies Lusin's property (N)*, i.e. $m(f(E)) = 0$ whenever $m(E) = 0$ for the planar Lebesgue measure m.*

If $f : D \to D'$ is K-quasiconformal, then inequality (1.1.9) can also be written

$$\max_{\alpha} |\partial_\alpha\, f(z)| \le K \min_{\alpha} |\partial_\alpha\, f(z)|.$$

If we assume also that f is sense-preserving, then

$$\max_{\alpha} |\partial_\alpha\, f| = |f_z| + |f_{\bar z}|,$$
$$\min_{\alpha} |\partial_\alpha\, f| = |f_z| - |f_{\bar z}|,$$

where f_z and $f_{\bar z}$ are the *complex derivatives*

$$f_z = \frac{1}{2}(f_x - i f_y) \qquad \text{and} \qquad f_{\bar z} = \frac{1}{2}(f_x + i f_y).$$

In this case (1.1.9) takes the form

$$|f_{\bar z}| \le \frac{K-1}{K+1}\,|f_z|. \tag{1.1.10}$$

Then since

$$|f_z|^2 - |f_{\bar z}|^2 = J_f > 0$$

a.e. in D, we may also consider the quotient

$$\mu_f = \frac{f_{\bar z}}{f_z}.$$

The function $\mu_f(z)$ is the *complex dilatation* of f at z. It satisfies the relations

$$\frac{1 + |\mu_f(z)|}{1 - |\mu_f(z)|} = H_f(z) \qquad \text{and} \qquad |\mu_f(z)| \le \frac{K-1}{K+1}$$

a.e. in D. Hence $\mu_f = 0$ a.e. in D if and only if f is conformal.

If $f : D \to D'$ and $g : D' \to D''$ are both sense-preserving and quasiconformal, then $\mu_{g \circ f} = \mu_f$ a.e. in D if and only if g is conformal.

It is possible to prescribe the complex dilatation $\mu_f(z)$, and hence the linear dilatation $H_f(z)$, at almost every point z of a domain D. This result, known as the *measurable Riemann mapping theorem*, has turned out to be a powerful tool in complex analysis. See Ahlfors-Bers [**9**], Lehto-Virtanen [**117**], Morrey [**133**], and Bojarski [**25**].

Theorem 1.1.11. *If μ is measurable with*

$$\|\mu\|_{L^\infty} = \operatorname*{ess\,sup}_D |\mu(z)| < 1,$$

then there exists a sense-preserving quasiconformal mapping f of D with $\mu_f = \mu$ a.e. in D. Moreover f is unique up to post composition with a conformal map.

1.2. Modulus of a curve family

The conditions for quasiconformality in Definition 1.1.3 and Theorem 1.1.8 involve the local behavior of a homeomorphism. We need a way to *integrate* the inequality in Theorem 1.1.8 in order to derive global properties of the mapping. When $K = 1$, f or its complex conjugate $\bar{f}$ is conformal and the Cauchy integral formula is available. The tool most often used to replace the Cauchy formula when $K > 1$ is the method of extremal length, first formulated by Ahlfors and Beurling in [**23**].

Suppose that Γ is a family of curves in $\overline{\mathbf{R}}^2$. We say that ρ is an *admissible density* for Γ, or is in $\mathrm{adm}(\Gamma)$, if ρ is nonnegative and Borel measurable in $\mathbf{R}^2$ and if

$$\int_\gamma \rho(z)|dz| \geq 1$$

for each locally rectifiable $\gamma \in \Gamma$. The *modulus* and *extremal length* of the family Γ are then given, respectively, by

$$\mathrm{mod}(\Gamma) = \inf_\rho \int_{\mathbf{R}^2} \rho(z)^2 \, dm \qquad \text{and} \qquad \lambda(\Gamma) = \frac{1}{\mathrm{mod}(\Gamma)},$$

where the infimum is taken over $\rho \in \mathrm{adm}(\Gamma)$.

Theorem 1.2.1. *If $f : D \to D'$ is conformal and if Γ is a family of curves in D, then*

$$\mathrm{mod}(f(\Gamma)) = \mathrm{mod}(\Gamma).$$

Proof. We consider the case where $D, D' \subset \mathbf{R}^2$. For each $\rho' \in \mathrm{adm}(f(\Gamma))$ let

$$\rho(z) = \begin{cases} \rho'(f(z))|f'(z)| & \text{if } z \in D, \\ 0 & \text{if } z \in \mathbf{R}^2 \setminus D. \end{cases}$$

