CHAPTER 1

Bessel Pairs and Sturm's Oscillation Theory

The ordinary differential equation associated to a non-negative real valued C^1-function P on $(0,+\infty)$,

$$y'' + \frac{1}{r}y' + P(r)y = 0,$$

as well as the equation

$$(V(r)y')' + W(r)y = 0,$$

associated to a pair (V, W) of non-negative functions on $(0,+\infty)$, are central to all results revolving around the inequalities of Hardy and Hardy-Rellich type. We summarize in this chapter the properties of these equations that will be used throughout this book. In particular, we give conditions on P (resp., V and W), which guarantee that the above equations have a positive solution on a non-trivial interval $(0, R)$.

1.1. The class of Hardy improving potentials

DEFINITION 1.1.1. We say that a non-negative real valued C^1-function P is a *Hardy improving potential* –abbreviated as "HI-potentials"– on $(0, R)$, if there exists $c > 0$ such that the equation

$$(\mathcal{B}_{cP}) \qquad y''(r) + \frac{1}{r}y'(r) + cP(r)y(r) = 0,$$

has a positive solution on $(0, R)$.

The class of HI-potentials on $(0, R)$ will be denoted by $\mathcal{B}(0, R)$. Here are a few immediate examples of such functions:

- $P \equiv 0$ is a HI-potential on $(0, R)$ for any $R > 0$. Indeed, It is clear that $\varphi(r) = -log(\frac{e}{R}r)$ is a positive solution of $(\mathcal{B}_0)$ on $(0, R)$.
- $P \equiv 1$ is a HI-potential on $(0, z_0)$, where $z_0 = 2.4048...$ is the first zero of the Bessel function J_0. Indeed, the latter is a positive solution of $y'' + \frac{1}{r}y' + y = 0$ until it reachs its first zero at z_0.
- For any $\rho > Re$, the function $P(r) = \frac{1}{4r^2(log\frac{\rho}{r})^2}$ is a HI-potential on $(0, R)$, with (B_P) having the explicit solution $\varphi_\rho(r) = (log\frac{\rho}{r})^{\frac{1}{2}}$.

We shall later need the following easy result regarding the behaviour of positive solutions of equation $(\mathcal{B}_P)$.

LEMMA 1.1.2. *Assume P is non-negative on $(0, R)$, $a \geq 1$ and that the equation*

$$y'' + \frac{a}{r}y' + P(r)y = 0, \tag{1.1}$$

has a positive solution φ on $(0, R)$. Then,

(1) *φ is decreasing on $(0, R)$.*

(2) *φ has the following limiting behavior on the boundary:*

$$\lim_{r\to 0} r\frac{\varphi'(r)}{\varphi(r)} = 0 \quad \text{and} \quad \limsup_{r\to R} \frac{\varphi'(r)}{\varphi(r)} \leq 0. \tag{1.2}$$

(3) *If $P(r) > 0$ on $(0, R)$, then φ is strictly decreasing on $(0, R)$.*
(4) *If $P(r) > 0$ on $(0, +\infty)$, then (1.1) has no positive solution on $(0, +\infty)$.*

Proof: Observe that the function $x(r) = r\frac{\varphi'(r)}{\varphi(r)}$ satisfies the ODE:

$$rx'(r) + x^2(r) = -F(r), \qquad \text{for } \ 0 < r \leq \delta, \tag{1.3}$$

where $F(r) = r^2P(r) \geq 0$. It follows that $\varphi'(t) \leq 0$ on $(0, R)$ and is therefore decreasing on that interval.

To prove 2), divide equation (1.3) by r and integrate once to obtain

$$x(r) \geq \int_r^R \frac{|x(s)|^2}{s} ds + x(R) + \int_r^R \frac{F(s)}{s} ds. \tag{1.4}$$

It follows that $\lim_{r\downarrow 0} x(r)$ exists. In order to prove that this limit is zero, it therefore suffices to prove that

$$G(r) := \int_r^R \frac{x^2(s)}{s} ds < +\infty. \tag{1.5}$$

Suppose not, that is $G(r) \to \infty$ as $r \to 0$. From (1.3) we have

$$(-rG'(r))^{\frac{1}{2}} \geq G(r) + x(1) + \int_r^R \frac{F(s)}{s} ds.$$

Since $F \geq 0$, and G goes to infinity as r goes to zero, then for r sufficiently small we have $-rG'(r) \geq \frac{1}{2}G^2(r)$, and hence, $(\frac{1}{G(r)})' \geq \frac{1}{2}(\ln(r))'$, which contradicts the fact that $G(r)$ goes to infinity as r tends to zero. It follows that indeed, $\lim_{r\downarrow 0} r\frac{\varphi'(r)}{\varphi(r)} = \lim_{r\downarrow 0} x(t) = 0$.

For 3) it suffices to note that if $P(r) > 0$ on $(0, R)$, then φ cannot have a local minimum in $(0, R)$. Indeed, if $\varphi'(x_0) = 0$ for some $R > x_0 > 0$, and $\varphi''(x_0) \geq 0$, then necessarily $\varphi''(x_0) = 0$ which contradicts the fact that φ is a positive solution of the above ODE. it follows that φ is strictly decreasing on the whole interval $(0, R)$.

4) Suppose that for any $R > 0$, the equation $y''(r) + \frac{a}{r}y' + P(r)y = 0$ has a positive solution φ_R, which is then necessarily strictly decreasing on $(0, R)$. It follows that $\frac{\varphi_R''(r)}{\varphi_R'(r)} \geq -\frac{a}{r}$ on $(0, R)$ which yields that for some $c > 0$, we have $\varphi_R'(r) \leq \frac{-c}{r}$ on $(0, R)$. We can also clearly assume that if $R < R'$, then $\varphi_R \geq \varphi_{R'}$ on $(0, R]$. It then follows that $\varphi_R(R) \leq \varphi_R(1) - c\ln R \leq \varphi_1(1) - c\ln R$ for any $R > 1$. This means that $\varphi_R(R) < 0$ for R large enough, which is clearly a contradiction. □

It is clear that if $P \in \mathcal{B}(0, R)$, then $P \in \mathcal{B}(0, R')$ for any $0 < R' < R$. We shall be interested by the largest such interval, i.e.,

$$\delta(P) := \sup\big\{R;\ (\mathcal{B}_P) \text{ has a positive solution on } (0, R)\big\}. \tag{1.6}$$

In view of the above lemma, $\delta(P)$ can be seen as the first time a particular positive solution of $(\mathcal{B}_P)$ reaches zero.

On the other hand, we shall consider for a given $R > 0$, the largest *HIP-constant* associated to a HI-potential $P \in \mathcal{B}(0,R)$, defined as:

$$\beta(P;R) = \sup\{c > 0;\ (\mathcal{B}_{cP}) \text{ has a positive solution on } (0,R)\}. \tag{1.7}$$

The following proposition will be frequently used in the sequel.

