CHAPTER 1

Variants

In this chapter we give six variants of Theorem 0.4, strengthening it in different ways. In Chapter 4, Theorem 0.4, Corollary 0.5 and the variants stated here will be reduced to yet another variant (Theorem 4.2) and we will spend most of the book proving the theorem in that form.

Our first variant is similar to the original theorem of Fekete and Szegö ([**25**]). In that theorem the sets $E_v \subset \mathbb{C}$ were compact, and the conjugates of the algebraic integers produced were required to lie in arbitrarily small open neighborhoods U_v of the E_v. In Theorem 1.2 below, we lift the assumption of compactness and replace the Cantor capacity with *inner Cantor Capacity* $\overline{\gamma}(\mathbb{E}, \mathfrak{X})$, which is defined for arbitrary adelic sets. We also replace the neighborhoods U_v with "quasi-neighborhoods", which are finite unions of open sets in $\mathcal{C}_v(\mathbb{C}_v)$ and open sets in $\mathbb{C}_v(F_w)$, for algebraic extensions F_w/K_v in $\mathbb{C}_v$.

The inner Cantor capacity $\overline{\gamma}(\mathbb{E}, \mathfrak{X})$ is similar to Cantor capacity except that it is defined in terms of upper Green's functions $\overline{G}(z, x_i; E_v)$. Here, we briefly recall the definitions of $\overline{G}(z, x_i; E_v)$ and $\overline{\gamma}(\mathbb{E}, \mathfrak{X})$ and some of their properties; they are studied in detail in §3.9 and §3.10 below.

Upper Green's functions are gotten by taking decreasing limits of Green's functions of compact sets. For an arbitrary $E_v \subset \mathcal{C}_v(\mathbb{C}_v)$, if $\zeta \notin E_v$ the upper Green's function is

$$\overline{G}(z, \zeta; E_v) \ = \ \inf_{\substack{H_v \subset E_v \\ H_v \text{ compact}}} G(z, \zeta; H_v) \ . \tag{1.1}$$

If ζ is not in the closure of E_v, the upper Robin constant $\overline{V}_\zeta(E_v)$ is finite and is defined by

$$\overline{V}_\zeta(E_v) \ = \ \lim_{z \to \zeta} \overline{G}(z, \zeta; E_v) + \log_v(|g_\zeta(z)|_v) \ , \tag{1.2}$$

where $g_\zeta(z)$ is the uniformizer from (0.3). By (0.6), if E_v is compact then by ([**51**], Theorem 4.4.4) $\overline{G}(z, \zeta; E_v) = G(z, \zeta; E_v)$ and $\overline{V}_\zeta(E_v) = V_\zeta(E_v)$. For nonarchimedean v, if E_v is assumed to be algebraically capacitable in the sense of ([**51**]), then $\overline{G}(z, \zeta; E_v) = G(z, \zeta; E_v)$ and $\overline{V}_\zeta(E_v) = V_\zeta(E_v)$. The upper Green's function is symmetric and nonnegative: for all $z, \zeta \notin E_v$, $\overline{G}(z, \zeta; E_v) = \overline{G}(\zeta, z; E_v) \geq 0$. It has functoriality properties under pullbacks and base change like those of $G(z, \zeta; E_v)$.

Now assume that each E_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$, and that $\mathbb{E} = \prod_v E_v$ is compatible with $\mathfrak{X}$. Let L/K be a finite normal extension containing $K(\mathfrak{X})$. For each place v of K and each place w of L with $w|v$, after fixing an isomorphism $\mathbb{C}_w \cong \mathbb{C}_v$, we can pull back E_v to a set $E_w \subset \mathcal{C}_w(\mathbb{C}_w)$, which is independent of the isomorphism chosen. If we identify $\mathcal{C}_v(\mathbb{C}_v)$ with $\mathcal{C}_w(\mathbb{C}_w)$, then for $z, \zeta \notin E_v$

$$\overline{G}(z, \zeta; E_w) \log(q_w) \ = \ [L_w : K_v] \cdot \overline{G}(z, \zeta; E_v) \log(q_v) \ . \tag{1.3}$$

For each $x_i \in \mathfrak{X}$, fix a global uniformizing parameter $g_{x_i}(x) \in L(\mathcal{C})$ and use it to define the upper Robin constants $\overline{V}_{x_i}(E_w)$ for all places w of L. For each w, the "local upper Green's matrix" is

$$\overline{\Gamma}(E_w, \mathfrak{X}) \;=\; \begin{pmatrix} \overline{V}_{x_1}(E_w) & \overline{G}(x_1, x_2; E_w) & \cdots & \overline{G}(x_1, x_m; E_w) \\ \overline{G}(x_2, x_1; E_w) & \overline{V}_{x_2}(E_w) & \cdots & \overline{G}(x_2, x_m; E_w) \\ \vdots & \vdots & \ddots & \vdots \\ \overline{G}(x_m, x_1; E_w) & \overline{G}(x_m, x_2; E_w) & \cdots & \overline{V}_{x_m}(E_w) \end{pmatrix} ,$$

and the 'global upper Green's matrix' is

$$\overline{\Gamma}(\mathbb{E}, \mathfrak{X}) \;=\; \frac{1}{[L:K]} \sum_{w \in \mathcal{M}_L} \overline{\Gamma}(E_w, \mathfrak{X}) \log(q_w) \ .$$

Since $\mathbb{E}$ is compatible with $\mathfrak{X}$, all but finitely many of the $\overline{\Gamma}(E_w, \mathfrak{X})$ are 0. By the product formula, $\overline{\Gamma}(\mathbb{E}, \mathfrak{X})$ is independent of the choice of the $g_{x_i}(z)$. By (0.7) it is independent of the choice of L. It is symmetric and non-negative off the diagonal; its entries are finite if and only if each E_v has positive inner capacity.

For each K-rational $\mathbb{E}$ compatible with $\mathfrak{X}$, the *inner Cantor capacity* is

$$\overline{\gamma}(\mathbb{E}, \mathfrak{X}) \;=\; e^{-\overline{V}(\mathbb{E}, \mathfrak{X})} ,$$

where $\overline{V}(\mathbb{E}, \mathfrak{X}) = \mathrm{val}(\overline{\Gamma}(\mathbb{E}, \mathfrak{X}))$ is the value of $\overline{\Gamma}(\mathbb{E}, \mathfrak{X})$ as a matrix game. When the sets E_v are compact or algebraically capacitable, the inner Cantor capacity coincides with the Cantor capacity $\gamma(\mathbb{E}, \mathfrak{X})$ defined in ([**51**]). It reduces to the classical logarithmic capacity when $\mathcal{C} = \mathbb{P}^1/\mathbb{Q}$, $\mathfrak{X} = \infty$, and all the nonarchimedean E_v are trivial.

The reason the inner Cantor capacity is the appropriate capacity to use in the Fekete-Szegö theorem is that one of the initial reductions in the proof is to replace each E_v which is not $\mathfrak{X}$-trivial by a compact set $H_v \subset E_v$. Since the Green's function is a limit of Green's functions of compact sets, this can be done in such a way that $\Gamma(\mathbb{E}, \mathfrak{X})$ remains negative definite.

Definition 1.1. Let v be a place of K. A set $U_v \subset \mathcal{C}_v(\mathbb{C}_v)$ will be called a *quasi-neighborhood* if there are open sets $U_{v,0}, U_{v,1}, \ldots, U_{v,M}$ in $\mathcal{C}_v(\mathbb{C}_v)$ and algebraic extensions $F_{w_1}/K_v, \ldots, F_{w_M}/K_v$ in $\mathbb{C}_v$ (of finite or infinite degree) such that

$$U_v \;=\; U_{v,0} \cup \bigcup_{\ell=1}^{M} \big(U_{v,\ell} \cap \mathcal{C}_v(F_{w_\ell})\big) \ .$$

We allow the possibility that one or more of the $U_{v,\ell}$ are empty. We will say that U_v is *K_v-symmetric* if it is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$, and that it is *separable* if each F_{w_ℓ}/K_v is separable. If U_v contains a set E_v, we will say that U_v is a quasi-neighborhood of E_v.

