Introduction

0.1. About this book. The purpose of this book is, first, to give a state of the art survey of the theory of birational rigidity and, second, to provide a readable introduction to that theory. Both tasks are not easy and need to be explained. In the past 40 years, starting from the pioneer paper of V.A. Iskovskikh and Yu.I.Manin on the three-dimensional quartics [**IM**], the theory of birational rigidity developed almost from scratch to an important area of birational algebraic geometry. In total, something close to a hundred research papers were published on the subject, some of them well known even beyond the community of experts in birational geometry. The area attracted more and more interest, so the need of a consistent and uniformly written modern survey became obvious. In response to this demand, in the past 15 years extensive survey papers or research papers containing significant survey-type parts, were written with increasing frequency [**IP, CPR, I01, Ch05c, Pukh07b, Pukh10b**]. However, none of them, due to their limited size could possibly cover the entire area, the variety of methods, techniques and ideas. This is what the present book does.

The origins of the theory of birational rigidity go back well into the nineteenth century, to the work of Clebsch, M. Noether and later, Italian geometers. In the 1900s young Gino Fano started his pioneer work on three-dimensional algebraic geometry and published several papers where, in particular, he claimed what is now known to be birational rigidity of several classes of algebraic three-folds (we now say Fano three-folds). However, Fano's arguments were not even an approximation of a rigorous proof and contained serious mistakes, although his intuition and the depth of his insights were amazing. For about 50 years (until 1970) Fano's work and his claims were a very special mysterious subject, not reliable but exciting and potentially promising. In the 1960s, the new foundations of algebraic geometry prompted a review of Fano's work.

The decisive step in the creation of the theory of birational rigidity was made by Manin and Iskovskikh in the above mentioned breakthrough paper (which followed their previous work on birational geometry of algebraic surfaces over non-closed fields [**M66, M67, I67, I70**]. They developed a beautiful scheme of arguments, consisting of several steps and making use of several geometric observations, that provided a rigorous proof of one of Fano's famous claims, that the group of birational self-maps of a non-singular three-dimensional quartic hypersurface in $\mathbb{P}^4$ coincides with the group of its biregular (= projective) automorphisms, being therefore finite, generically trivial. It follows immediately that the quartic hypersurface is non-rational, since the group of birational self-maps is obviously a birational invariant of an algebraic variety and the group of birational transformations of the projective space (the Cremona group) is very large, certainly infinite (it should be noted that non-rationality of the non-singular quartics follows directly from the proof given in [**IM**], without the intermediate reference to the size of the group of birational self-maps, we discuss this point in detail in Chapter 2). Since some non-singular quartic three-folds were known to be unirational, the theorem of Iskovskikh and Manin meant a solution to the famous Lüroth problem ("does unirationality imply rationality?"; the answer is positive for curves and for surfaces over an algebraically closed field; the problem was open for three-folds and higher-dimensional varieties for several decades), which also put the paper [**IM**] on the list of the most remarkable research papers of the twentieth century in algebraic geometry.

0.2. Birational rigidity. It is appropriate now to say a few words about some terminology which is firmly associated with the theory of birational rigidity. First of all, it is the very term "rigidity". Its origin is in the same paper on the three-dimensional quartics: having formulated their main result, the authors remark that the finiteness of the group of birational self-maps demonstrates the extreme "rigidity" of the quartics. However, this was an informal image, when the word was used for the first time. As a rigorous definition, "birational rigidity" appeared first in [**Pukh95**] (and in spoken language, in the author's talk at the conference on algebraic geometry in Warwick in September 1991).

The system of arguments, computational techniques and geometric observations, employed in the theory of birational rigidity, is now known as "the method of maximal singularities", since the concept of a "maximal singularity" of a movable linear system plays the central role in the proofs. To say that a system has a maximal singularity is the same as to say that a certain pair is not canonical; therefore, we can say that the concept of a canonical singularity (of a movable linear system on a non-singular variety) was also first given in [**IM**], much ahead of its time. It should be noted that the method of maximal singularities is about 15 years older than the minimal model program, and it developed independently of the latter, although in the last ten years the influence of ideas and methods of the minimal model program in the theory of birational rigidity was visibly increasing.