PROPOSITION 1.1.1. *Let P be a non-negative C^1-function on an interval $(0,R)$. Then,*

1. *$(\mathcal{B}_P)$ has a positive solution on $(0,R)$ if and only if it has a positive supersolution φ on $(0,R)$, i.e., if*

$$\varphi'' + \tfrac{1}{r}\varphi' + P(r)\varphi \leq 0 \text{ on } (0,R). \tag{1.8}$$

2. *Consequently, if $\mathcal{B}_P$ has a positive solution on an interval $(0,R)$ for some non-negative C^1-potential $P \geq 0$, then for any C^1-function Q such that $0 \leq Q \leq P$, the equation $(\mathcal{B}_Q)$ has also a positive solution on $(0,R)$.*
3. *The class $\mathcal{B}(0,R)$ of HI-potentials on $(0,R)$ is a closed convex and solid subset of $C^1(0,R)$.*
4. *Moreover, for every $P \in \mathcal{B}(0,R)$, the equation (B_{cP}) has a positive solution on $(0,R)$, for all $c \leq \beta(P;R)$.*

Proof: Statement 1) is a direct consequence of proposition 1.2.1. The proofs of 2), 3), and 4) are straightforward and are left to the interested reader. □

For a given HI-potential on an interval $(0,R)$, we shall often be interested in computing its *HIP-constant* $\beta(P;R)$. This is often closely related to finding $\delta(P)$. Indeed, if (B_P) has a positive solution φ on $(0,\delta)$ for some $\delta > 0$, then $\psi(r) = \varphi(\frac{\delta r}{R})$ is a solution for $y''(r) + \frac{1}{r}y' + \frac{\delta^2}{R^2}P(\frac{\delta}{R}r)y = 0$ on $(0,R)$. In other words, the scaled potential $V(x) = \frac{\delta^2}{R^2}P(\frac{\delta}{R}x)$ is then a HI-potential on $(0,R)$. We therefore have the following relations.

PROPOSITION 1.1.2. *If P is a C^1-function such that $(\mathcal{B}_P)$ has a positive solution on $(0,\delta)$ for some $\delta > 0$, then for any $R > 0$, the function Q defined by $Q(x) := P(\frac{\delta}{R}x)$ belongs to $\mathcal{B}(0,R)$, and*

$$\beta(Q;R) = \frac{\delta(P)^2}{R^2}\beta(P;\delta(P)). \tag{1.9}$$

In particular, if P is also α-homogeneous (i.e. $P(\lambda x) = \lambda^{-\alpha}P(x)$ for some $\alpha > 0$), then

$$\beta(P;R) = \frac{\delta(P)^{2-\alpha}}{R^{2-\alpha}}. \tag{1.10}$$

□

We now exhibit a few explicit HI-potentials and compute their HIP-constants. We use the following notation.

$$log^{(1)}(.) = log(.) \quad \text{and} \quad log^{(k)}(.) = log(log^{(k-1)}(.)) \quad \text{for} \quad k \geq 2. \tag{1.11}$$

and

$$X_1(t) = (1 - \log(t))^{-1}, \quad X_k(t) = X_1(X_{k-1}(t)) \quad k = 2, 3, ..., \tag{1.12}$$

THEOREM 1.1.3. Explicit HI-potentials

(1) *$P \equiv 0$ is a HI-potential on $(0,R)$ for any $R>0$.*
(2) *$P \equiv 1$ is a HI-potential on $(0,R)$ for any $R>0$. Moreover,*

$$\delta(1) = z_0, \quad \text{and} \quad \beta(1;R) = \tfrac{z_0^2}{R^2} \text{ for every } R>0, \tag{1.13}$$

(3) *If $0 \le a < 2$, then $P(r) = r^{-a}$ is a HI-potential on $(0,R)$ for any $R>0$. Moreover, there is $z_a > 0$ such that*

$$\delta(r^{-a}) = z_a, \quad \text{and} \quad \beta(r^{-a};R) = \tfrac{z_a^{2-\alpha}}{R^{2-\alpha}} \text{ for every } R>0, \tag{1.14}$$

where z_a is the first root of the largest solution of the equation $y'' + \frac{1}{r}y' + r^{-a}y = 0$.

(4) *For each $k \ge 1$ and $R>0$, the function $P_{k,\rho}(r) = \frac{1}{r^2}\sum\limits_{j=1}^{k}\big(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{r}\big)^{-2}$, where $\rho > R(e^{e^{e^{\cdot^{\cdot^{\cdot^{e(k-times)}}}}}})$, is a HI-potential on $(0,R)$ and*

$$\beta(P_{k;\rho}, R) = \tfrac{1}{4}. \tag{1.15}$$

(5) *For $k \ge 1$ and $R>0$, the function*

$$\tilde{P}_{k,R}(r) = \frac{1}{r^2}\sum_{j=1}^{k} X_1^2(\frac{r}{R})X_2^2(\frac{r}{R})\ldots X_{j-1}^2(\frac{r}{R})X_j^2(\frac{r}{R})$$

is a HI-potential on $(0,R)$ and

$$\beta(\tilde{P}_{k,R};R) = \tfrac{1}{4}. \tag{1.16}$$

Proof: 1) As noted above, for any $R>0$ the function $\varphi(r) = -log(\frac{e}{R}r)$ is a positive solution of $(\mathcal{B}_0)$ on $(0,R)$.

2) The Bessel function J_0 is a positive solution for equation $\mathcal{B}_P$ with $P \equiv 1$, on $(0,z_0)$, where $z_0 = 2.4048...$ is the first zero of J_0. Moreover, z_0 is larger than the first root of any other solution for $(\mathcal{B}_1)$. Indeed if α is the first root of the an arbitrary solution of the Bessel equation $y'' + \frac{y'}{r} + y(r) = 0$, then we have $\alpha \le z_0$. To see this let $x(t) = aJ_0(t) + bY_0(t)$, where J_0 and Y_0 are the two standard linearly independent solutions of Bessel equation, and a and b are constants. Assume the first zero of $x(t)$ is larger than z_0. Since the first zero of Y_0 is smaller than z_0, we have $a \ge 0$. Also $b \le 0$, because $Y_0(t) \to -\infty$ as $t \to 0$. Finally note that $Y_0(z_0) > 0$, so if $b<0$, then $x(z_0+\epsilon) < 0$ for ϵ sufficiently small. Therefore, $b=0$ which is a contradiction. The rest follows from Proposition 1.1.2.

3) will follow from the integral criteria below –as applied in Corollary 1.3.2– while formula (1.14) also follows from Proposition 1.1.2.