Equivalently, a quasi-neighborhood $U_v \subset \mathcal{C}_v(\mathbb{C}_v)$ is the union of finitely many sets, each of which is either open in $\mathcal{C}_v(\mathbb{C}_v)$ or is open in $\mathcal{C}_v(F_{w_\ell})$ for some algebraic extension F_{w_ℓ}/K_v in $\mathbb{C}_v$. These sets need not be disjoint. For example, take $\mathcal{C} = \mathbb{P}^1$ and identify $\mathbb{P}^1(\mathbb{C}_v)$ with $\mathbb{C}_v \cup \infty$. Suppose v is nonarchimedean; let $F_{w_1}, \ldots, F_{w_M}$ be algebraic extensions of K_v contained in $\mathbb{C}_v$, and let $\mathcal{O}_{w_1}, \ldots, \mathcal{O}_{w_M}$ be their rings of integers. Then the set $U_v = \mathcal{O}_{v_1} \cup \cdots \cup \mathcal{O}_{w_M}$ is a quasi-neighborhood of the origin in $\mathbb{P}^1(\mathbb{C}_v)$.

If $\mathbb{E} = \prod_v E_v \subseteq \prod_v \mathcal{C}_v(\mathbb{C}_v)$ is an adelic set, we will call a set $\mathbb{U} = \prod_v U_v \subseteq \prod_v \mathcal{C}_v(\mathbb{C}_v)$ a *K-rational separable quasi-neighborhood of* $\mathbb{E}$ if each U_v is a separable quasi-neighborhood of E_v, stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$.

THEOREM 1.2 (FSZ with LR for Quasi-neighborhoods). *Let K be a global field, and let $\mathcal{C}/K$ be a smooth, connected, projective curve. Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite set of points stable under $\mathrm{Aut}(\widetilde{K}/K)$, and let $\mathbb{E} = \prod_v E_v \subset \prod_v \mathcal{C}_v(\mathbb{C}_v)$ be a K-rational adelic set compatible with $\mathfrak{X}$, so each E_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$ and all but finitely many E_v are $\mathfrak{X}$-trivial.*

Suppose $\overline{\gamma}(\mathbb{E}, \mathfrak{X}) > 1$. Then for any K-rational separable quasi-neighborhood $\mathbb{U}$ of $\mathbb{E}$, there are infinitely many points $\alpha \in \mathcal{C}(\widetilde{K}^{\mathrm{sep}})$ such that for each $v \in \mathcal{M}_K$, the $\mathrm{Aut}(\widetilde{K}/K)$-conjugates of α all belong to U_v.

Our next variant is a stronger, but more technical, version of Theorem 0.4, which produces points with all their conjugates in $\mathbb{E}$. It uses the inner capacity, and weakens the conditions on the sets E_v.

Write $\mathrm{cl}(E_v)$ for the closure of E_v in $\mathcal{C}_v(\mathbb{C}_v)$. If v is an archimedean place of K, and a set $E_v \subset \mathcal{C}_v(\mathbb{C})$ and a subset $E'_v \subset E_v$ are given, we will say that a point $z_0 \in E_v$ is *analytically accessible* from E'_v if for some $r > 0$, there is a non-constant analytic map $f : D(0,r)^- \to \mathcal{C}_v(\mathbb{C})$ with $f(0) = z_0$, such that $f((0,r)) \subset E'_v$. (See Definition 3.29.)

THEOREM 1.3 (Strong FSZ with LR). *Let K be a global field, and let $\mathcal{C}/K$ be a smooth, geometrically integral projective curve. Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite set of points stable under $\mathrm{Aut}(\widetilde{K}/K)$, and let $\mathbb{E} = \prod_v E_v \subset \prod_v \mathcal{C}_v(\mathbb{C}_v)$ be a K-rational adelic set compatible with $\mathfrak{X}$, so each E_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$ and all but finitely many E_v are $\mathfrak{X}$-trivial. Let $S \subset \mathcal{M}_K$ be a finite set of places v, containing all archimedean v, such that E_v is $\mathfrak{X}$-trivial for each $v \notin S$.*

Assume that $\overline{\gamma}(\mathbb{E}, \mathfrak{X}) > 1$. Assume also that for each $v \in S$, there is a (*possibly empty*) *$\mathrm{Aut}_c(\mathbb{C}_v/K_v)$-stable Borel subset $e_v \subset \mathcal{C}_v(\mathbb{C}_v)$ of inner capacity 0 such that*

(*A*) *If v is archimedean and $K_v \cong \mathbb{C}$, then each point of $\mathrm{cl}(E_v)\backslash e_v$ is analytically accessible from the $\mathcal{C}_v(\mathbb{C})$-interior of E_v.*

(*B*) *If v is archimedean and $K_v \cong \mathbb{R}$, then each point of $\mathrm{cl}(E_v)\backslash e_v$ is*

(1) *analytically accessible from the $\mathcal{C}_v(\mathbb{C})$-interior of E_v, or*

(2) *is an endpoint of an open segment contained in $E_v \cap \mathcal{C}_v(\mathbb{R})$.*

(*C*) *If v is nonarchimedean, then E_v is the disjoint union of e_v and finitely many sets $E_{v,1}, \ldots, E_{v,M_v}$, where each $E_{v,\ell}$ is*

(1) *open in $\mathcal{C}_v(\mathbb{C}_v)$, or*

(2) *of the form $U_{v,\ell} \cap \mathcal{C}_v(F_{w,\ell})$, where $U_{v,\ell}$ is open in $\mathcal{C}_v(\mathbb{C}_v)$ and $F_{w,\ell}$ is a separable algebraic extension of K_v contained in $\mathbb{C}_v$* (*possibly of infinite degree*)*.*

Then there are infinitely many points $\alpha \in \mathcal{C}(\widetilde{K}^{\mathrm{sep}})$ such that for each $v \in \mathcal{M}_K$, the $\mathrm{Aut}(\widetilde{K}/K)$-conjugates of α all belong to E_v.

Note that if v is archimedean, then the set e_v in Theorem 1.3 can be taken to belong to ∂E_v, since trivially each point of the $\mathcal{C}_v(\mathbb{C})$-interior of E_v or the $\mathcal{C}_v(\mathbb{R})$-interior of $E_v \cap \mathcal{C}_v(\mathbb{R})$ is analytically accessible. Any countable set has inner capacity 0, so the conditions in Theorem 0.4 imply those in Theorem 1.3.

If v is nonarchimedean, note that RL-domains and balls $B(a,r)^-$, $B(a,r)$, are both open and closed in the $\mathcal{C}_v(\mathbb{C}_v)$-topology. Thus if E_v is a finite union of sets which are RL-domains, open or closed balls, or their intersections with $\mathcal{C}_v(F_{w,\ell})$ for separable algebraic extensions $F_{w,\ell}/K_v$ in $\mathbb{C}_v$, the theorem applies with $e_v = \phi$.

For an example of an archimedean set satisfying the conditions of Theorem 1.3 but not Theorem 0.4, take $K = \mathbb{Q}$, $\mathcal{C} = \mathbb{P}^1$, and let v be the archimedean place of $\mathbb{Q}$. Identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \infty$, and take $E_v = \{0\} \cup \big(\bigcup_{n=2}^{\infty} D(2/n, 1/n^2)\big)$. Then each point of $E_v \backslash \{0\}$ is analytically accessible from E_v^0. For an example where the conditions of Theorem 1.3 fail, let E_v be the union of a circle $C(0,r)$ and countably many pairwise disjoint discs $D(a_i, r_i)$ contained in $D(0,r)^-$ chosen in such a way that each point of $C(0,r)$ is a limit point of those discs.

Our third variant is a version of Theorem 0.4 which adds side conditions concerning ramification. It says that at a finite number of places outside S we can require that the algebraic numbers produced are ramified or unramified, "for free".

