Preface

This book is not meant to be another compendium of select inequalities, nor does it claim to contain the latest or the slickest ways of proving them. It is rather an attempt at describing how most functional inequalities are not merely the byproduct of ingenious guess work by a few wizards among us, but are often manifestations of natural mathematical structures and physical phenomena. Our main goal here is to show how this point of view leads to "systematic" approaches for proving the most basic functional inequalities, but also for understanding and improving them, and for devising new ones - sometimes at will, and often on demand.

Our aim is therefore to describe how a few general principles are behind the validity of large classes – and often "equivalence classes" – of functional inequalities, old and new. As such, Hardy and Hardy-Rellich type inequalities involving radially symmetric weights are variational manifestations of Sturm's theory on the oscillatory behavior of certain ordinary differential equations. Similarly, allowable non-radial weights in Hardy-type inequalities for more general uniformly elliptic operators are closely related to the resolution of certain linear PDEs in divergence form with either a prescribed boundary condition or with prescribed singularities in the interior of the domain.

On the other hand, most geometric inequalities including those of Sobolev and Log-Sobolev type, are simply expressions of the convexity of certain free energy functionals along the geodesics of the space of probability measures equipped with the optimal mass transport (Wasserstein) metric. Hardy-Sobolev and Hardy-Rellich-Sobolev type inequalities are then obtained by interpolating the above two classes of inequalities via the classical ones of Hölder.

Besides leading to new and improved inequalities, these general principles offer novel ways for estimating their best constants, and for deciding whether they are attained or not in the appropriate function space. In the improved versions of Hardy-type inequalities, the best constants are related to the largest parameters for which certain linear ODEs have non-oscillatory solutions. Duality methods, which naturally appear in the new "geodesic convexity" approach to geometric inequalities, allow for the evaluation of the best constants from first order equations via the limiting case of Legendre-Fenchel duality, as opposed to the standard method of solving second order Euler-Lagrange equations.

Whether a "best constant" on specific domains is attained or not, is often dependent on how it compares to related best constants on limiting domains, such as the whole space or on half-space. These results are based on delicate blow-up analysis, and are reminiscent of the prescribed curvature problems initiated by Yamabe and Nirenberg. The exceptional case of the Sobolev inequalities in two dimensions initiated by Trudinger and Moser can also be linked to mass transport methods, and some of their recent improvements by Onofri, Aubin and others are both interesting and still challenging. They will be described in the last part of the monograph.

The parts dealing with Hardy and Hardy-type inequalities represent a compendium of an approach mostly developed by –and sometimes with– my (now former) students Amir Moradifam and Craig Cowan. The weighted second order inequalities have some overlap with the books of B. Opic and A. Kufner and the one by A. Kufner and L-E Persson, which have been the standard references on the subject. The part on fourth order inequalities reflect more recent developments.

The “mass transport” approach to geometric inequalities follows closely work with my former student X. Kang and postdoctoral fellow Martial Agueh. It is largely based on the pioneering work of Cedric Villani, Felix Otto, Robert McCann, Wilfrid Gangbo, Dario Cordero-Erausquin, Bruno Nazareth, Christian Houdré and many others. Unfortunately, we do not include here another related “entropy-energy” approach to functional inequalities that is closely intertwined with the study of large time asymptotics of evolution equations, typically diffusive or hypocoercive kinetic equations. This approach has been developed and used extensively by A. Arnold, J. Carillo, L. Desvillettes, J. Dolbeault, A. Jungel, P. Markowich, G. Toscani, A. Unterreiter and C. Villani, to name a few.

The chapters dealing with Hardy-Sobolev type inequalities follow work done with my students Chaogui Yuan, and Xiaosong Kang, as well as my collaborator Frederic Robert. Finally, much of the progress on the –still unresolved– best constant in Moser-Onofri-Aubin inequalities on the 2-dimensional sphere was done with my friends and collaborators, Joel Feldman, Richard Froese, Changfeng Gui, and Chang-Shou Lin. I owe all these people a great deal of gratitude.

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