4) Note first that $\varphi_{k,\rho}(r) = (\prod_{i=1}^{k} log^{(i)}\frac{\rho}{r})^{\frac{1}{2}}$ is an explicit solution of the equation $(\mathcal{B}_{\frac{1}{4}P_k})$ on $(0,R)$ provided $\rho > R(e^{e^{e^{\cdot^{\cdot^{\cdot^{e(k-times)}}}}}})$. This readily implies that $\beta(P_{k;\rho}, R) \ge \frac{1}{4}$. To establish equality, we need to show that equation $(\mathcal{B}_{(\frac{1}{4}+\lambda)P_{k,\rho}})$ has no positive solution for any $\lambda > 0$. Since $P_{k,\rho}(r) = \Sigma_{j=1}^{k} U_j$ where $U_j(r) = \frac{1}{r^2}\big(\prod_{i=1}^{j} log^{(i)}\frac{\rho}{r}\big)^{-2}$, it suffices to show that equation $(\mathcal{B}_{\frac{1}{4}P_{k-1,\rho}+\lambda U_k})$ which corresponds to the smaller weight $\frac{1}{4}P_{k-1,\rho} + \lambda U_k$ has no positive solution for any $\lambda > 0$.

To do that, we assume that there exists a positive function φ on $(0,R)$ such that

$$-\frac{\varphi'(r)+r\varphi''(r)}{\varphi(r)}=\frac{1}{4}\sum_{j=1}^{k-1}\frac{1}{r}(\prod_{i=1}^{j}log^{(i)}\frac{\rho}{r})^{-2}+(\frac{1}{4}+\lambda)\frac{1}{r}(\prod_{i=1}^{k}log^{(i)}\frac{\rho}{r})^{-2},$$

and work towards a contradiction.

Set $f(r)=\frac{\varphi(r)}{\varphi_k(r)}>0$, where $\varphi_k=\varphi_{k,\rho}$ defined above, and calculate,

$$\frac{\varphi'(r)+r\varphi''(r)}{\varphi(r)}=\frac{\varphi_k'(r)+r\varphi_k''(r)}{\varphi_k(r)}+\frac{f'(r)+rf''(r)}{f(r)}-\frac{f'(r)}{f(r)}\sum_{i=1}^{k}\frac{1}{\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}.$$

Thus,

$$\frac{f'(r)+rf''(r)}{f(r)}-\frac{f'(r)}{f(r)}\sum_{i=1}^{k}\frac{1}{\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}=-\lambda\frac{1}{r}(\prod_{i=1}^{k}log^{(i)}\frac{\rho}{r})^{-2}. \tag{1.17}$$

If now $f'(\alpha_n)=0$ for some sequence $\{\alpha_n\}_{n=1}^{\infty}$ that converges to zero, then there exists a sequence $\{\beta_n\}_{n=1}^{\infty}$ that also converges to zero, such that $f''(\beta_n)=0$, and $f'(\beta_n)>0$. But this contradicts (1.17), which means that f is eventually monotone for r small enough. We consider the two cases according to whether f is increasing or decreasing:

Case I: Assume $f'(r)>0$ for $r>0$ sufficiently small. Then we will have

$$\frac{(rf'(r))'}{rf'(r)}\leq\sum_{i=1}^{k}\frac{1}{r\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}.$$

Integrating once we get $f'(r)\geq\frac{c}{r\prod_{j=1}^{k}\log^j(\frac{\rho}{r})}$, for some $c>0$. Hence, $\lim_{r\to0}f(r)=-\infty$ which is a contradiction.

Case II: Assume $f'(r)<0$ for $r>0$ sufficiently small. Then

$$\frac{(rf'(r))'}{rf'(r)}\geq\sum_{i=1}^{k}\frac{1}{r\prod_{j=1}^{i}\log^j(\frac{\rho}{r})}.$$

Thus,

$$f'(r)\geq-\frac{c}{r\prod_{j=1}^{k}\log^j(\frac{\rho}{r})}, \tag{1.18}$$

for some $c>0$ and $r>0$ sufficiently small. On the other hand

$$\frac{f'(r)+rf''(r)}{f(r)}\leq-\lambda\sum_{j=1}^{k}\frac{1}{r}(\prod_{i=1}^{j}log^{(i)}\frac{R}{r})^{-2}\leq-\lambda(\frac{1}{\prod_{j=1}^{k}\log^j(\frac{\rho}{r})})'.$$

Since $f'(r)<0$, there exists l such that $f(r)>l>0$ for $r>0$ sufficiently small. From the above inequality we then have

$$bf'(b)-af'(a)<-\lambda l(\frac{1}{\prod_{j=1}^{k}\log^j(\frac{\rho}{b})}-\frac{1}{\prod_{j=1}^{k}\log^j(\frac{\rho}{a})}).$$

From (1.18) we have $\lim_{a\to0}af'(a)=0$. Hence, $bf'(b)<-\frac{\lambda l}{\prod_{j=1}^{k}\log^j(\frac{\rho}{b})}$, for every $b>0$, and $f'(r)<-\frac{\lambda l}{r\prod_{j=1}^{k}\log^j(\frac{\rho}{r})}$, for $r>0$ sufficiently small. Therefore,

$\lim_{r\to 0} f(r) = +\infty$, and by choosing l large enouph (e.g., $l > \frac{c}{\lambda}$) we get to contradict (1.18).

The proof of 5) is similar. Indeed, let $D \geq \sup_{x\in\Omega} |x|$, and define

$$\varphi_k(r) = (X_1(\frac{r}{D})X_2(\frac{r}{D})\ldots X_{i-1}(\frac{r}{D})X_i(\frac{r}{D}))^{-\frac{1}{2}}, \quad i = 1, 2, \ldots.$$

Using the fact that $X_k'(r) = \frac{1}{r}X_1(r)X_2(r)\ldots X_{k-1}(r)X_k^2(r)$ for $k = 1, 2, \ldots$, we get

$$-\frac{\varphi_k'(r) + r\varphi_k''(r)}{\varphi_k(r)} = \frac{1}{4r}\Sigma_{j=1}^k X_1^2(\frac{r}{D})X_2^2(\frac{r}{D})\ldots X_{j-1}^2(\frac{r}{D})X_j^2(\frac{r}{D}).$$

This means that $\beta(\tilde{P}_{k;R}, R) \geq \frac{1}{4}$.