Theorem 1.4 (FSZ with LR and Ramification Side Conditions). *Let K be a global field, and let $\mathcal{C}/K$ be a smooth, connected, projective curve. Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite, Galois-stable set of points, and let $\mathbb{E} = \prod_v E_v \subset \prod_v \mathcal{C}_v(\mathbb{C}_v)$ be a K-rational adelic set compatible with $\mathfrak{X}$, so each E_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$ and all but finitely many E_v are $\mathfrak{X}$-trivial.*

Let $S, S', S'' \subset \mathcal{M}_K$ be finite (possibly empty) sets of places of K which are pairwise disjoint, such that the places in $S' \cup S''$ are nonarchimedean. Assume that $\overline{\gamma}(\mathbb{E}, \mathfrak{X}) > 1$, and that

(A) for each $v \in S$, E_v satisfies the conditions of Theorem 0.4 or Theorem 1.3.

(B) for each $v \in S'$, either E_v is $\mathfrak{X}$-trivial, or E_v is a finite union of closed isometrically parametrizable balls $B(a_i, r_i)$ whose radii belong to the value group of $K_v^{\times}$ and whose centers belong to an unramified extension of K_v;

(C) for each $v \in S''$, either E_v is $\mathfrak{X}$-trivial and $E_v \cap \mathcal{C}_v(K_v)$ is nonempty, or E_v is a finite union of closed and/or open isometrically parametrizable balls $B(a_i, r_i)$, $B(a_j, r_j)^-$ with centers in $\mathcal{C}_v(K_v)$.

Then there are infinitely many points $\alpha \in \mathcal{C}(\widetilde{K}^{\mathrm{sep}})$ such that

(1) for each $v \in \mathcal{M}_K$, the $\mathrm{Aut}(\widetilde{K}/K)$-conjugates of α all belong to E_v;

(2) for each $v \in S'$, each place of $K(\alpha)/K$ above v is unramified over v;

(3) for each $v \in S''$, each place of $K(\alpha)/K$ above v is totally ramified over v.

Our fourth variant involves a partial converse to the Fekete-Szegö theorem, known as Fekete's theorem, which asserts that if $\gamma(\mathbb{E}, \mathfrak{X}) < 1$ then for a sufficiently small neighborhood $\mathbb{U}$ of $\mathbb{E}$, there are only finitely many points $\alpha \in \mathcal{C}(\widetilde{K})$ whose conjugates all belong to $\mathbb{U}$. Fekete's theorem on curves is proved in ([**51**], Theorem 6.3.1). However, Fekete's theorem requires a different notion of capacity than we have been using here: it concerns the "outer capacity" $\underline{\gamma}(\mathbb{E}, \mathfrak{X})$, rather than the inner capacity $\overline{\gamma}(\mathbb{E}, \mathfrak{X})$.

Extending the definition of algebraic capacitability in ([**51**]) to both archimedean and nonarchimedean sets, we will say that E_v *algebraically capacitable* if it is closed in $\mathcal{C}_v(\mathbb{C}_v)$ and $\overline{\gamma}_\zeta(E_v) = \underline{\gamma}_\zeta(E_v)$ for each $\zeta \notin E_v$. If each E_v is algebraically capacitable, then $\overline{\gamma}(\mathbb{E}, \mathfrak{X})$ and $\underline{\gamma}(\mathbb{E}, \mathfrak{X})$ are equal, and coincide with the capacity $\gamma(\mathbb{E}, \mathfrak{X})$ in ([**51**]). Here

$$\overline{\gamma}_\zeta(E_v) = \sup_{\substack{H_v \subset E_v \\ H_v \text{ compact}}} \gamma(H_v) , \qquad \underline{\gamma}_\zeta(E_v) = \inf_{\substack{U_v \supset E_v \\ U_v \text{ a } PL_\zeta\text{-domain}}} \overline{\gamma}(U_v) .$$

A set U_v is a PL_ζ-domain if there is a nonconstant rational function $f(z) \in \mathbb{C}_v(\mathcal{C})$, whose only poles are at ζ, for which $U_v = \{z \in \mathcal{C}_v(\mathbb{C}_v) : |f(z)|_v \le 1\}$. In the nonarchimedean case, the compatibility of this definition with the one given in ([**51**], p.259) follows from ([**51**], Propositions 4.3.1 and 4.3.16). In ([**51**]), algebraic capacitability was not defined in the archimedean case, but all archimedean sets were required to be compact.

If v is archimedean, it follows from ([**51**], Proposition 3.3.3) that every compact set is algebraically capacitable. If v is nonarchimedean, it is shown in ([**51**], Theorem 4.3.13) that any set E_v which can be expressed as a finite combination of unions and intersections of compact sets and RL-domains, is algebraically capacitable.

Assuming algebraic capacitability for the sets E_v, the following result describes the dichotomy provided by Fekete's theorem and the Fekete-Szegö theorem in terms of the Green's matrix $\Gamma(\mathbb{E}, \mathfrak{X})$. Recall (see [**51**], §5.1) that $\gamma(\mathbb{E}, \mathfrak{X}) > 1$ if and only if $\Gamma(\mathbb{E}, \mathfrak{X})$ is negative definite, and that $\gamma(\mathbb{E}, \mathfrak{X}) < 1$ if and only if when the rows and columns of $\Gamma(\mathbb{E}, \mathfrak{X})$ are permuted to bring $\Gamma(\mathbb{E}, \mathfrak{X})$ into block diagonal form, then some eigenvalue of each block is positive.

THEOREM 1.5 (Fekete/FSZ with LR for Algebraically Capacitable Sets). *Let K be a global field and let $\mathcal{C}/K$ be a smooth, connected, projective curve. Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite, Galois-stable set of points, and let $\mathbb{E} = \prod_v E_v \subset \prod_v \mathcal{C}_v(\mathbb{C}_v)$ be a K-rational adelic set compatible with $\mathfrak{X}$, so each E_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$ and all but finitely many E_v are $\mathfrak{X}$-trivial. Assume that each E_v is algebraically capacitable. Then*

(A) If all the eigenvalues of $\Gamma(\mathbb{E}, \mathfrak{X})$ are non-positive (that is, $\Gamma(\mathbb{E}, \mathfrak{X})$ is either negative definite or negative semi-definite), let $\mathbb{U} = \prod_v U_v$ be a separable K-rational quasi-neighborhood of $\mathbb{E}$ such that there is at least one place v_0 where E_{v_0} is compact and the quasi-neighborhood U_{v_0} properly contains E_{v_0}. If v_0 is archimedean, assume also that U_{v_0} meets each component of $\mathcal{C}_{v_0}(\mathbb{C}) \backslash E_{v_0}$ containing a point of $\mathfrak{X}$. Then there are infinitely many $\alpha \in \mathcal{C}(\widetilde{K}^{\mathrm{sep}})$ such that all the conjugates of α belong to $\mathbb{U}$.

(B) If some eigenvalue of $\Gamma(\mathbb{E}, \mathfrak{X})$ is positive (that is, $\Gamma(\mathbb{E}, \mathfrak{X})$ is either indefinite, nonzero and positive semi-definite, or positive definite), there is an adelic neighborhood $\mathbb{U}$ of $\mathbb{E}$ such that only finitely many points $\alpha \in \mathcal{C}(\widetilde{K})$ have all their conjugates in $\mathbb{U}$.

Finally, we formulate two Berkovich versions of the Fekete-Szegö Theorem with local rationality.

For each nonarchimedean place v of K, let $\mathcal{C}_v^{\mathrm{an}}$ be the Berkovich analytic space associated to $\mathcal{C}_v \times_{K_v} \mathrm{Spec}(\mathbb{C}_v)$ (see [**10**]). This is a locally ringed space whose underlying topological space is a compact, path connected Hausdorff space with $\mathcal{C}_v(\mathbb{C}_v)$ as a dense subset; it has the same sheaf of functions as the rigid analytic space associated to $\mathcal{C}_v \times_{K_v} \mathrm{Spec}(\mathbb{C}_v)$. In his doctoral thesis, Amaury Thuillier ([**64**]) constructed a potential theory on $\mathcal{C}_v^{\mathrm{an}}$ which includes a dd^c operator, harmonic functions, subharmonic functions, capacities, and Green's functions. When $\mathcal{C} \cong \mathbb{P}^1$, Baker and Rumely ([**7**]) constructed a similar theory in an elementary way.