Apart from birational (super)rigidity and maximal singularities, there are a few other concepts in this theory which are known in the wider community, such as "maximal subvariety", "K^2-condition", "K-condition", and some general constructions, such as, in the first place, the graph of a sequence of blow ups, the latter seems to have been re-invented several times in different areas of geometry, which is hardly surprising, given its natural definition and essential information accumulated in that graph.

Speaking informally, the phenomenon of birational rigidity is that an algebraic variety that has no non-zero regular differential forms behaves as if it had plenty of them. Indeed, even for a beginner who just started to study algebraic geometry it is obvious why a non-singular hypersurface $V_m \subset \mathbb{P}^d$ is non-rational for $m \geq d+1$ or why if two such hypersurfaces V_{m_1} and V_{m_2} are birational, then $m_1 = m_2$ and they are projectively isomorphic: one has to consider the global differential forms on V_m, V_{m_1} and V_{m_2}. However, it is unexpected and mysterious, why a non-singular hypersurface $V_m \subset \mathbb{P}^d$ for $m \leq d$ should behave in the same way, being rationally connected and for that reason admitting no non-zero differential-geometric birational invariants. A birationally superrigid Fano variety is a Fano variety that behaves in many crucial respects as a variety of general type.

The reasons of this mysterious behaviour are not yet known. However, it is known that birational rigidity is a very typical phenomenon in higher-dimensional algebraic geometry: the most natural constructions of Fano varieties, such as complete intersections in weighted projective space, yield birationally rigid varieties (whenever the study was successfully completed). The underlying reasons are yet to be understood: the area is still full of mysteries!

0.3. The contents of this book. The structure of this book is as follows. Chapter 1, which is aimed at a wide audience, provides an introduction to the subject, tracing the history of birational geometry of rationally connected varieties from its very first steps in the nineteenth century to the present. The chapter is written informally; no special knowledge of algebraic geometry is required to read it through. We show how the straightforward and natural *rationality problem* developed, following its own logic, into the modern problem of describing the structures of a rationally connected fibre space on a given rationally connected variety. Following this course, we mention and sometimes briefly discuss the milestone results that made the area what it is now.

Starting from Chapter 2, the book is a standard mathematical text. To be able to read it, one needs to be acquainted with the basics of algebraic geometry including the intersection theory. R. Hartshorne's "Algebraic Geometry" plus the first three chapters of W. Fulton's "Intersection Theory" are sufficient to read this book without any difficulties, but for an active reader a sufficient background could be reduced to the first volume of I.R. Shafarevich's "Basic Algebraic Geometry". In other words, only some acquaintance with algebraic geometry is assumed, in order to make this book accessible for graduate students on their way into algebraic geometry.

Chapter 2 contains the basics of the method of maximal singularities, illustrated by numerous examples, many of them elementary. A Fano variety V has no global differential-geometric invariants (that is, no non-zero global regular covariant tensors), its canonical class K_V is negative; therefore, the natural idea is to use "the measure of negativity" of K_V instead of the global sections of the sheaves $\mathcal{O}_V(nK_V)$, $n \in \mathbb{Z}_+$, as in the case of varieties of general type. This leads to the classical concept of *termination of canonical adjunction* and eventually, via a natural construction, to the concept of a *maximal singularity* of a movable linear system. We discuss the possible types of maximal singularities and develop techniques to study them. As the first applications, we are able to prove birational (super)rigidity of certain Fano varieties of dimension three and higher. In particular, we give a complete computation of the group of birational self-maps of the complete intersection $V_{2\cdot 3} \subset \mathbb{P}^5$ of quadric and cubic hypersurfaces in $\mathbb{P}^5$, one of the most non-trivial such groups known today.