One can again show that $\frac{1}{4}$ is the best constant by assuming in contradiction that for some $\lambda > 0$, there exists a positive function φ such that

$$\begin{aligned} -\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)} &= \frac{1}{4}\sum_{j=1}^{m-1}\frac{1}{r}X_1^2(\frac{r}{D})X_2^2(\frac{r}{D})\ldots X_{j-1}^2(\frac{r}{D})X_j^2(\frac{r}{D}) \\ &+ (\frac{1}{4} + \lambda)\frac{1}{r}X_1^2(\frac{r}{D})X_2^2(\frac{r}{D})\ldots X_{m-1}^2(\frac{r}{D})X_m^2(\frac{r}{D}). \end{aligned}$$

Setting $f(r) = \frac{\varphi(r)}{\varphi_m(r)} > 0$, we have

$$\frac{\varphi'(r) + r\varphi''(r)}{\varphi(r)} = \frac{\varphi_m'(r) + r\varphi_m''(r)}{\varphi_m(r)} + \frac{f'(r) + rf''(r)}{f(r)} - \frac{f'(r)}{f(r)}\sum_{i=1}^m\prod_{j=1}^i X_j(\frac{r}{D}).$$

Thus,

$$\frac{f'(r) + rf''(r)}{f(r)} - \frac{f'(r)}{f(r)}\sum_{i=1}^m\prod_{j=1}^i X_j(\frac{r}{D}) = -\lambda\frac{1}{r}\prod_{j=1}^m X_j^2(\frac{r}{D}). \tag{1.19}$$

Arguing as before, we deduce that f is eventually monotone for r small enough, and we consider two cases:

Case I: If $f'(r) > 0$ for $r > 0$ sufficiently small, then we will have

$$\frac{(rf'(r))'}{rf'(r)} \leq \sum_{i=1}^m\frac{1}{r}\prod_{j=1}^i X_j(\frac{r}{D}).$$

Integrating once we get $f'(r) \geq \frac{c}{r}\prod_{j=1}^m X_j(\frac{r}{D})$, for some $c > 0$, and therefore $\lim_{r\to 0} f(r) = -\infty$ which is a contradiction.

Case II: Assume $f'(r) < 0$ for $r > 0$ sufficiently small. Then

$$\frac{(rf'(r))'}{rf'(r)} \geq \sum_{i=1}^m\frac{1}{r}\prod_{j=1}^i X_j(\frac{r}{D})$$

Thus,

$$f'(r) \geq -\frac{c}{r}\prod_{j=1}^m X_j(\frac{r}{D}), \tag{1.20}$$

for some $c > 0$ and $r > 0$ sufficiently small. On the other hand

$$\frac{f'(r) + rf''(r)}{f(r)} \leq -\lambda \sum_{j=1}^{m} \frac{1}{r} \prod_{i=1}^{j} X_j^2 \leq -\lambda (\prod_{j=1}^{m} X_j(\frac{r}{D}))'.$$

Since $f'(r) < 0$, we may assume $f(r) > l > 0$ for $r > 0$ sufficiently small, and from the above inequality we have

$$bf'(b) - af'(a) < -\lambda l (\prod_{j=1}^{m} X_j(\frac{b}{D}) - \prod_{j=1}^{m} X_j(\frac{a}{D})).$$

From (1.20) we have $\lim_{a\to 0} af'(a) = 0$. Hence, $f'(r)) < -\frac{\lambda l}{r} \prod_{j=1}^{m} X_j(\frac{r}{D})$, for $r > 0$ sufficiently small. Therefore, $\lim_{r\to 0} f(r) = +\infty$, and by choosing l large enouph (i.e. $l > \frac{c}{\lambda}$) we contradict (1.20) and the proof of Theorem 1.1.3 is complete. □

1.2. Sturm theory and integral criteria for HI-potentials

The existence of zeros for the solutions of linear ordinary differential equations of the following type

$$(a(x)y')' + b(x)y = 0, \tag{1.21}$$

is of central importance for the identification of the class of HI-potentials – as well as the class of Bessel pairs that will be studied in the next section. There is fortunately a well developed theory to deal with that, starting with Sturm's first comparison principle, whose proof can be found in [**178**]. The rest of the chapter is self-contained.

THEOREM 1.2.1. **(Sturm's First Comparison Theorem)** *Suppose a, b_1, and b_2 are continuous functions on $I := [x_0, x^0]$ and $b_2(x) \geq b_1(x)$ on I. Let $y_1 \not\equiv 0$ be a solution of*

$$(a(x)y')' + b_1(x)y = 0 \quad on \quad I,$$

and assume $y_1(x)$ has exactly n (≥ 1) zeroes on I. Let $y_2(x)$ be a solution of

$$(a(x)y')' + b_2(x)y = 0 \quad on \quad I,$$

satisfying

$$\frac{a(x)y_1'(x)}{y_1(x)} \leq \frac{a(x)y_2'(x)}{y_2(x)} \quad \left(resp., \frac{a(x)y_1'(x)}{y_1(x)} \geq \frac{a(x)y_2'(x)}{y_2(x)} \right) \tag{1.22}$$

at $x = x^0$ (resp., at $x = x_0$). Then $y_2(x)$ has at least n zeroes on I.

Note that the expression of the right [or left] of (1.22) at $x = x^0$ is considered to be ∞ if $y_2(x^0) = 0$ [or $y_1(x^0) = 0$]. In particular, (1.22) holds if $y_2(x^0) = 0$.

PROPOSITION 1.2.1. *Assume $a(x)$ and $b(x)$ are continuous functions on $I := [x_0, x^0]$. If the ordinary differential equation*

$$(a(x)y')' + b(x)y = 0, \tag{1.23}$$

has a positive supersolution φ on I, then it has a positive solution ψ on I with

$$\frac{a(x)\varphi'(x^0)}{\varphi(x^0)} = \frac{a(x)\psi'(x^0)}{\psi(x^0)} \quad if \quad \varphi(x^0) \neq 0, \tag{1.24}$$

and $\psi(x^0) \geq \varphi(x^0) = 0$, otherwise.

Proof: Let φ be a supersolution of the equation (1.23), i.e.

$$(a(x)\varphi')' + b(x)\varphi \leq 0,$$

and assume first that $\varphi(x^0) > 0$. Define

$$q(x) := -\frac{(a(x)\varphi')' + b(x)\varphi}{\varphi(x)} \geq 0.$$

Then φ is a positive solution of the equation

$$(a(x)\varphi')' + (b(x) + q(x))\varphi = 0, \quad x \in (x_0, x^0).$$

Let ψ be the unique solution of the following ordinary differential equation on I

$$(a(x)\psi')' + b(x)\psi = 0, \quad \psi(x^0) = \varphi(x^0), \quad \psi'(x^0) = \varphi'(x^0).$$

Since $q(x) + b(x) \geq b(x)$ on I and since φ does not have any zeroe on I, it follows from Proposition 1.2.1, that $\psi > 0$ on I.

Now assume $\varphi(x^0) = 0$ and let ψ be the unique solution of the following ordinary differential equation on (x_0, x^0)

$$(a(x)\psi')' + b(x)\psi = 0, \quad \psi(\bar{x}) = \varphi(\bar{x}), \quad \psi'(\bar{x}) = \varphi'(\bar{x}),$$

for some $\bar{x} \in (x_0, x^0)$. It follows from Proposition 1.2.1 that $\psi > 0$ on the interval (x_0, x^0) and consequently that $\psi(x^0) \geq 0$. □

DEFINITION 1.2.2. A non-trivial solution of (1.21) is said to be *oscillatory* if there is a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$. Otherwise, it is said to be *non-oscillatory*.