Below, we assume familiarity with Berkovich analytic spaces and Thuillier's theory. For each compact, non-polar subset $\mathbf{E}_v \subset \mathcal{C}_v^{\mathrm{an}}$ and each $\zeta \in \mathcal{C}_v^{\mathrm{an}} \backslash \mathbf{E}_v$, Thuillier ([**64**], Théorème 3.6.15) has constructed a Green's function $g_{\zeta, \mathbf{E}_v}(z)$ which is non-negative, vanishes on $\mathbf{E}_v$ except possibly on a set of capacity 0, is subharmonic in $\mathcal{C}_v^{\mathrm{an}}$, harmonic in $\mathcal{C}_v^{\mathrm{an}} \backslash (\mathbf{E}_v \cup \{\zeta\})$, and satisfies the distributional equation

$dd^c g_{\zeta,\mathbf{E}_v} = \mu - \delta_\zeta$ where μ is a probability measure supported on K. We will write $G(z,\zeta;\mathbf{E}_v)^{\mathrm{an}}$ for $g_{\zeta,\mathbf{E}_v}(z)$, and regard it as a function of two variables. By Proposition 4.3 below, for all $z, \zeta \in \mathcal{C}_v^{\mathrm{an}} \backslash \mathbf{E}_v$ with $z \neq \zeta$,

$$G(z,\zeta;\mathbf{E}_v)^{\mathrm{an}} \;=\; G(\zeta,z;\mathbf{E}_v)^{\mathrm{an}} \;,$$

and for each $\zeta \in \mathcal{C}_v(\mathbb{C}_v)\backslash\mathbf{E}_v$, the Robin constant

$$V_\zeta(\mathbf{E}_v)^{\mathrm{an}} \;=\; \lim_{\substack{z\to\zeta \\ z\in\mathcal{C}_v^{\mathrm{an}}}} G(z,\zeta;\mathbf{E}_v)^{\mathrm{an}} + \log(|g_\zeta(z)|_v)$$

exists. The group $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$ acts on $\mathcal{C}_v^{\mathrm{an}}$ in a natural way, and for all $\sigma \in \mathrm{Aut}_c(\mathbb{C}_v/K_v)$

$$G(\sigma(z),\sigma(\zeta);\sigma(\mathbf{E}_v))^{\mathrm{an}} \;=\; G(z,\zeta;\mathbf{E}_v)^{\mathrm{an}} \;.$$

By Proposition 4.4 below, the Green's functions $G(z,\zeta;\mathbf{E}_v)^{\mathrm{an}}$ and the functions $G(z,\zeta;E_v)$ from this work are compatible up to a normalizing factor, in the sense that if $E_v \subset \mathcal{C}_v(\mathbb{C}_v)$ is algebraically capacitable (in particular, if E_v is a finite union of RL-domains and compact sets), and if $\mathbf{E}_v$ is the closure of E_v in $\mathcal{C}_v^{\mathrm{an}}$ for the Berkovich topology, then for all $z,\zeta \in \mathcal{C}_v(\mathbb{C}_v)\backslash E_v$,

$$G(z,\zeta;\mathbf{E}_v)^{\mathrm{an}} \;=\; G(z,\zeta;E_v)\log(q_v) \;.$$

If v is an archimedean place of K, we take $\mathcal{C}_v^{\mathrm{an}}$ to be the Riemann surface $\mathcal{C}_v(\mathbb{C})$, and for a set $\mathbf{E}_v = E_v \subset \mathcal{C}_v(\mathbb{C})$ we put $G(z,\zeta;\mathbf{E}_v)^{\mathrm{an}} = G(z,\zeta;E_v)$ and $V_\zeta(\mathbf{E}_v)^{\mathrm{an}} = V_\zeta(E_v)$.

Let $\mathfrak{X} = \{x_1,\ldots,x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite, Galois-stable set of points. We will now define the notion of a *compact Berkovich adelic set compatible with* $\mathfrak{X}$. For each place v of K, let $\mathbf{E}_v \subset \mathcal{C}_v^{\mathrm{an}}$ be a compact, nonpolar set disjoint from $\mathfrak{X}$. (A set $\mathbf{E}_v \subset \mathcal{C}_v^{\mathrm{an}}$ is nonpolar if and only if it has positive capacity: see ([**64**], §3.4.2 and Theorem 3.6.11).) We will say that $\mathbf{E}_v$ is $\mathfrak{X}$-trivial if v is nonarchimedean and the model $\mathfrak{C}_v/\operatorname{Spec}(\mathcal{O}_v)$ from Definition 0.2 has good reduction, the points of $\mathfrak{X}$ specialize to distinct points in the special fibre $r_v(\mathfrak{C}_v)$, and $\mathbf{E}_v$ consists of all points $z \in \mathcal{C}_v^{an}$ whose specialization $r_v(z) \in r_v(\mathfrak{C}_v)$ is distinct from $\{r_v(x_1),\ldots,r_v(x_m)\}$. Equivalently, $\mathbf{E}_v$ is $\mathfrak{X}$-trivial if it is the closure of the $\mathfrak{X}$-trivial set $E_v = \mathcal{C}_v(\mathbb{C}_v)\backslash(\bigcup_{i=1}^m B(x_i,1)^-)$ in $\mathcal{C}_v(\mathbb{C}_v)$. Then

$$\mathbb{E} \;:=\; \prod_v \mathbf{E}_v \;\subset\; \prod_v \mathcal{C}_v^{\mathrm{an}}$$

is a *compact Berkovich adelic set compatible with* $\mathfrak{X}$ if each $\mathbf{E}_v$ satisfies the conditions above, and $\mathbf{E}_v$ is $\mathfrak{X}$-trivial for all but finitely many v.

If $\mathbb{E}$ is a compact Berkovich adelic set compatible with $\mathfrak{X}$, we define the local and global Green's matrices $\Gamma(\mathbf{E}_w,\mathfrak{X})^{\mathrm{an}}$ and $\Gamma(\mathbb{E},\mathfrak{X})^{\mathrm{an}}$ as in (0.8), (0.9), replacing $G(z,\zeta;E_v)$ by $G(z,\zeta;\mathbf{E}_v)^{\mathrm{an}}$ and $V_\zeta(E_v)$ by $V_\zeta(\mathbf{E}_v)^{\mathrm{an}}$, but omitting the weights $\log(q_v)$ at nonarchimedean places. We then define the global Robin constant $V(\mathbb{E},\mathfrak{X})^{\mathrm{an}}$ using the minimax formula (0.11) taking $\Gamma = \Gamma(\mathbb{E},\mathfrak{X})^{\mathrm{an}}$, and the global capacity by

$$\gamma(\mathbb{E},\mathfrak{X})^{\mathrm{an}} \;=\; e^{-V(\mathbb{E},\mathfrak{X})^{\mathrm{an}}} \;.$$

We will call a set

$$\mathbb{U} \;=\; \prod_v \mathbf{U}_v \;\subset\; \prod_v \mathcal{C}_v^{\mathrm{an}}$$

a *Berkovich adelic neighborhood* of $\mathbb{E}$ if $\mathbf{U}_v$ contains $\mathbf{E}_v$ for each v, and either $\mathbf{U}_v$ is an open set in $\mathcal{C}_v^{\mathrm{an}}$, or $\mathbf{E}_v$ is $\mathfrak{X}$-trivial and $\mathbf{U}_v = \mathbf{E}_v$. We will call $\mathbb{U}$ a *separable Berkovich quasi-neighborhood* of $\mathbb{E}$ if $\mathbf{U}_v$ contains $\mathbf{E}_v$ for each v, and either $\mathbf{U}_v$ is the union of a Berkovich open set and finitely many open sets in $\mathcal{C}_v(F_w)$ for finite separable extensions F_w/K_v, or $\mathbf{E}_v$ is $\mathfrak{X}$-trivial and $\mathbf{U}_v = \mathbf{E}_v$. We will say that $\mathbb{U}$ is *K-rational* if each U_v is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$.