Chapter 3 is about the technique of hypertangent divisors and linear systems. For Fano varieties of high degree this technique is now the standard and most efficient tool in the proof of their birational (super)rigidity. As a warm-up example, we start with hypersurfaces $V_d \subset \mathbb{P}^d$, $d \geq 5$, providing a series of superrigid Fano varieties in arbitrary dimension. After that we proceed to more sophisticated and technically difficult classes of higher dimensional Fano varieties: complete intersections and cyclic covers. As an additional application, we describe the structures of non-maximal Kodaira dimension on Fano complete intersections; it turns out that all such structures are just pencils of hyperplane sections. However surprising it may look, for any rational map $V \overset{\beta}{\dashrightarrow} S$ onto a positive-dimensional variety S, where V is a generic Fano complete intersection, if β is not a linear projection onto $\mathbb{P}^1$, then the generic fibre of β is of general type!

Chapter 4 is an informal introduction to the method of maximal singularities for the class of *fibre spaces*. By elementary examples we illustrate the new phenomena that occur in the relative case compared to the absolute one (that is, when we study birational geometry of a fibre space $V \to S$ over a positive dimensional variety S rather than that of a primitive Fano variety V with S a point). In particular, we discuss the fibrewise and non-fibrewise birational self-maps and changing the structure of a rationally connected fibre space. After that, we briefly discuss the *Sarkisov Program* — a general theory of factorizing birational maps into elementary modifications (*links*). Finally, we summarize the main results on birational rigidity of Fano fibre spaces, obtained by the method of maximal singularities, and introduce the important K^2-*condition* and K-*condition* that ensure birational (super)rigidity of Fano fibre spaces over $\mathbb{P}^1$.

Chapters 5–8 contain a detailed exposition of the theory outlined in the introductory Chapter 4. Namely, Chapter 5 deals with Fano fibre spaces $V \to \mathbb{P}^1$ with $\dim V \geq 4$. We prove general sufficient conditions of birational superrigidity in terms of numerical geometry of the fibres. This general theory is similar to the methods developed in Chapter 2, but now the relative case requires additional constructions. After that, we develop further the technique of hypertangent divisors, in order to show that generic Fano fibre spaces over $\mathbb{P}^1$ satisfy the needed numerical properties and the general theory applies, giving a proof of their birational rigidity. We consider several classes of Fano fibre spaces, including the two most general ones: fibrations into Fano complete intersections in $\mathbb{P}^n$ and fibrations into Fano cyclic covers. Proofs for the latter two classes are technically the most difficult; they are given in full detail.

Chapter 6 covers three-dimensional fibrations into del Pezzo surfaces. This is by now a very well studied class of varieties, where birational geometry has been exhaustively described even for some of the hardest families. The chapter provides a complete proof of birational rigidity of del Pezzo fibrations of degrees 1, 2 and 3, satisfying the K^2-condition. Furthermore, we explain how the techniques of the proof should be extended (by adding some geometric constructions and observations) in order to make it efficient when the K^2-condition is not satisfied. An "overwhelming majority" of del Pezzo fibrations (for degrees 1 and 2, all but finitely many families) satisfy the K^2-condition, but the extended technique also works in many of the remaining cases. After reading this chapter, one should be able to go through any of the research papers on birational geometry of del Pezzo fibrations.

Chapter 7 introduces and develops the new ideas and constructions of the "linear method", as opposed to the "quadratic method" of the previous chapters. The quadratic constructions were used in the theory of birational rigidity starting from the pioneer paper of Iskovskikh and Manin: if Σ is a movable linear system, then its *self-intersection* $Z = (D_1 \circ D_2)$, where $D_1, D_2 \in \Sigma$ are generic divisors, is an effective cycle of codimension two, and the key principle of the method of maximal singularities was to study the singularities of the system Σ by looking at the induced singularities of the self-intersection Z. It has turned out recently that in certain cases it is much more profitable to study the singularities of the generic divisor $D \in \Sigma$ without taking the self-intersection, for example, by restricting D onto a suitable subvariety and applying the Connectedness Principle of Shokurov and Kollár. Realizing this approach, we prove in this chapter the theorem on Fano direct products and certain other results that give an alternative approach to proving birational rigidity of Fano fibre spaces over $\mathbb{P}^1$.