The first important result in the study of the oscillatory behaviour of solutions of ODEs is the celebrated comparison theorem of Sturm, which deals with second order self-adjoint equations of the form:

$$lu \equiv \frac{d}{dx}[a(x)\frac{du}{dx}] + c(x)u = 0 \tag{1.25}$$

$$Lv \equiv \frac{d}{dx}[a(x)\frac{dv}{dx}] + C(x)v = 0 \tag{1.26}$$

on a bounded open interval $\alpha < x < \beta$, where a, c, and C are real-valued continuous functions and $a(x) > 0$ on $[\alpha, \beta]$.

THEOREM 1.2.3. **(Sturm's Comparison Theorem)** *Suppose $c(x) < C(x)$ in the bounded interval $\alpha < x < \beta$. If there exists a nontrivial real solution u of $lu = 0$ such that $u(\alpha) = u(\beta) = 0$, then every real solution of $Lv = 0$ has at least one zero in (α, β).*

Proof: Suppose to the contrary that v does not vanish in (α, β). It may be supposed without loss of generality that $v(x) > 0$ and also $u(x) > 0$ in (α, β). Multiplying the above equations with v and u, subtracting the resulting equations, and integrating over (α, β) yields

$$\int_\alpha^\beta [(au')'v - (av')'u]dx = \int_\alpha \beta(C - c)uvdx. \tag{1.27}$$

Since the integrand on the left hand side is the derivative of $a(u'v - uv')$ and since $C(x) - c(x) > 0$, it follows that

$$[a(x)(u'(x)v(x) - u(x)v'(x))]_\alpha^\beta > 0. \tag{1.28}$$

However, $u(\alpha) = u(\beta) = 0$ by hypothesis, and $u(x) > 0$ in (α, β), while $u'(\alpha) > 0$ and $u'(\beta) < 0$. This contradicts (1.28). □

The following criterion is a consequence of Sturm's Comparison Theorem.

THEOREM 1.2.4. **(Hille-Kneser)** *Set*

$$\omega^* = \limsup_{x\to\infty} x^2 c(x) \quad \text{and} \quad \omega_* = \liminf_{x\to\infty} x^2 c(x).$$

Then the equation

$$u'' + c(x)u = 0, \tag{1.29}$$

is oscillatory if $\omega_* > \frac{1}{4}$ *and nonoscillatory if* $\omega^* < \frac{1}{4}$. *On the other hand, the equation can be oscillatory or nonoscillatory if either* ω_* *or* ω^* *equals* $\frac{1}{4}$.

Proof: If $\omega_* > \frac{1}{4}$, there exists a $\gamma > \frac{1}{4}$ and a positive number x_0 such that $c(x) - \gamma x^{-2} > 0$ for $x \geq x_0$. Since the Euler equation $v'' + \gamma x^{-2} v = 0$ is oscillatory for $\gamma > \frac{1}{4}$, the Sturm's comparison theorem shows that every solution of (1.29) has arbitrary large zeros.

If $\omega^* < \frac{1}{4}$, there exists $\gamma < \frac{1}{4}$ and a number $x_0 > 0$ such that $c(x) - \gamma x^{-2} < 0$ for $x \geq x_0$. If a solution of (1.29) had arbitrary large zeros, then every solution of $v'' + \gamma x^{-2} v = 0$ would have arbitrary large zeros by Sturm's comparison theorem, which is a contradiction. □

Now we are ready to establish oscillation criteria for equation (1.21).

THEOREM 1.2.5. *Assume a satisfies the condition*

$$\int_\alpha^\infty \frac{1}{a(\tau)} d\tau = \infty. \tag{1.30}$$

(1) *If* $a(t)$ *and* $b(t)$ *satisfy for* t *sufficiently large,*

$$a(t)b(t)\left(\int_\alpha^t \frac{1}{a(\tau)} d\tau\right)^2 > \frac{1}{4}, \tag{1.31}$$

then equation (1.21) *is oscillatory.*

(2) *On the other hand, if* $a(t)$ *and* $b(t)$ *satisfy for* t *sufficiently large,*

$$a(t)b(t)\left(\int_\alpha^t \frac{1}{a(\tau)} d\tau\right)^2 < \frac{1}{4}, \tag{1.32}$$

then equation (1.21) *is non-oscillatory.*

Proof: Set

$$s(t) = \int_\alpha^t \frac{1}{a(\tau)} d\tau \quad \text{and} \quad u(s) = x(t(s)),$$

where $t(s)$ is the inverse function of $s(t)$. Then

$$x'(t) = \frac{ds}{dt}\frac{du}{ds} = \frac{1}{a(t)}\frac{du}{ds},$$

and equation (1.21) is transformed into the equation

$$u'' + a(t(s))b(t(s))u = 0, \tag{1.33}$$

which has the form of (1.29). Since $a(t)$ is positive and satisfies for $t > \alpha$ and satisfies (1.30) the functions $s(t)$ and $t(s)$ are increasing and $s(t) \to \infty$ as $t \to \infty$. Hence equation (1.21) is oscillatory (non-oscillatory) if and only if equation (1.29) is oscillatory (non-oscillatory). Let $c(s) = a(t(s))b(t(s))$. Then conditions (1.31) and (1.32) coincide with those of the Hille-Kneser theorem. The proof is complete. □

We now state another integral criteria for the oscillatory behavior of equation (1.29) which is also due to E. Hille.

THEOREM 1.2.6. *Let c be a continuous function on $\mathbb{R}$.*

(1) *If* $\limsup_{t\to\infty} t\int_t^\infty c(s)ds < \frac{1}{4}$, *then Eq.* (1.29) *is non-oscillatory,*
(2) *If* $\liminf_{t\to\infty} t\int_t^\infty c(s)ds > \frac{1}{4}$, *then Eq.* (1.29) *is oscillatory.*

In order to prove Theorem 1.2.6 we first need to establish some preliminary results. Here the key idea is to study the existence of positive solutions of the associated Riccati equation

$$v' + v^2 + c(x) = 0. \tag{1.34}$$

We start with the following simple lemma.

LEMMA 1.2.7. *Suppose that equation* (1.29) *is non-oscillatory and let u be a solution of* (1.29) *that is positive for $x \geq \alpha$, then u is monotone increasing and concave downwards for $x > \alpha$. Furthermore, u' is positive and monotone decreasing towards a non-negative limit.*

Proof: From (1.29), we have for $x_2 > x_1 > \alpha$ that

$$u'(x_2) - u'(x_1) = -\int_{x_1}^{x_2} c(t)u(t)dt \leq 0.$$

It follows that u' is non-increasing and that u is concave downwards for $x > \alpha$. Since the graph of u does not intersect the $x-$axis for $x > \alpha$, we must have that $u'(x) > 0$ for $x > \alpha$. □

The following lemma shows the connection between the oscillatory behavior of solutions of (1.29) and the existence of solutions of the associated Riccati equation (1.34).