The following is the Berkovich analogue of Theorem 0.4:

THEOREM 1.6 (Berkovich FSZ with LR). *Let K be a global field, and let $\mathcal{C}/K$ be a smooth, geometrically integral, projective curve. Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite set of points stable under $\mathrm{Aut}(\widetilde{K}/K)$, and let $\mathbb{E} = \prod_v \mathbf{E}_v \subset \prod_v \mathcal{C}_v^{\mathrm{an}}$ be a compact Berkovic adelic set compatible with $\mathfrak{X}$, such that each $\mathbf{E}_v$ is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$. Let $S \subset \mathcal{M}_K$ be a finite set of places, containing all archimedean v, such that $\mathbf{E}_v$ is $\mathfrak{X}$-trivial for each $v \notin S$.*

Assume that $\gamma(\mathbb{E}, \mathfrak{X}) > 1$. Assume also that for each $v \in S$,

(A) If v is archimedean and $K_v \cong \mathbb{C}$, then $\mathbf{E}_v$ is compact, and is a finite union of sets $E_{v,i}$, each of which is compact, connected, and bounded by finitely many Jordan curves.

(B) If v is archimedean and $K_v \cong \mathbb{R}$, then $\mathbf{E}_v$ is compact and is a finite union of sets $E_{v,\ell}$, where each $E_{v,\ell}$ is either

(1) compact, connected, and bounded by finitely many Jordan curves, or

(2) is a closed subinterval of $\mathcal{C}_v(\mathbb{R})$ with nonempty interior.

(C) If v is nonarchimedean, then $\mathbf{E}_v$ is compact and is a finite union of sets $E_{v,\ell}$, where each $E_{v,\ell}$ is either

(1) a strict closed Berkovich affinoid, or

(2) is a compact subset of $\mathcal{C}_v(\mathbb{C}_v)$ and has the form $\mathcal{C}_v(F_{w,\ell}) \cap B(a_\ell, r_\ell)$ for some finite separable extension $F_{w,\ell}/K_v$ in $\mathbb{C}_v$, and some ball $B(a_\ell, r_\ell)$.

Then there are infinitely many points $\alpha \in \mathcal{C}(\widetilde{K}^{\mathrm{sep}})$ such that for each $v \in \mathcal{M}_K$, the $\mathrm{Aut}(\widetilde{K}/K)$-conjugates of α all belong to $\mathbf{E}_v$.

Last, we give a Berkovich version of the Fekete-Szegö Theorem with local rationality for quasi-neighborhoods, generalizing Theorem 1.2 and ([**7**], Theorem 7.48):

THEOREM 1.7 (Berkovich Fekete/FSZ with LR for Quasi-neighborhoods). *Let K be a global field, and let $\mathcal{C}/K$ be a smooth, connected, projective curve. Let $\mathfrak{X} = \{x_1, \ldots, x_m\} \subset \mathcal{C}(\widetilde{K})$ be a finite set of points stable under $\mathrm{Aut}(\widetilde{K}/K)$, and let $\mathbb{E} = \prod_v \mathbf{E}_v \subset \prod_v \mathcal{C}_v^{\mathrm{an}}$ be a compact Berkovich adelic set compatible with $\mathfrak{X}$, such that each $\mathbf{E}_v$ is stable under $\mathrm{Aut}_c(\mathbb{C}_v/K_v)$.*

(A) If $\gamma(\mathbb{E}, \mathfrak{X})^{\mathrm{an}} < 1$, there is a K-rational Berkovich neighborhood $\mathbb{U} = \prod_v \mathbf{U}_v$ of $\mathbb{E}$ such that there are only finitely many points of $\mathcal{C}(\widetilde{K})$ whose $\mathrm{Aut}(\widetilde{K}/K)$-conjugates are all contained in $\mathbf{U}_v$, for each $v \in \mathcal{M}_K$.

(B) If $\gamma(\mathbb{E}, \mathfrak{X})^{\mathrm{an}} > 1$, then for any K-rational separable Berkovich quasi-neighborhood $\mathbb{U}$ of $\mathbb{E}$, there are infinitely many points $\alpha \in \mathcal{C}(\widetilde{K}^{\mathrm{sep}})$ such that for each $v \in \mathcal{M}_K$, the $\mathrm{Aut}(\widetilde{K}/K)$-conjugates of α all belong to $\mathbf{U}_v$.

CHAPTER 2

Examples and Applications

In this chapter we illustrate the Fekete-Szegö theorem with local rationality. We first apply it on $\mathbb{P}^1$, using it to construct algebraic integers and algebraic units satisfying various conditions. We then apply it on elliptic curves, Fermat curves, and modular curves.

1. Local Capacities and Green's Functions of Archimedean Sets

Suppose $K_v = \mathbb{R}$ or $K_v = \mathbb{C}$. In this section we give formulas for local capacities and Green's functions of sets in $\mathbb{P}^1(\mathbb{C})$ which arise naturally in arithmetic applications. Some involve closed formulas, others require numerical computations. Most of the formulas appear in the literature. Further examples, mainly concerning sets in $\mathbb{C}$ with geometric symmetry, are given in ([**51**], pp. 348-351).

For archimedean sets, the most effective way of determining capacities is by "guessing" the Green's function: given E and $\zeta \notin E$, if a function can be found which is continuous, 0 on E, and harmonic in the complement of E except for a positive logarithmic pole at ζ, then by the maximum modulus principle, it must be the Green's function. Then, given a uniformizing parameter $g_\zeta(z)$, the Robin constant and capacity of E with respect to ζ can be read off by

$$(2.1) \qquad V_\zeta(E) \;=\; \lim_{z\to\zeta} G(z,\zeta;E) + \log(|g_\zeta(z)|) \;, \quad \gamma_\zeta(E) = e^{-V_\zeta(E)} \;.$$

For the sets we are dealing with here, which are compact unions of continua, the upper Green's function $\overline{G}(z,\zeta;E)$ coincides with the usual Green's function $G(z,\zeta;E)$.

In the discussion below, we will identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C} \cup \{\infty\}$. When $\zeta = \infty$, we take $g_\zeta(z) = 1/z$; when $\zeta \in \mathbb{C}$, we take $g_\zeta(z) = z - \zeta$.

The Disc. The most basic example is when E is the disc $D(0,r) \subset \mathbb{C}$. Here

$$(2.2) \qquad G(z,\infty;E) \;=\; \log^+(|z/r|) \;=\; \begin{cases} \log(|z/r|) & \text{if } |z| > r \\ 0 & \text{if } |z| \le r \end{cases} \;.$$

Computing capacities relative to the parameter $g_\infty(z) = 1/z$, we find

$$(2.3) \qquad V_\infty(E) = \lim_{z\to\infty} G(z,\infty;E) - \log(|z|) = -\log(r) \;, \qquad \gamma_\infty(E) = e^{-V_\infty(E)} = r \;.$$

By applying a linear fractional transformation, one can find the Green's function of $D(0,r)$ with respect to an arbitrary point $\zeta \in \mathbb{C}$:

$$(2.4) \qquad G(z,\zeta;E) \;=\; \log^+\left(\left|\frac{r^2 - \overline{\zeta}z}{r(z-\zeta)}\right|\right) \;.$$

Computing capacities relative to $g_\zeta(z) = z - \zeta$, one has

$$(2.5) \qquad V_\zeta(E) \;=\; \lim_{z\to\zeta} G(z,\infty;E) + \log(|z-\zeta|) \;=\; \log(\frac{|\zeta|^2 - r^2}{r}) \ ,$$

$$(2.6) \qquad \gamma_\zeta(E) = e^{-V_\zeta(E)} = \frac{r}{|\zeta|^2 - r^2} \ .$$

The Segment. Another basic example is when E is a segment $[a,b] \subset \mathbb{R}$. Choosing the branch of $\sqrt{z}$ which is positive on the positive real axis and cut along the negative real axis, the map $z \mapsto w = \sqrt{(z-a)/(z-b)}$ takes $\mathbb{P}^1(\mathbb{C})\backslash[a,b]$ to the right half-plane and takes ∞ to 1; then $w \mapsto (w+1)/(w-1)$ takes the right half-plane to the exterior of the unit disc, and takes 1 to ∞. It follows that

$$(2.7) \qquad G(z,\infty;E) \;=\; \log^+\left(\left|\frac{\sqrt{(z-a)/(z-b)}+1}{\sqrt{(z-a)/(z-b)}-1}\right|\right) \ .$$