Then ρ is nonnegative and Borel measurable in $\mathbf{R}^2$. If γ is locally rectifiable, then $f(\gamma) \in f(\Gamma)$ is locally rectifiable and

$$\int_\gamma \rho(z)|dz| = \int_\gamma \rho'(f(z))|f'(z)||dz| = \int_{f(\gamma)} \rho'(w)|dw| \geq 1.$$

Thus $\rho \in \mathrm{adm}(\Gamma)$,

$$\begin{aligned}\mathrm{mod}(\Gamma) &\le \int_{\mathbf{R}^2} \rho(z)^2 dm = \int_D \rho'(f(z))^2 |f'(z)|^2 dm \\ &= \int_{D'} \rho'(w)^2 dm \le \int_{\mathbf{R}^2} \rho'(w)^2 dm,\end{aligned}$$

whence

$$\mathrm{mod}(\Gamma) \le \inf_{\rho'} \int_{\mathbf{R}^2} \rho'(w)^2 dm = \mathrm{mod}(f(\Gamma)).$$

Now take the infimum over all such ρ.

Finally we obtain

$$\mathrm{mod}(\Gamma) = \mathrm{mod}(f(\Gamma))$$

by applying the above argument to f^{-1}. □

If the curves $\gamma \in \Gamma$ are disjoint arcs, we may think of them as homogeneous electric wires. Then the modulus $\mathrm{mod}(\Gamma)$ is a conformally invariant electrical transconductance for the family of wires γ and the extremal length $\lambda(\Gamma)$ is the total electrical resistance of the system. In particular, $\mathrm{mod}(\Gamma)$ is big if the curves $\gamma \in \Gamma$ are short and plentiful and small if the curves γ are long or scarce.

The following properties show that $\mathrm{mod}(\Gamma)$ is also an outer measure on the curve families Γ in $\overline{\mathbf{R}}^2$:

1° $\mathrm{mod}(\emptyset) = 0$.
2° $\mathrm{mod}(\Gamma_1) \le \mathrm{mod}(\Gamma_2)$ if $\Gamma_1 \subset \Gamma_2$.
3° $\mathrm{mod}(\bigcup_j \Gamma_j) \le \sum_j \mathrm{mod}(\Gamma_j)$.

Finally the conformal invariant $\mathrm{mod}(\Gamma)$ yields a third characterization for quasiconformal mappings.

THEOREM 1.2.2 (Ahlfors [**7**]). *A homeomorphism $f : D \to D'$ is K-quasiconformal if and only if*

$$\frac{1}{K}\,\mathrm{mod}(\Gamma) \le \mathrm{mod}(f(\Gamma)) \le K\,\mathrm{mod}(\Gamma)$$

for each family Γ of curves in D.

1.3. Modulus estimates

Estimates for the moduli of various curve families are useful tools for studying geometric properties of conformal and quasiconformal mappings. We derive here three simple modulus estimates and a distortion theorem for quasiconformal mappings of the plane which we will need later.

LEMMA 1.3.1. *Suppose that $R = R(0, a, a+i, i)$ is the rectangle with vertices at $0, a, a+i, i$ where $a > 0$ and suppose that Γ is the family of curves which join the horizontal sides of ∂R in R. Then*

$$\mathrm{mod}(\Gamma) = a.$$

PROOF. The segment $\gamma = \{z : x + i\,y : 0 < y < 1\}$ is in Γ for $0 < x < a$. Hence if $\rho \in \mathrm{adm}(\Gamma)$, then by the Cauchy-Schwarz inequality,

$$1 \le \left(\int_0^1 \rho(x + i\,y)\,dy\right)^2 \le \int_0^1 \rho(x + i\,y)^2\,dy$$

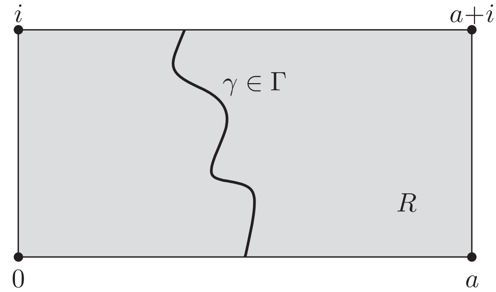

FIGURE 1.2

for $0 < x < a$. Thus

$$\int_{\mathbf{R}^2} \rho(z)^2 \, dm \geq \int_0^a \left(\int_0^1 \rho(x + i\,y)^2 \, dy \right) dx \geq a$$

and

$$\operatorname{mod}(\Gamma) = \inf_\rho \int_{\mathbf{R}^2} \rho(z)^2 \, dm \geq a.$$