Finally, Chapter 8 describes birational geometry of Fano double spaces of index two and dimension at least 6. The study of this class of varieties makes the first step outside the realm of birationally rigid ones. However, it comes naturally in this book, since it requires essentially the same techniques as birationally rigid Fano varieties and fibre spaces. Fano double spaces fit perfectly into the general ideology of the method of maximal singularities and demonstrate that the method potentially applies to a much wider world of algebraic varieties than that of birationally rigid ones. The main result of this chapter is that every non-trivial structure of a rationally connected fibre space on a double space V of index 2 is a map $V \dashrightarrow \mathbb{P}^1$ given by an arbitrary pencil in the half-anticanonical linear system of V. This is precisely what one should expect from a Fano variety of index two and makes an evidence of further potential of the method of maximal singularities.

This completes our brief description of the contents of this book. In more details the contents of each chapter is summarized in the introductory section at the beginning of the chapter. Historical remarks tracing the papers where the particular facts were proven and particular ideas introduced and developed, are collected in special sections at the end of Chapters 2–8; for that reason, we give very few references in the body of the text (e.g. when some claim is stated but not proved in this book, the reference is given immediately after the statement), collecting them in the "Notes and references" section at the end of the chapter.

0.4. References and cross-references. Now we explain the system of enumeration and cross-references. All claims (theorems, propositions, lemmas, corollaries) as well as definitions, remarks and examples are numbered independently in each section and independently of each other. A reference to Theorem (Proposition, Lemma, ...) $a.b$ means a reference to a theorem (proposition,...) in the *current* chapter, which is to be found in Section a (under the number $a.b$) and is the b-th theorem (proposition,...) in that section. A references to Theorem (Proposition, Lemma,...) $a.b.c$ means a reference to a theorem (Proposition,...) in Section b of Chapter a, which has the number $b.c$ in that section; such a reference occurs if Chapter a is *not* the current one. The same principle of numeration and references applies to definitions, remarks and examples.

A reference to Section a means a reference to the section number a in the *current* chapter. If we refer to Section b in Chapter a, which is *not* the current one, we write Section $a.b$. Sections are divided into subsections which are numbered consecutively: Section a consists of Subsections $a.1$, $a.2$,.... For references, we use the same principle as above: a reference to Subsection $a.b$ means the b'th subsection of Section a of the current chapter, whereas a reference to Subsection $a.b.c$ means the Subsection $b.c$ in Chapter a, which is different from the current one. With such a system of enumeration and references any misunderstanding is excluded.

Selection of bibliography also needs some explanation. Providing a comprehensive survey of the theory of birational rigidity, this book is not an encyclopedia of the area. The main achievements are all covered, most of the significant results are either covered or at least mentioned and discussed. It was not possible to include every result on birational rigidity and, the more so, every related result, as the book would have grown at least twice longer in that case and become unreadable.

This applies to the bibliography as well. All the main research and survey papers on birational rigidity are there. All or almost all significant research papers on birational rigidity are there, too. However, including every paper on birational rigidity would have made the list of references unnecessarily long. (It is also possible that I overlooked or forgot some items.) As for the papers related to, but not dealing with, birational rigidity, I had to be very selective there. I would explicitly name three very important areas, which are almost completely missing in the bibliography, although the problems they are studying (but not the methods they are employing, which was the decisive argument for me) are quite close to the problems that are solved or discussed in this book: the investigation of the Cremona group of higher (≥ 3) rank, the (biregular) classification of higher-dimensional Fano varieties and the study of rational curves on algebraic varieties. These areas are getting increasingly popular today and there are some indications that their interaction with the theory of birational rigidity will increase in the future.

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