For an arbitrary $\zeta \in \mathbb{C}$, a similar computation (see [**16**], p.165) gives

$$(2.8) \qquad G(z,\zeta;E) \;=\; \log^+\left(\left|\frac{\sqrt{(z-a)/(z-b)} + \overline{\sqrt{(\zeta-a)/(\zeta-b)}}}{\sqrt{(z-a)/(z-b)} - \overline{\sqrt{(\zeta-a)/(\zeta-bB)}}}\right|\right) \ .$$

With $g_\infty(z) = 1/z$ and $g_\zeta(z) = z - \zeta$ for $\zeta \in \mathbb{C}\backslash E$, one finds

$$(2.9) \qquad V_\infty(E) \;=\; -\log((b-a)/4) \ , \qquad \gamma_\infty(E) \;=\; (b-a)/4 \ ,$$

$$(2.10) \qquad \gamma_\zeta(E) = e^{-V_\zeta(E)} = \frac{b-a}{4 \cdot \mathrm{Re}(\sqrt{(\zeta-a)|\zeta-a| \cdot (\overline{\zeta-b})|\zeta-b|})}$$

When $\zeta = \infty$ there is another expression for $G(z,\infty,E)$ which makes its geometric behavior clearer. For simplicity, assume $E = [-2r, 2r]$ where $0 < r \in \mathbb{R}$. It is well known, and easy to verify, that the Joukowski map

$$(2.11) \qquad z \;=\; J_r(w) \;=\; w + \frac{r^2}{w}$$

maps $\mathbb{C}\backslash D(0,r)$ conformally onto $\mathbb{C}\backslash[-2r,2r]$. For each $R > r$, it takes the circle $C(0,R)$ parametrized by $w = R\cos(\theta) + iR\sin(\theta)$ to the ellipse $\mathcal{E}(R + \frac{r^2}{R}, R - \frac{r^2}{R})$ parametrized by

$$(2.12) \quad z \;=\; x+iy \;=\; (R+\frac{r^2}{R})\cos(\theta)+i(R-\frac{r^2}{R})\sin(\theta) \;=\; J_r(R\cos(\theta)+iR\sin(\theta)) \ .$$

It maps the circle $C(0,R)$ in a $2-1$ manner to the interval $[-2r,2r]$, and takes ∞ to ∞.

The function $G_r(z) = \log(|J_r^{-1}(z)|/r)$ is harmonic on $\mathbb{C}\backslash E$, with a logarithmic pole at ∞; it has a continuous extension to $\mathbb{C}$ which takes the value 0 on E. By the characterization of Green's functions, $G(z,\infty;[-2r,2r]) = G_r(z)$. Thus, for each $R > r$,

$$(2.13) \qquad \{z \in \mathbb{C} : G(z,\infty;[-2r,2r]) = \log(R/r)\} \;=\; \mathcal{E}(R + \frac{r^2}{R}, R - \frac{r^2}{R}) \ .$$

Two Segments. When $E = [a,b] \cup [c,d] \subset \mathbb{R}$, there are closed formulas for the Green's function and capacity. When the segments have the same length, $G(z,\infty;E)$ and $\gamma_\infty(E)$ are given by elementary formulas. In general, they can be expressed in terms of theta-functions.

First suppose $E = [-b, -a] \cup [a, b] \subset \mathbb{R}$. Put $f(z) = z^2$; then $f^*((\infty)) = 2(\infty)$ and $E = f^{-1}([a^2, b^2])$. By the pullback formula for Green's functions (see (2.61) below),

$$G(z, \infty; E) \;=\; \frac{1}{2}\log^+\left(\left|\frac{\sqrt{(z^2-a^2)/(z^2-b^2)}+1}{\sqrt{(z^2-a^2)/(z^2-b^2)}-1}\right|\right) \;. \tag{2.14}$$

Using this, we find

$$V_\infty(E) = \frac{1}{2}\log(4/(b^2-a^2)) \;, \quad \gamma_\infty(E) = \frac{\sqrt{b^2-a^2}}{2} \;, \tag{2.15}$$

$$G(0, \infty; E) \;=\; G(\infty, 0, E) \;=\; \frac{1}{2}\log(\frac{b+a}{b-a}) \;. \tag{2.16}$$

Similarly, when $\zeta = 0$, pulling back $[1/b^2, 1/a^2]$ by $f(z) = 1/z^2$, we get

$$G(z, 0; E) \;=\; -\frac{1}{2}\log\left(\left|\frac{\sqrt{(z^2-b^2)/(z^2-a^2)}+b/a}{\sqrt{(z^2-b^2)/(z^2-a^2)}-b/a}\right|\right) \;, \tag{2.17}$$

$$V_0(E) = \frac{1}{2}\log(4a^2b^2/(b^2-a^2)) \;, \quad \gamma_0(E) = \frac{\sqrt{b^2-a^2}}{2ab} \;. \tag{2.18}$$

Before dealing with a general set $E = [a, b] \cup [c, d] \subset \mathbb{R}$, and arbitrary ζ, it will be useful to recall some of the properties of classical theta-functions (see Shimura, [**60**], [**67**]). For $u \in \mathbb{C}$, $\tau \in \mathfrak{H} = \{\mathrm{Im}(z) > 0\}$, and $r, s \in \mathbb{R}$, write $e(z) = e^{2\pi i z}$ and put

$$\theta(u, \tau; r, s) \;=\; \sum_{n\in\mathbb{Z}} e(\frac{1}{2}(n+r)^2\tau + (n+r)(u+s)) \;. \tag{2.19}$$

Because of the quadratic dependence on n in (2.19), the series defining $\theta(u, \tau; r, s)$ converges very rapidly. $\theta(u, \tau; r, s)$ is continuous in all four variables and is jointly holomophic in u and τ.

We will be particularly interested in $\theta(u, \tau; \frac{1}{2}, \frac{1}{2})$. When $r, s \in \{0, 1/2\}$, the functions $\theta(u, \tau; r, s)$ appear in the literature with several names. Our notation follows that of Krazer-Prym and Shimura; it differs from that of Riemann and Mumford (and of Whittaker-Watson [**67**]). The equivalences (KPS = RM = WW) are as follows:

$$\theta(u, \tau; \tfrac{1}{2}, \tfrac{1}{2}) = \theta_{11}(u, \tau) = \vartheta_1(\pi u|\tau) \;, \qquad \theta(u, \tau; \tfrac{1}{2}, 0) = \theta_{10}(u, \tau) = \vartheta_2(\pi u|\tau) \;,$$
$$\theta(u, \tau; 0, 0) = \theta_{00}(u, \tau) = \vartheta_3(\pi u|\tau) \;, \qquad \theta(u, \tau; 0, \tfrac{1}{2}) = \theta_{01}(u, \tau) = \vartheta_4(\pi u|\tau) \;.$$

Courant and Hilbert ([**13**]) use a fourth notation; the equivalence (KPS = CH) is:

$$\theta(u, \tau; 0, \frac{1}{2}) = \theta_0(u) \qquad \text{and} \qquad \theta(u, \tau; \frac{1}{2}, \frac{1}{2}) = \theta_1(u) \;.$$

Considering $\theta(u, \tau; r, s)$ as a function of u and using the definition, one sees that for all $a, b \in \mathbb{Z}$

$$\theta(u + za + b, \tau; r, s) = e(rb - as) \cdot e(-\frac{1}{2}a^2\tau - au) \cdot \theta(u, z; r, s) \;. \tag{2.20}$$

Applying the Argument Principle, it follows that $\theta(u, \tau, r, s)$ has a simple zero in each period parallelogram for the lattice $\langle 1, \tau\rangle \subset \mathbb{C}$; the zero occurs at $u \equiv (\frac{1}{2} - r)\tau + (\frac{1}{2} - s) \pmod{\langle 1, \tau\rangle}$ (see [**67**], p.465-466, and [**60**], formula (11), p.675).