Next the function

$$\rho(z) = \begin{cases} 1 & \text{if } z \in R, \\ 0 & \text{otherwise} \end{cases}$$

is in $\operatorname{adm}(\Gamma)$ and

$$\int_{\mathbf{R}^2} \rho(z)^2 \, dm = a,$$

completing the proof for Lemma 1.3.1. □

LEMMA 1.3.2. *If Γ is a family of curves and if for each t with $a < t < b$ the circle $\{z : |z| = t\}$ contains a curve $\gamma \in \Gamma$, then*

$$\operatorname{mod}(\Gamma) \geq \frac{1}{2\pi} \log \frac{b}{a}.$$

PROOF. See Figure 1.3. If $\rho \in \operatorname{adm}(\Gamma)$, then

$$1 \leq \left(\int_\gamma \rho(z) \, |dz| \right)^2 \leq \left(\int_0^{2\pi} \rho(t\,e^{i\,\theta}) \, t \, d\theta \right)^2 \leq 2\pi t \int_0^{2\pi} \rho(t\,e^{i\,\theta})^2 \, t \, d\theta,$$

whence

$$\frac{1}{2\pi} \log \frac{b}{a} = \int_a^b \frac{1}{2\pi t} \, dt \leq \int_a^b \left(\int_0^{2\pi} \rho(t\,e^{i\,\theta})^2 t \, d\theta \right) dt \leq \int_{\mathbf{R}^2} \rho(z)^2 \, dm.$$

Now take the infimum over all such ρ. □

LEMMA 1.3.3. *If Γ is a family of curves which join continua C_1 and C_2 where*

$$\operatorname{dist}(C_1, C_2) \geq a > 0, \qquad \operatorname{diam}(C_1) \leq b,$$

then

$$\operatorname{mod}(\Gamma) \leq \pi \left(\frac{b}{a} + 1 \right)^2.$$

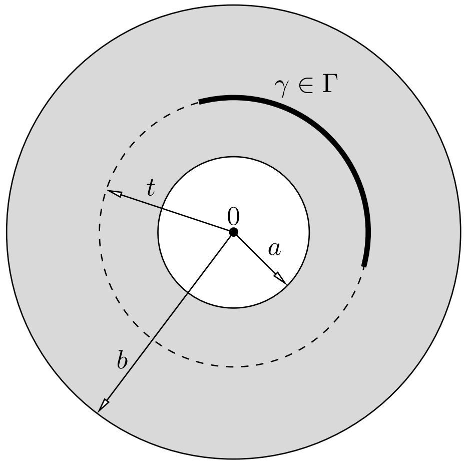

FIGURE 1.3

PROOF. Choose $z_1 \in C_1$ and $z_2 \in C_2$ so that $|z_1 - z_2| = \mathrm{dist}(C_1, C_2)$ and set

$$\rho(z) = \begin{cases} 1/a & \text{if } z \in \mathbf{B}(z_1, a+b), \\ 0 & \text{otherwise.} \end{cases}$$

Then each $\gamma \in \Gamma$ either joins C_1 to C_2 in $\mathbf{B}(z_1, a+b)$ or joins $\partial\mathbf{B}(z_1, b)$ to $\partial\mathbf{B}(z_1, a+b)$. In either case γ contains a subarc of length at least a which lies in $\mathbf{B}(z_1, a+b)$. Thus $\rho \in \mathrm{adm}(\Gamma)$ and

$$\mathrm{mod}(\Gamma) \leq \int_{\mathbf{R}^2} \rho(z)^2 \, dm = \pi\Big(\frac{b}{a} + 1\Big)^2.$$

□

We now apply the modulus estimate established above in Lemma 1.3.2 to prove an elementary distortion theorem for quasiconformal mappings which we will need in what follows.