LEMMA 1.2.8. *Equation* (1.29) *is non-oscillatory if and only if the integral equation*

$$v(x) = \int_x^\infty [v^2(t) + c(t)]dt \tag{1.35}$$

has a solution for sufficiently large x.

Proof: If there exists a solution $u(x)$ of (1.29) such that $u(x) \neq 0$ for sufficiently large x, then $v(x) := \frac{u'(x)}{u(x)}$ is a solution of (1.35) for sufficiently large x.

Now suppose there is a finite $\alpha > 0$ such that (1.35) has a solution for $x \geq \alpha$. It follows from the form of the equation that $v^2 \in L^1(\alpha, \infty)$ and $v(x)$ is positive, monotone decreasing, absolutely continuous function. Differentiating with respect to x shows that $v(x)$ satisfies (1.34) for almost all x. Hence if we put $u(x) = e^{\int_\alpha^x v(t)dt}$, then u satisfies (1.29) for almost all $x \geq \alpha$ and $u(x) \geq 1$. Equation (1.29) is therefore non-oscillatory. □

Introduce now the notation

$$y(x) = xv(x), \quad d(x) = x\int_x^\infty c(t)dt,$$

in terms of which (1.35) becomes

$$y(x) = x \int_x^\infty y^2(t) \frac{dt}{t^2} + d(x). \tag{1.36}$$

The following comparison lemma will be needed for the proof of Theorem 1.2.6.

LEMMA 1.2.9. *Consider the ordinary differential equations*

$$U'' + C(x)U = 0, \tag{1.37}$$

and

$$u'' + c(x)u = 0. \tag{1.38}$$

Set $D(x) = x\int_x^\infty C(t)dt$ *and* $d(x) = x\int_x^\infty c(t)dt$. *If equation (*1.37*) is non-oscillatory and* $D(x) \geq d(x)$ *for* $x \geq \alpha$, *then equation (*1.38*) is also non-oscillatory.*

Proof: By Lemma 1.2.8, the integral equation

$$Y(x) = x \int_x^\infty Y^2(t) \frac{dt}{t^2} + D(x),$$

has a solution $Y(x)$ for $x \geq \beta$ for some $\beta > 0$. We now consider equation (1.36) for $x \geq \gamma = \max\{\alpha, \beta\}$, and define successive approximations by writing

$$y_0(x) = Y(x), \qquad y_n(x) = x \int_x^\infty y_{n-1}^2(t) \frac{dt}{t^2} + d(x).$$

We have

$$y_1(x) = x \int_x^\infty Y^2(t) \frac{dt}{t^2} + d(x) \leq x \int_x^\infty Y^2(t) \frac{dt}{t^2} + D(x) = Y(x) = y_0(x).$$

Since

$$y_n(x) - y_{n-1}(x) = x \int_x^\infty [y_{n-1}^2(t) - y_{n-2}^2(t)] \frac{dt}{t^2},$$

we see that $y_{n-1}(x) \geq y_n(x) \geq d(x)$ for all x and all n. Hence $\lim y_n(x) = y(x)$ exists and satisfies (1.36). It then follows from Lemma 1.2.8 that equation (1.38) is non-oscillatory as claimed. □

Proof of Theorem 1.2.6: To prove (1) we apply Lemma 1.2.9. Indeed $D(x) = \frac{1}{4}$ corresponds to $C(x) = \frac{1}{4}x^{-2}$ and $U(x) = x^{1/2}\log(x)$ so that the corresponding equation (1.37) in non-oscillatory. Since, $\limsup_{t\to\infty} t\int_t^\infty c(s)ds < \frac{1}{4}$, it follows from Theorem 1.2.6 that equation (1.29) is non-oscillatory.

(2) We shall show that if the equation (1.29) is non-oscillatory, then we necessarily have that $\liminf_{t\to\infty} t\int_t^\infty c(t) \leq \frac{1}{4}$. Since (1.29) is assumed to be non-oscillatory, equation (1.36) has a solution $y(x)$ for x sufficiently large. Define

$$y_* := \liminf_{t\to\infty} y(t), \qquad d_* := \liminf_{t\to\infty} t \int_t^\infty c(t).$$

Since for every $\epsilon > 0$ $(y_*)^2 - \epsilon \leq t\int_t^\infty y^2(s)\frac{ds}{s^2}$ for x sufficiently large, it follows from (1.36) that $y_* \geq (y_*)^2 + d_*$, which is possible only if $d_* \leq \frac{1}{4}$. The proof is now complete. □

The following summarizes the connection between the oscillatory behavior of equation (1.29) and the existence of positive solutions of $(\mathcal{B}_P)$ on a finite interval.

COROLLARY 1.2.1. *Let P be a positive locally integrable function on $\mathbb{R}$.*

(1) *If* $\liminf_{r\to 0}\ \mathrm{logr}\int_0^{\mathrm{r}} \mathrm{sP(s)ds} > -\infty$, *then for every* $R > 0$, *there exists* $\alpha := \alpha(R) > 0$ *such that the scaled function* $P_\alpha(r) := \alpha^2 P(\alpha r)$ *is a HI-potential on* $(0, R)$.

(2) *If* $\lim_{r\to 0}\ \mathrm{logr}\int_0^{\mathrm{r}} \mathrm{sP(s)ds} = -\infty$, *then there are no* $\alpha, c > 0$, *for which* $P_{\alpha,c}(r) := cP(\alpha r)$ *is a HI-potential on* $(0, R)$.

Proof: The change of variable $s = -\mathrm{logr}$, $z(s) = \varphi(e^{-s})$ maps a solution φ of the equation $\mathcal{B}_P$ (i.e., $\varphi'' + \frac{1}{r}\varphi' + P(r)\varphi = 0$) to a solution of the

$$(\mathcal{B}'_P) \qquad z'' + e^{-2s}P(e^{-s})z(s) = 0.$$

It is clear that the equation $z''(s) + a(s)z(s) = 0$ where $a(s) = e^{-2s}P(e^{-s})$ is non-oscillatory (i.e., has a positive solution on some interval (b, ∞)) if and only if $(\mathcal{B}_P)$ has a positive solution on some interval $(0, R)$. The criteria of the preceeding corollary, coupled with the scaling property in Proposition 1.1.2, yield the result.