Again using the definitions, one sees that $\theta(u,\tau;\frac{1}{2},\frac{1}{2})$ is an odd function of u, that $\theta(u+\frac{1}{2},\tau;\frac{1}{2},\frac{1}{2}) = -\theta(u,\tau;\frac{1}{2},0)$, and if τ is pure imaginary, then $\theta(\overline{u},\tau;\frac{1}{2},\frac{1}{2}) = \overline{\theta(u,\tau;\frac{1}{2},\frac{1}{2})}$. Similarly $\theta(u,\tau;\frac{1}{2},0)$ is an even function of u, and if τ is pure imaginary, then $\theta(\overline{u},\tau;\frac{1}{2},0) = \overline{\theta(u,\tau;\frac{1}{2},0)}$.

With these facts, one can check that if τ is pure imaginary, then for each $M \in \mathbb{C}$ with $\mathrm{Re}(M) \notin \frac{1}{2}\mathbb{Z}$, the function

$$\mathcal{G}(u) \;=\; \frac{\theta(u-M,\tau;\frac{1}{2},\frac{1}{2})}{\theta(u+\overline{M},\tau;\frac{1}{2},\frac{1}{2})} \tag{2.21}$$

satisfies $|\mathcal{G}(u+\tau)| = |\mathcal{G}(u+1)| = |\mathcal{G}(u)|$, and if $\mathrm{Re}(u) = 0$ or if $\mathrm{Re}(u) = \frac{1}{2}$ then $|\mathcal{G}(u)| = 1$. It has simple zeros at points $u \equiv M \pmod{\langle 1,\tau\rangle}$, simple poles at $u \equiv -\overline{M} \pmod{\langle 1,\tau\rangle}$, and no other zeros or poles.

Now consider a set $E = [a,b] \cup [c,d] \subset \mathbb{R}$, where $a < b < c < d$. We will give a (multivalued, periodic) conformal mapping of $\mathbb{C}\backslash E$ onto a vertical strip, which will enable us to express $G(z,\zeta;E)$ in terms of the function $\mathcal{G}(u)$ in (2.21). We follow Akhiezer ([**2**]) and Falliero and Sebbar ([**22**], [**23**]), but obtain a different expression for the capacity.

First, put

$$w \;=\; T(z) \;=\; \sqrt{\frac{(z-a)}{(z-b)}\frac{(d-b)}{(d-a)}}\;. \tag{2.22}$$

where $\sqrt{z}$ is positive for $z > 0$ and is slit along the negative real axis. $T(z)$ maps $\mathbb{C}\backslash E$ conformally onto the right half-plane with the segment $[1, 1/k]$ removed, where

$$k \;=\; \frac{1}{T(c)} \;=\; \sqrt{\frac{(c-b)}{(c-a)}\frac{(d-a)}{(d-b)}}\;. \tag{2.23}$$

$T(z)$ takes $a \mapsto 0$, $b \mapsto \infty$, $d \mapsto 1$, and $c \mapsto 1/k$. Since the linear fractional transformation $F(z) = (z-a)(d-b)/(z-b)(d-a)$ maps $\mathbb{R} \cup \infty$ to itself and preserves the cyclic order of a, b, c, d, ones sees that $T(c) > 1$ and $0 < k < 1$. Note that $1/k^2$ is the crossratio $(a, b; c, d)$.

Follow $T(z)$ with the elliptic integral

$$u \;=\; S(w) \;=\; \int_0^w \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}\;. \tag{2.24}$$

Here $S(w)$ is the Schwarz-Christoffel map which sends the upper half-plane to the rectangle with corners $\pm K, \pm K + iK'$, where

$$K \;=\; \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2.25}$$

$$iK' \;=\; \int_1^{1/k} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{2.26}$$

and $K, K' > 0$. It takes the imaginary axis to itself, and sends $0 \mapsto 0$, $1 \mapsto K$, $1/k \mapsto K + iK'$, and $\infty \mapsto iK'$. By the Schwarz Reflection Principle, $S(w)$ extends to a multivalued holomorphic function taking $\{\mathrm{Re}(w) > 0\}\backslash[1, 1/k]$ to the vertical strip $0 < \mathrm{Re}(u) < K$, with period $2iK'$. The inverse function to $S(w)$ is the Jacobian elliptic function $w = \mathrm{sn}(u,k)$ (see [**67**], §22, and [**44**], §VI.3).

Now let $\tau = iK'/K$. Fix $\zeta \notin E$; put $u = S(T(z))$, $M = M(\zeta) = S(T(\zeta))$. Scaling $u \mapsto v = u/(2K)$ takes $0 < \mathrm{Re}(u) < K$ to the strip $0 < \mathrm{Re}(v) < 1/2$, with $2iK' \mapsto \tau$. We claim that

$$G(z,\zeta;E) \;=\; -\log\left(\left|\frac{\theta(\frac{u-M}{2K},\tau;\frac{1}{2},\frac{1}{2})}{\theta(\frac{u+\overline{M}}{2K},\tau;\frac{1}{2},\frac{1}{2})}\right|\right). \tag{2.27}$$

Indeed, by our discussion of theta-functions, the function on the right has the properties characterizing $G(z,\zeta;E)$: it is well-defined and continuous, vanishes on E, is harmonic on $\mathbb{P}^1(\mathbb{C})\backslash(E \cup \zeta)$, and has a positive logarithmic pole as $z \to \zeta$. This formula is one given by Falliero and Sebbar ([**22**]; [**23**], p.416).

Numerically, K and K' can be found using the hypergeometric function

$$F(a,b,c;z) \;=\; 1 + \frac{a\cdot b}{1\cdot c}z + \frac{a(a+1)\cdot b(b+1)}{1\cdot 2\cdot c(c+1)}z^2 + \cdots$$

with $K = \frac{1}{2}\pi F(\frac{1}{2},\frac{1}{2},1;k^2)$, $K' = \frac{1}{2}\pi F(\frac{1}{2},\frac{1}{2},1;1-k^2)$ (see [**67**], pp.499, 501); then $\tau = iK'/K$. Another way to determine τ, K and K' is by first solving for $q = e^{i\pi\tau}$ using the relation

$$\frac{(c-b)}{(c-a)}\frac{(d-a)}{(d-b)} = k^2 = \frac{\theta(0,\tau,\frac{1}{2},0)^4}{\theta(0,\tau,0,0)^4} = \frac{16(q^{1/4}+q^{9/4}+q^{25/4}+\cdots)^4}{(1+2q^4+2q^9+\cdots)^4} \tag{2.28}$$

and then using the formulas

$$K = \frac{1}{2}\pi\theta(0,\tau,0,0)^2\ , \qquad K' = -i\tau K\ . \tag{2.29}$$

Finally, M can be determined by solving

$$T(\zeta) \;=\; \mathrm{sn}(M,k) \;=\; \frac{1}{k}\frac{\theta(M/2K,\tau;\frac{1}{2},\frac{1}{2})}{\theta(M/2K,\tau;0,\frac{1}{2})}\ . \tag{2.30}$$

(See [**67**], pp.492, 501.)