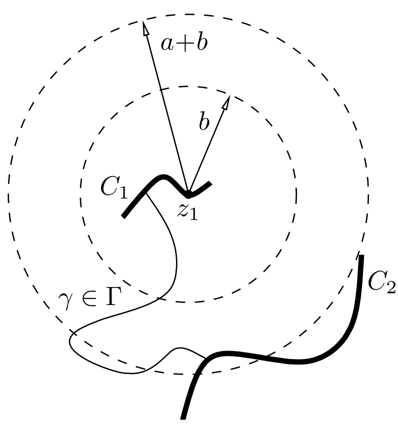

FIGURE 1.4

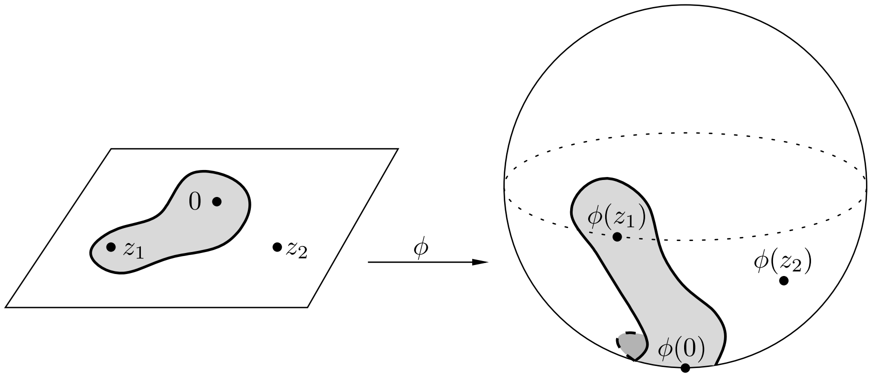

FIGURE 1.5

THEOREM 1.3.4. *If $f : \mathbf{R}^2 \to \mathbf{R}^2$ is K-quasiconformal and if*

$$|z_2 - z_0| \le |z_1 - z_0|,$$

then

$$|f(z_2) - f(z_0)| \le c\,|f(z_1) - f(z_0)| \tag{1.3.5}$$

where $c = e^{8K}$.

PROOF. By means of preliminary similarity transformations, we may assume that $z_0 = f(z_0) = 0$ and that $|z_1| = 1$, whence $|z_2| \le 1$. We may also assume that $|f(z_1)| < |f(z_2)|$ since otherwise there is nothing to prove.

Let Γ' be the family of circles $\{w : |w| = t\}$ where $|f(z_1)| < t < |f(z_2)|$. Then

$$\frac{1}{2\pi} \log \frac{|f(z_2)|}{|f(z_1)|} \le \mathrm{mod}(\Gamma')$$

by Lemma 1.3.2.

To estimate the modulus of $\Gamma = f^{-1}(\Gamma')$, let ϕ denote the stereographic projection of $\overline{\mathbf{R}}^2$ onto the Riemann sphere $\mathbf{S}^2 = \{x \in \mathbf{R}^3 : |x| = 1\}$. If $\gamma \in \Gamma = f^{-1}(\Gamma')$, then γ separates the points 0 and z_1 from ∞ and z_2; hence $\phi(\gamma)$ is a closed curve on $\mathbf{S}^2$ which separates the points $\phi(0)$ and $\phi(z_1)$ from $\phi(\infty)$ and $\phi(z_2)$. Since each arc on $\mathbf{S}^2$ which joins $\phi(0)$ to $\phi(z_1)$ or $\phi(\infty)$ to $\phi(z_2)$ has length at least $\pi/2$,

$$\int_\gamma \frac{2}{1+|z|^2}\,|dz| = \mathrm{length}(\phi(\gamma)) \ge \pi$$

and hence the density

$$\rho(z) = \frac{1}{\pi}\,\frac{2}{1+|z|^2}$$

is admissible for Γ. Thus

$$\mathrm{mod}(\Gamma) \le \int_{\mathbf{R}^2} \rho(z)^2\,dm = \frac{1}{\pi^2}\int_{\mathbf{R}^2} \frac{4}{(1+|z|^2)^2}\,dm = \frac{4}{\pi}$$

and we obtain

$$\frac{1}{2\pi} \log \frac{|f(z_2)|}{|f(z_1)|} \le \mathrm{mod}(\Gamma') \le K\,\mathrm{mod}(\Gamma) \le \frac{4K}{\pi}$$

from which (1.3.5) follows. $\square$

A more detailed reasoning yields the following sharp estimate for the constant c in (1.3.5), namely $c = \lambda(K)$ where

$$\lambda(K) = \left(\frac{1}{4}\, e^{\pi K/2} - e^{-\pi K/2}\right)^2 + \delta(K), \qquad 0 < \delta(K) < e^{-\pi K}. \tag{1.3.6}$$

See Anderson-Vamanamurthy-Vuorinen [**11**] and Lehto-Virtanen-Väisälä [**118**].

COROLLARY 1.3.7. *If* $f : \mathbf{R}^2 \to \mathbf{R}^2$ *is* K*-quasiconformal and if*

$$|z_2 - z_0| \le 2^k\, |z_1 - z_0| \tag{1.3.8}$$

where k *is an integer,* $k \ge 0$*, then*

$$|f(z_2) - f(z_0)| \le c\,(c+1)^k\, |f(z_1) - f(z_0)| \tag{1.3.9}$$

where $c = e^{8K}$.