1.3. The class of Bessel pairs

We shall say that a couple of C^1-functions (V, W) is a *n-dimensional Bessel pair on* $(0, R)$, provided there exists a scalar $c > 0$ such that the ordinary differential equation

$$(\mathcal{B}_{V,cW}) \qquad y''(r) + (\frac{n-1}{r} + \frac{V_r(r)}{V(r)})y'(r) + \frac{cW(r)}{V(r)}y(r) = 0,$$

has a positive solution on the interval $(0, R)$. The *weight* of such a pair is then defined as

$$(1.39) \qquad \beta(V, W; R) = \sup\big\{c;\ (\mathcal{B}_{V,cW}) \text{ has a positive solution on } (0, R)\big\}.$$

Note that we can rewrite $(\mathcal{B}_{V,cW})$ as

$$(1.40) \qquad (r^{n-1}V(r)y')' + cr^{n-1}W(r)y = 0,$$

which means that (V, W) is a *n-dimensional Bessel pair on* $(0, R)$ if and only if the pair $(\tilde{V}, \tilde{W}) := (r^{n-1}V, r^{n-1}W)$ is a 1*-dimensional Bessel pair –or simply a Bessel pair on* $(0, R)$– meaning that the ODE

$$(1.41) \qquad (\tilde{V}(r)y')' + c\tilde{W}(r)y = 0$$

has a positive solution on the interval $(0, R)$.

A simple change of variables in the corresponding ODEs, gives the following relationship between the HI-potentials defined in the last section and Bessel pairs.

PROPOSITION 1.3.1. *Assume $n \geq 3$. The following assertions are then equivalent:*

(1) *P is a HI-potential on $(0, R)$ with $\beta(P, R) = 1$,*

(2) *For any $0 \leq \lambda \leq n-2$, the pair $\big(r^{-\lambda}, (\frac{n-\lambda-2}{2})^2 r^{-\lambda-2} + r^{-\lambda}P(r)\big)$ is a n-dimensional Bessel pair on $(0, R)$.*

(3) *For any $1 \leq \alpha \leq n-1$, the couple $\Big(r^\alpha, \frac{(\alpha-1)^2}{4}r^{\alpha-2} + r^\alpha P(r)\Big)$ is a Bessel pair on $(0, R)$.*

Proof: It follows from a straightforward calculation that $y(r)$ is a solution of $(\mathcal{B}_P)$ if and only if $\varphi := r^{-\frac{n-\lambda-2}{2}}y(r)$ is a solution of

$$\varphi'' + \frac{n-\lambda-1}{r}\varphi' + (P(r) + \frac{(n-\lambda-2)^2}{4r^2})\varphi = 0.$$

Note that the above equation is the corresponding ordinary differential equation $(\mathcal{B}_{V,W})$ for the n-dimensional Bessel pair $(V,W) = \big(r^{-\lambda}, (\frac{n-\lambda-2}{2})^2 r^{-\lambda-2} + r^{-\lambda}P(r)\big)$. Therefore 1) and 2) are equivalent. That they are equivalent to 3) follows from (1.40). □

COROLLARY 1.3.1. *(Explicit Bessel pairs) Assume $n \geq 3$, and $0 \leq \lambda \leq n-2$. We then have the following:*

(1) *For any $R > 0$, $\big(r^{-\lambda}, r^{-\lambda-2}\big)$ is a n-dimensional Bessel pair on $(0,R)$, and*

$$\beta\big(r^{-\lambda}, r^{-\lambda-2}\big) = (\frac{n-\lambda-2}{2})^2. \tag{1.42}$$

(2) *For any $R > 0$, the couple $\big(r^{-\lambda}, (\frac{n-\lambda-2}{2})^2 r^{-\lambda-2} + \frac{z_0^2}{R^2} r^{-\lambda}\big)$ is a n-dimensional Bessel pair on $(0,R)$, and*

$$\beta\big(r^{-\lambda}, (\frac{n-\lambda-2}{2})^2 r^{-\lambda-2} + \frac{z_0^2}{R^2} r^{-\lambda}\big) = 1. \tag{1.43}$$

(3) *For any $R > 0$, the couple*

$$(V, W_\rho) := \big(r^{-\lambda}, (\frac{n-\lambda-2}{2})^2 r^{-\lambda-2} + \frac{1}{4} r^{-2-\lambda}(log\frac{\rho}{r})^{-2}\big), \tag{1.44}$$

where $\rho > Re$, is a n-dimensional Bessel pair on $(0,R)$, and

$$\beta(V, W_\rho) = 1. \tag{1.44}$$

(4) *The couple $\big(r^{-\lambda}, (\frac{n-\lambda-2}{2})^2 r^{-\lambda-2} + r^{-\lambda-\alpha}\big)$ is a n-dimensional Bessel pair on some $(0, R_\alpha)$, whenever $0 \leq \alpha < 2$.*

Proof: Statements 1), 2), and 3) follow directly from Theorem 1.1.3 and Proposition 1.3.1. To prove 4) notice that by Corollary 1.2.1, r^α is a Bessel potential for any $\alpha < 2$. The proof of 4) follows form Proposition 1.3.1. □

We now make an important connection between Bessel pairs and the oscillatory behavior of the following related equations. For that, we rewrite again $(\mathcal{B}_{V,W})$ as

$$(r^{n-1}V(r)y')' + r^{n-1}W(r)y = 0,$$

and then by setting $s = \frac{1}{r}$ and $x(s) = y(r)$, we see that y is a solution of $(\mathcal{B}_{V,W})$ on an interval $(0,\delta)$ if and only if x is a positive solution for the equation

$$(s^{-(n-3)}V(\tfrac{1}{s})x'(s))' + s^{-(n+1)}W(\tfrac{1}{s})x(s) = 0 \quad \text{on} \quad (\tfrac{1}{\delta}, \infty). \tag{1.45}$$

As in the previous section, the fact that (V,W) is a Bessel pair or not is closely related to the oscillatory behavior of the equation (1.45).

THEOREM 1.3.1. *Let V and W be positive C^1-functions on $(0,R)$. Assume*

$$\textstyle\int_0^R \frac{1}{\tau^{n-1}V(\tau)}d\tau = +\infty \text{ and } \int_0^R \tau^{n-1}V(\tau)d\tau < \infty. \tag{1.46}$$

(1) *If*

$$\limsup_{r\to 0} r^{2(n-1)}V(r)W(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2 < \frac{1}{4}, \tag{1.47}$$

then (V,W) is a n-dimensional Bessel pair on $(0,\rho)$ for some $\rho > 0$.

(2) *On the other hand, if*

$$\liminf_{r\to 0} r^{2(n-1)}V(r)W(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2 > \frac{1}{4}, \tag{1.48}$$

then there is no interval $(0,\rho)$ on which (V,W) is a n-dimensional Bessel pair.

Proof: The proof follows from Theorem 1.2.5 applied to the ordinary differential equation (1.45). □

The above integral criterium allows to show the following extension of Proposition 1.3.1.