PROOF. By Theorem 1.3.4, (1.3.8) implies (1.3.9) when $k = 0$. Suppose this implication is true for some $k \ge 0$ and set $z = \frac{1}{2}(z_2 + z_0)$. Then

$$|z_2 - z| = |z - z_0| \le 2^k\, |z_1 - z_0|$$

and

$$|f(z_2) - f(z)| \le c\, |f(z) - f(z_0)|$$

again by Theorem 1.3.4. Since

$$|f(z) - f(z_0)| \le c(c+1)^k\, |f(z_1) - f(z_0)|$$

by hypothesis, we obtain

$$\begin{aligned} |f(z_2) - f(z_0)| &\le |f(z_2) - f(z)| + |f(z) - f(z_0)| \\ &\le (c+1)\, |f(z) - f(z_0)| \\ &\le c(c+1)^{k+1}\, |f(z_1) - f(z_0)|. \end{aligned}$$

Thus (1.3.8) implies (1.3.9) for $k+1$ and hence for all k by induction. □

Theorem 1.3.4 and its corollary are also consequences of the following general result (Gehring-Hag [**57**]), the proof of which is less elementary and depends on theorems due to Teichmüller [**157**] and Agard [**1**].

THEOREM 1.3.10. *If* $f : \mathbf{R}^2 \to \mathbf{R}^2$ *is* K*-quasiconformal, then*

$$\frac{|f(z_2) - f(z_0)|}{|f(z_1) - f(z_0)|} + 1 \le 16^{K-1} \left(\frac{|z_2 - z_0|}{|z_1 - z_0|} + 1\right)^K$$

for $z_0, z_1, z_2 \in \mathbf{R}^2$. *The coefficient* 16^{K-1} *cannot be replaced by any smaller constant.*

The property in Theorem 1.3.10 is called *quasisymmetry* (Heinonen [**80**], Astala-Iwaniec-Martin [**16**]).

We conclude by listing two properties of quasiconformal mappings that we will need in what follows. See, for example, Lehto-Virtanen [**117**].

THEOREM 1.3.11. *If* $f : D \to D'$ *is quasiconformal and if* D *and* D' *are Jordan domains, then* f *has a homeomorphic extension which maps* $\overline{D}$ *onto* $\overline{D'}$.

THEOREM 1.3.12. *Suppose that* $E \subset D$ *is closed and contained in a countable union of rectifiable curves. If* $f : D \to D'$ *is a homeomorphism which is* K*-quasiconformal in each component of* $D \setminus E$*, then* f *is* K*-quasiconformal in* D.

1.4. Quasidisks

We come now to the principal object of study in this book.

DEFINITION 1.4.1. A domain D is a K-*quasidisk* if it is the image of a Euclidean disk or half-plane under a K-quasiconformal self-mapping of $\overline{\mathbf{R}}^2$. D is a *quasidisk* if it is a K-quasidisk for some K.

We present next three Jordan domains that we will use in what follows to illustrate various properties of quasidisks. The first of these is an angular sector.

EXAMPLE 1.4.2. For $0 < \alpha < 2\,\pi$ let $\mathbf{S}(\alpha)$ denote the angular sector

$$\mathbf{S}(\alpha) = \{z = r\,e^{i\,\theta} : 0 < r < \infty,\ |\theta| < \frac{\alpha}{2}\}.$$

Then $\mathbf{S}(\alpha)$ is a K-quasidisk where

$$K = \max\left(\sqrt{\frac{2\,\pi - \alpha}{\alpha}}, \sqrt{\frac{\alpha}{2\,\pi - \alpha}}\right). \tag{1.4.3}$$

The bound in (1.4.3) is sharp.

To prove this, let

$$f(r\,e^{i\,\theta}) = r^p\,e^{i\,\phi(\theta)}$$

for $0 < r < \infty$ and $|\theta| \le \pi$ where

$$p = \frac{\pi}{\sqrt{(2\,\pi - \alpha)\,\alpha}}$$

and

$$\phi(\theta) = \begin{cases} \dfrac{\pi\,\theta}{\alpha} & \text{if} \quad 0 \le \theta \le \dfrac{\alpha}{2}, \\ \pi - \dfrac{\pi\,(\pi - \theta)}{2\,\pi - \alpha} & \text{if} \quad \dfrac{\alpha}{2} \le \theta \le \pi, \\ -\phi(-\theta) & \text{if} \quad -\pi \le \theta \le 0. \end{cases}$$

An elementary calculation shows that f is K-quasiconformal, where K is as in (1.4.3), and that f maps $\mathbf{S}(\alpha)$ onto the right half-plane $\mathbf{S}(\pi)$.

To show that the bound in (1.4.3) is best possible, suppose that f is a K-quasiconformal mapping of $\overline{\mathbf{R}}^2$ which maps $\mathbf{S}(\alpha)$ onto the right half-plane $\mathbf{S}(\pi)$ and let $h = f^{-1} \circ g \circ f$ where g denotes reflection in the imaginary axis. Then h is a K^2-quasiconformal mapping of $\overline{\mathbf{R}}^2$ which maps $\mathbf{S}(\alpha)$ onto its exterior $\mathbf{S}^*(\alpha)$.

Next fix $0 < a < b < \infty$ and let Γ denote the family of arcs which join the circles $\{z : |z| = a\}$ and $\{z : |z| = b\}$ in

$$\{z : a \le |z| \le b,\ |\arg(z)| < \alpha/2\}.$$

Then it is not difficult to check that

$$\operatorname{mod}(\Gamma) = \frac{\alpha}{\log(b/a)}.$$

Similarly,

$$\operatorname{mod}(\Gamma') = \frac{2\,\pi - \alpha}{\log(b/a) + 2\log(c)}$$

where Γ' is the family of arcs which join $\{z : |z| = a/c\}$ and $\{z : |z| = bc\}$ in

$$\{z : a/c \le |z| \le bc,\ \alpha/2 < |\arg(z)| \le \pi\}.$$

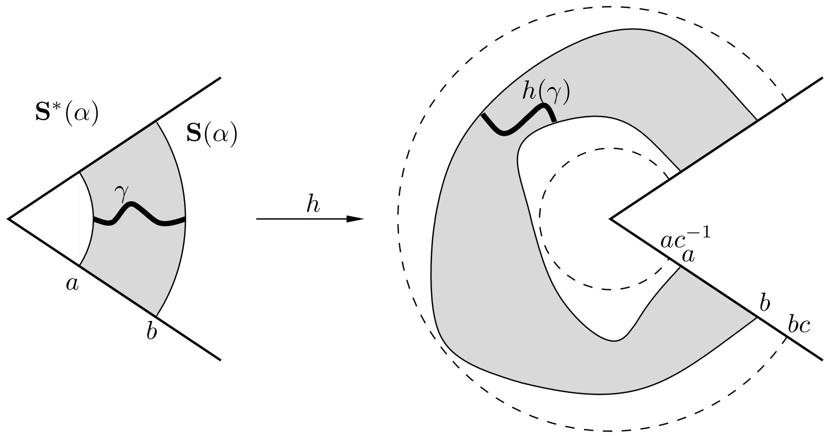

FIGURE 1.6

Hence if $c = 8\,e^{K^2}$, then Theorem 1.3.4 implies that for each arc $\gamma' \subset \Gamma'$ there exists an arc $\gamma \in \Gamma$ such that $h(\gamma) \subset \gamma'$. Thus $\mathrm{adm}(h(\Gamma)) \subset \mathrm{adm}(\Gamma')$, whence

$$\mathrm{mod}(h(\Gamma)) \geq \mathrm{mod}(\Gamma')$$

and

$$K^2 \geq \frac{\mathrm{mod}(h(\Gamma))}{\mathrm{mod}(\Gamma)} \geq \frac{2\,\pi - \alpha}{\alpha}\,\frac{\log(b/a)}{\log(b/a) + 2\log(c)}.$$

We conclude that

$$K^2 \geq \frac{2\,\pi - \alpha}{\alpha}$$

by letting $b/a \to \infty$.

Finally reversing the roles of $\mathbf{S}(\alpha)$ and $\mathbf{S}^*(\alpha)$ in the above argument yields

$$K^2 \geq \frac{\alpha}{2\,\pi - \alpha}$$

and hence (1.4.3).

DEFINITION 1.4.4. A domain D is a *sector of angle* α if it is the image of $\mathbf{S}(\alpha)$ under a similarity mapping.

Our second example is a simple Jordan domain that is not a quasidisk.

EXAMPLE 1.4.5. The half-strip

$$D = \{z = x + i\,y : 0 < x < \infty,\ |y| < 1\}$$

is not a quasidisk.

We shall show that there exists no quasiconformal self-mapping f of $\overline{\mathbf{R}}^2$ which maps $\mathbf{H}$ onto D. By performing a preliminary Möbius transformation, we need only consider the case where $f(\infty) = \infty$.