THEOREM 1.3.2. *Let V be a strictly positive C^1-function on $(0,R)$ such that*

$$\tfrac{rV_r(r)}{V(r)} + \lambda \geq 0 \text{ on } (0,R) \text{ and } \lim_{r\to 0} \tfrac{rV_r(r)}{V(r)} + \lambda = 0, \tag{1.49}$$

where $\lambda \leq n-2$. Then for any HI-potential P on $(0,R)$ and any $c \leq \beta(P;R)$, the couple $(V, W_{\lambda,c})$ is a n-dimensional Bessel pair, where

$$W_{\lambda,c}(r) = V(r)\big(\big(\frac{n-\lambda-2}{2}\big)^2 r^{-2} + cP(r)\big). \tag{1.50}$$

Moreover, $\beta\big(V, W_{\lambda,c}; R\big) = 1$ for all $c \leq \beta(P;R)$.

Proof: Write $\frac{V_r(r)}{V(r)} = -\frac{\lambda}{r} + f(r)$ where $f(r) \geq 0$ on $(0,R)$ and $\lim_{r\to 0} rf(r) = 0$. In order to prove that $\big(V(r), V(r)((\frac{n-\lambda-2}{2})^2 r^{-2} + cP(r))\big)$ is a n-dimensional Bessel pair, we need to show that the equation

$$y'' + \big(\frac{n-\lambda-1}{r} + f(r)\big)y' + \left(\big(\frac{n-\lambda-2}{2}\big)^2 r^{-2} + cP(r)\right) y(r) = 0, \tag{1.51}$$

has a positive solution on $(0,R)$. But first we note that the equation

$$x'' + \big(\frac{n-\lambda-1}{r}\big)x' + \big(\big(\frac{n-\lambda-2}{2}\big)^2 r^{-2} + cP(r)\big)x(r) = 0,$$

has a positive solution on $(0,R)$ whenever $c \leq \beta(P;R)$. Since $f(r) \geq 0$ and since, by Lemma 1.1.2, $x'(r) \leq 0$, we get that x is a positive subsolution for the equation (1.51) on $(0,R)$, and thus it has a positive solution of $(0,R)$. This means that $\beta(V, W_{\lambda,c}; R) \geq 1$.

For the reverse inequality, we shall use the criterium in Theorem 1.3.1. Indeed apply (1.47) to $V(r)$ and $W_1(r) = C\frac{V(r)}{r^2}$ to get

$$\begin{aligned}
\lim_{r\to 0} r^{2(n-1)}V(r)W_1(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2 &= C\lim_{r\to 0} r^{2(n-2)}V^2(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2\\
&= C\Big(\lim_{r\to 0} r^{(n-2)}V(r)\int_r^R \frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2\\
&= C\Big(\lim_{r\to 0}\frac{\frac{1}{r^{n-1}V(r)}}{\frac{(n-2)r^{n-3}V(r)+r^{n-2}V_r(r)}{r^{2(n-2)}V^2(r)}}\Big)^2\\
&= C\Big(\lim_{r\to 0}\frac{1}{(n-2)+r\frac{V_r(r)}{V(r)}}\Big)^2\\
&= \frac{C}{(n-\lambda-2)^2}.
\end{aligned}$$

For $\big(V, CV(r^{-2}+cP)\big)$ to be a n-dimensional Bessel pair, it is necessary that $\frac{C}{(n-\lambda-2)^2} \le \frac{1}{4}$, and the proof for the best constant is complete. $\square$

Corollary 1.3.2. *Let V and W be positive C^1-functions on $(0,+\infty)$. Assume that*

$$\lim_{r\to 0} r\frac{V_r(r)}{V(r)} = -\lambda \text{ and } \lambda \le n-2. \tag{1.52}$$

(1) *If $\limsup_{r\to 0} r^2\frac{W(r)}{V(r)} < (\frac{n-\lambda-2}{2})^2$, then (V,W) is a n-dimensional Bessel pair on some interval $(0,\rho)$.*
(2) *On the other hand, if $\liminf_{r\to 0} r^2\frac{W(r)}{V(r)} > (\frac{n-\lambda-2}{2})^2$, then there is no $R>0$ such that (V,W) is a n-dimensional Bessel pair on the interval $(0,R)$.*

Proof: To prove 1) assume

$$\limsup_{r\to 0} r^2\frac{W(r)}{V(r)} < (\frac{n-\lambda-2}{2})^2.$$

With an argument similar to that of Theorem 1.3.2 we have

$$\limsup_{r\to 0} r^{2(n-1)}V(r)W_1(r)\Big(\int_r^R \frac{1}{\tau^{n-1}V(\tau)}d\tau\Big)^2 < \frac{1}{4},$$

and 1) then follows from Theorem 1.3.1. Proof of 2) follows from a similar argument. $\square$

By applying the above to $V \equiv 1$ and $W(r) := \frac{(n-2)^2}{4}r^{-2} + \alpha r^{-a}$, we get the following result.

Corollary 1.3.3. *If $n \ge 2$, $a \ge 2$ and $\alpha > 0$, then there is no $R>0$ such that the couple $\left(1, \frac{(n-2)^2}{4}r^{-2}+\alpha r^{-a}\right)$ is a n-dimensional Bessel pair on $(0,R)$.*

1.4. Further comments

The book by Agarwal-Bohner-Li [**12**] is a good reference on the oscillatory theory of first and second order differential equations. It addresses delay and ordinary differential equations as well as non-linear differential systems. Another good source on the oscillatory behaviour of ordinary differential equations and Sturm theory is Hartman's book [**178**]. Theorems 1.2.4 and 1.2.6 were proved by E. Hille in [**180**]. The criterium at infinity for studying the oscillatory behavior of equation (1.21) (Theorem 1.2.5) is more recent and is due to Sugie-Kita-Yamaoka [**254**]. See [**13**] and [**186**] for more recent results about the oscillatory behaviour of solutions of second order differential equations. The notions of HI-potential (originally named Bessel potential) and Bessel pairs were introduced by Ghoussoub-Moradifam [**162, 163**] in their work that connected improvements of Hardy inequalities – studied in the next chapters– to the oscillatory behaviour of associated differential equations.

It is important to relate the above notion of Bessel pairs to other notions introduced for the same purpose of extending Hardy's inequality. Say that (V, W) is a *Muckenhought pair* [**225**] on the interval $(0, R)$ if

$$\gamma(V,W,R) := \sup_{0<r<R} (\int_0^r W(t)\,dt)(\int_r^R \frac{1}{V(t)}\,dt) < \infty. \tag{1.53}$$

Problem: Describe the relationship between Bessel pairs and Muckenhought pairs, as well as the constants $\beta(V, W, R)$ and $\gamma(V, W, R)$.