Suppose that f is a K-quasiconformal self-mapping of $\mathbf{R}^2$ with $f(\mathbf{H}) = D$, set $w_1 = x + i$, $w_2 = 0$, $w_3 = x - i$, and let $z_i = f^{-1}(w_i)$ for $i = 1, 2, 3$. Then z_1, z_2, z_3 is an ordered triple of points on $\partial\mathbf{H}$ with

$$|z_1 - z_2| < |z_1 - z_3|$$

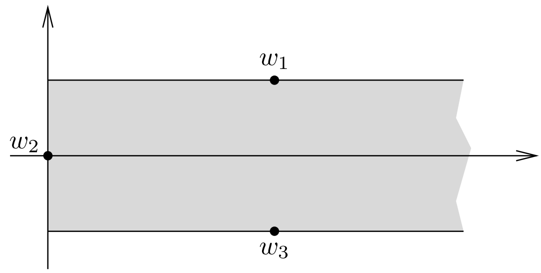

FIGURE 1.7

for each choice of x in $(0, \infty)$. On the other hand,

$$x < |f(z_1) - f(z_2)| \le c\,|f(z_1) - f(z_3)| = 2\,c$$

by Theorem 1.3.4 where $c = c(K)$ and we have a contradiction.

If D is a K-quasidisk, then ∂D is the image of a circle under a self-homeomorphism f of $\overline{\mathbf{R}}^2$ which is differentiable a.e. Thus ∂D is a Jordan curve which is a circle or line when $K = 1$. Hence it is natural to ask if ∂D has any nice analytic properties when $1 < K < \infty$. For example, is ∂D locally rectifiable?

Our third example shows that the answer is no and that, from the standpoint of Euclidean geometry, the boundary of a quasidisk can be quite wild. See Gehring-Väisälä [**70**].

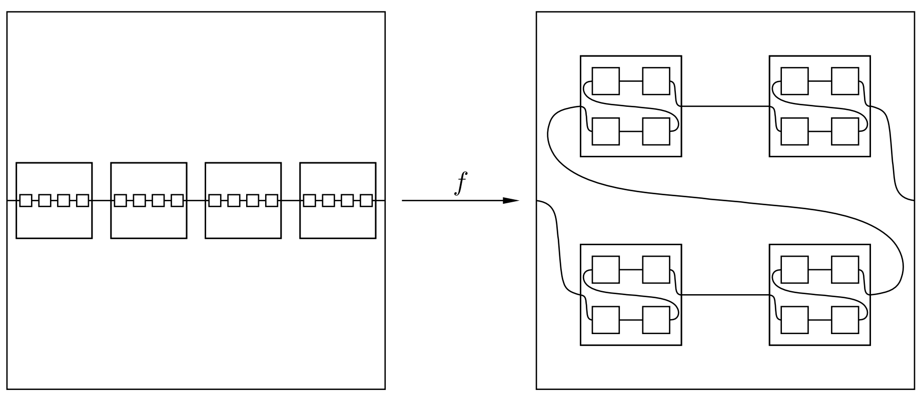

FIGURE 1.8

EXAMPLE 1.4.6. For each $1 < a < 2$ there exists a quasidisk D such that

$$\dim(\partial D) \ge a$$

where dim denotes Hausdorff dimension.

We will sketch a proof of this. We say that a square is *oriented* if its sides are parallel to the coordinate axes and we let Q and Q' denote the open squares

$$Q = Q' = \{z = x + i\,y : |x| < 1, |y| < 1\}.$$

Next set

$$z_1 = \frac{3}{4}, \qquad z_2 = \frac{1}{4}, \qquad z_3 = -\frac{1}{4}, \qquad z_4 = -\frac{3}{4}$$

and

$$w_1 = \frac{1+i}{2}, \qquad w_2 = \frac{-1+i}{2}, \qquad w_3 = \frac{1-i}{2}, \qquad w_4 = \frac{-1-i}{2},$$

and fix $0 < r < 1/2$. Then choose $0 < s < 1$ so that

$$\frac{\log 4}{\log(2/s)} = a.$$

Finally for $j = 1, 2, 3, 4$ let Q_j denote the open oriented square with center z_j and side length r and let Q'_j be the open oriented square with center w_j and side length s. Then we can choose a piecewise linear homeomorphism

$$f_0 : \overline{Q} \setminus \bigcup_{j=1}^{4} Q_j \to \overline{Q'} \setminus \bigcup_{j=1}^{4} Q'_j$$

such that f_0 is the identity on ∂Q and is of the form $a_j z + b_j, a_j > 0$, on ∂Q_j with $f_0(\partial Q_j) = \partial Q'_j$. Then f_0 is K-quasiconformal in $Q \setminus \bigcup_j \overline{Q}_j$ where $K = K(r, s)$.

Next for each j choose oriented squares $Q_{j,k}$ in Q_j and $Q'_{j,k}$ in Q'_j in the same way as the squares Q_j and Q'_j were chosen in Q and Q', respectively. By scaling we can extend f_0 to obtain a piecewise linear homeomorphism

$$f_1 : \overline{Q} \setminus \bigcup_{j,k=1}^{4} Q_{j,k} \to \overline{Q'} \setminus \bigcup_{j,k=1}^{4} Q'_{j,k}$$

which is K-quasiconformal in $Q \setminus \bigcup_{j,k} \overline{Q}_{j,k}$.

Continuing in this way, we obtain a homeomorphism

$$f : \overline{Q} \setminus E \to \overline{Q'} \setminus E'$$

where E and E' are Cantor sets. Then f can be extended by continuity to give a K-quasiconformal mapping which maps $\overline{Q}$ onto $\overline{Q'}$ and is the identity on ∂Q.

Set $f(z) = z$ in $\overline{\mathbf{R}}^2 \setminus \overline{Q}$. Then f is a K-quasiconformal self-mapping of $\overline{\mathbf{R}}^2$ which maps the upper half-plane $\mathbf{H}$ onto a quasidisk D with Hausdorff dimension

$$\dim(\partial D) \geq \frac{\log 4}{\log(2/s)} = a.$$

See Beardon [**17**] or page 67 in Mattila [**127**].

Although the Hausdorff dimension of the boundary ∂D of a quasidisk D can be arbitrarily close to 2, it always satisfies $m(\partial D) = 0$ where m is planar Lebesgue measure. This follows from Lusin's property (N) of quasiconformal mappings in Theorem 1.1.8. On the other hand, a result due to Astala [**13**] gives the estimate

$$\dim(\partial D) \leq \frac{2K}{K+1}$$

for the Hausdorff dimension of the boundary ∂D of a K-quasidisk D.

Our final example, or rather class of examples, in this section illustrates how quasidisks arise naturally in complex dynamics.

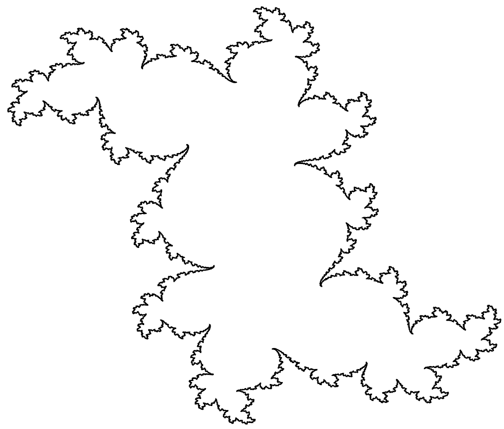

FIGURE 1.9

EXAMPLE 1.4.7. For a nonconstant meromorphic function $f\colon \overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}^2$, the *iterates*

$$f^n(z) = f \circ f^{n-1}(z),\ n \geq 2, \qquad f^1(z) = f(z)$$

are all defined and meromorphic. The *Fatou set* F_f of f is the largest open set where the sequence (f^n) is a normal family, while its complement, $J_f = \overline{\mathbf{R}}^2 \setminus F_f$, is called the *Julia set*.

If p is a polynomial function of degree two, we may assume without loss of generality that it has the form

$$p_c(z) = z^2 + c.$$

If $c = 0$, the Julia set is the unit circle, and if $|c| < 1/4$, it can be shown that the Fatou set has exactly two components F_0 and F_∞, with $0 \in F_0$ and $\infty \in F_\infty$. See Beardon [**19**] or Carleson-Gamelin [**31**]. Arguments using Theorem 1.1.11 in an ingenious way reveal that in fact F_0 is a quasidisk. See e.g. Carleson-Gamelin [**31**].

1.5. What is ahead

Though quasidisks can be quite pathological domains, they occur very naturally in surprisingly many branches of analysis and geometry. We will describe in what follows some thirty different properties of quasidisks which generalize corresponding properties of Euclidean disks and which characterize this class of domains. See also Gehring [**51**] and [**54**].

The properties of a quasidisk D that we will discuss fall into the following categories:

1° geometric properties of D,
2° conformal invariants defined in D,
3° injectivity criteria for functions defined in D,
4° criteria for extension of functions defined in D,
5° two-sided criteria for D and D^*,
6° miscellaneous properties.

In the remainder of Part 1 (Chapters 2 to 7) we will consider properties in each of these categories. A number of them can be used to characterize Euclidean disks or half-planes. We will indicate when this is the case.

In Part 2 (Chapters 8 to 11) we will present proofs for some of the characterizations mentioned above. Many of the arguments follow a series of implications. There are four main series of implications, as well as some additional equivalences proved. Some proofs not in the main series of implications belong naturally to the discussion of the results and are presented in Part 1.