We now determine the capacity of E. If $\zeta = \infty$, put $\widehat{z} = 1/z$; otherwise put $\widehat{z} = z - \zeta$. Then as $\widehat{z} \to 0$, we have $z \to \zeta$, $w \to T(\zeta)$, and $u \to M$. Using (2.27), it follows that

$$\begin{aligned} V_\zeta(E) \;&=\; \lim_{\widehat{z}\to 0} G(z,\zeta;E) + \log(|\widehat{z}|) \\ &=\; -\log\left(\left|\frac{\frac{d}{du}\theta(0,\tau;\frac{1}{2},\frac{1}{2})}{\theta(\frac{M+\overline{M}}{2K},\tau;\frac{1}{2},\frac{1}{2})}\cdot\frac{1}{2K}\cdot\frac{dw}{du}(T(\zeta))\cdot\frac{dT}{d\widehat{z}}(0)\right|\right) \end{aligned} \tag{2.31}$$

The last two terms can be computed in terms of ζ and a,b,c,d; the expression can then be simplified using the Jacobi identity

$$\frac{d}{du}\theta(0,\tau;\frac{1}{2},\frac{1}{2}) = \pi\theta(0,\tau;0,0)\theta(0,\tau;\frac{1}{2},0)\theta(0,\tau;0,\frac{1}{2}) \tag{2.32}$$

(see [**67**], p.470), together with (2.29) and (2.28). If $\zeta = \infty$ one obtains

$$\gamma_\infty(E) = e^{-V_\infty(E)} = \frac{\sqrt[4]{(c-a)(c-b)(d-a)(d-b)}}{2\left|\frac{\theta(\mathrm{Re}(M(\infty))/K,\tau;\frac{1}{2},\frac{1}{2})}{\theta(0,\tau;0,\frac{1}{2})}\right|}\ ; \tag{2.33}$$

if $\zeta \in \mathbb{C}\backslash E$, then

$$\gamma_\zeta(E) = \frac{\sqrt[4]{(c-a)(c-b)(d-a)(d-b)}}{2\left|\frac{\theta(\mathrm{Re}(M(\zeta))/K,\tau;\frac{1}{2},\frac{1}{2})}{\theta(0,\tau;0,\frac{1}{2})}\right|\cdot|(\zeta-a)(\zeta-b)(\zeta-c)(\zeta-d)|^{1/2}}\ . \tag{2.34}$$

Numerical examples confirm the compatibility of (2.33) with (2.15). However the formula of Akhiezer reported in ([**23**], p.422) seems to be incorrect.

Three Segments. When $E = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3] \subset \mathbb{R}$ and $\zeta = \infty$, Thérèse Falliero has given formulas for the Green's function and capacity of E using theta-functions of genus 2; for these, we refer the reader to Falliero ([**22**]) and Falliero-Sebbar ([**23**]).

Multiple Segments. When $E = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_n, b_n] \subset \mathbb{R}$ with $a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n$ and n arbitrary, Harold Widom ([**68**], pp.224ff) has given formulas for $G(z, \zeta; E)$ and $V_\zeta(E)$ which we recall below.

Let $q(z) = \prod_{j=1}^n (z - a_j)(z - b_j)$, and fix the branch of the square root

$$q(z)^{1/2} \;=\; \sqrt{\prod_{j=1}^{n}(z - a_j)(z - b_j)}$$

on $\mathbb{C}\backslash E$ which is positive as $z \to \infty$ along the real axis. This branch is well-defined throughout $\mathbb{C}\backslash E$ and positive on $\mathbb{R}\backslash E$. For each $x \in E$, the limiting values of $q(z)^{1/2}$ as $z \to x+i0^+$ and $z \to x+i0^-$ are pure imaginary, and are negatives of each other. For convenience, we will extend $q(z)^{1/2}$ to E by defining $q(x)^{1/2} = q(x + i0^-)^{1/2}$ when $x \in E$. Thus, $q(z)^{1/2}$ is pure imaginary on E.

First take $\zeta = \infty$. Fix a point $z_0 \in E$, and let $h(z) = h_0+h_1z+\cdots+h_{n-1}z^{n-1} \in \mathbb{R}(z)$ be a polynomial of degree $\le n-1$ with real coefficients. Consider the multiple-valued function

$$G_h(z) \;=\; \int_{z_0}^{z} h(w)/q(w)^{1/2}\, dw$$

on $\mathbb{C}$, where the integral is taken over any path from z_0 to z which is disjoint from E except for one or both of its endpoints. Since $G_h(z)$ has pure imaginary periods around ∞ and around each component $[a_j, b_j]$ of E, the function $\mathrm{Re}(G_h(z))$ is well-defined and continuous, and constant on each component of E. Since $G(z)$ has a holomorphic branch in a neighborhood of each point $w \in \mathbb{C}\backslash E$, $\mathrm{Re}(G_h(z))$ is harmonic in $\mathbb{C}\backslash E$.

Clearly $\mathrm{Re}(G_h(z)) \equiv 0$ on the component of E containing z_0. If

$$\int_{b_j}^{a_{j+1}} h(x)/q(x)^{1/2}\, dx \;=\; 0 \tag{2.35}$$

for each 'gap' (b_j, a_{j+1}), then $\mathrm{Re}(G_h(z)) \equiv 0$ on E. If in addition $h(z)$ is monic, there is a number $V \in \mathbb{R}$ such that $\mathrm{Re}(G_h(z))$ is asymptotic to $\log(|z|) + V$ as $z \to \infty$. In this setting, the characterization of Green's functions shows that

$$G(z, \infty; E) \;=\; \mathrm{Re}(G_h(z)) \;. \tag{2.36}$$

We will now show that such an $h(z)$ exists.

Following Widom, put $A_{jk} = \int_{b_j}^{a_{j+1}} x^k/q(x)\, dx$ for $j = 1, \ldots, n-1$, $k = 0, \ldots, n-1$. Note that $1/q(z)^{1/2}$ has singularities at b_j and a_{j+1} of order $z^{-1/2}$, so each A_{jk} is finite and belongs to $\mathbb{R}$. We claim that there is a unique solution $h_0, h_1, \ldots, h_{n-1}$ to the system of linear equations

$$\begin{cases} \sum_{k=0}^{n-1} A_{jk}h_k \;=\; 0 \qquad \text{for } j = 1, \ldots, n-1 \;, \\ \qquad\quad h_{n-1} \;=\; 1 \;. \end{cases} \tag{2.37}$$

If $h_0, \ldots, h_{n-1}$ satisfy (2.37), and $h(z)$ is the corresponding polynomial, then $h(z)$ is monic and the conditions (2.35) hold. Thus $G(z, \infty; E) = \mathrm{Re}(G_h(z))$. Solving the system (2.37) is called the 'Jacobi Inversion Problem'.

To see that (2.37) has a unique solution, it suffices to show that the $n \times n$ matrix associated to the system has rank n, or equivalently, that $h_0 = \cdots = h_{n-1} = 0$ is the only solution to the corresponding homogeneous system. Let $h_0, \ldots, h_{n-1} \in \mathbb{R}$ be any solution to the homogeneous system, and let $h(z)$ be the corresponding polynomial. Then $G_h(z)$ is harmonic on $\mathbb{C}\backslash E$, vanishes on E, and remains bounded as $z \to \infty$ since $h_{n-1} = 0$, so it extends to a function on $\mathbb{P}^1(\mathbb{C})\backslash E$ harmonic at ∞. By the maximum principle for harmonic functions, $G_h(z) \equiv 0$. Restricting $G_h(z)$ to $\mathbb{R}$, differentiating, and using the Fundamental Theorem of Calculus, we see that $h(z)/q(z)^{1/2} \equiv 0$ on $\mathbb{R}\backslash E$. Since $q(z)$ is nonzero except at the endpoints of E, it follows that $h(z) \equiv 0$, and hence that $h_0 = \cdots = h_{n-1} = 0$.

We remark that $h(z)$ has one zero in each gap (b_j, a_{j+1}), and no other zeros. Indeed $G(x, \infty; E)$ vanishes at b_j and a_{j+1}, and is real-valued and differentiable on (b_j, a_{j+1}), so by Rolle's Theorem there is a point $x_j^* \in (b_j, a_{j+1})$ where $G'(x_j^*, \infty; E) = 0$. The argument above shows that $h(x_j^*) = 0$. Since $h(z)$ has degree $n-1$, it has a unique zero in each gap, and these are its only zeros. Thus, $h(z) = \prod_{j=1}^{n-1}(z - x_j^*)$, and $h(z)$ has constant sign on each component of E.

Now consider the case when $\zeta \in \mathbb{C}\backslash E$. Again, fix $z_0 \in E$. We claim that for a suitable polynomial $h(z) = h_0 + h_1 z + \cdots h_{n-1} z^{n-1} \in \mathbb{C}[z]$, we have

$$G(z, \zeta; E) \;=\; \mathrm{Re}(G_h(z)) \;, \tag{2.38}$$

where now

$$G_h(z) \;=\; \int_{z_0}^{z} \frac{h(w)}{q(w)^{1/2}(w-\zeta)}\, dw \;. \tag{2.39}$$

Here $h(z)$ must be chosen so that the following properties are satisfied:

(1) The periods of $G_h(z)$ are pure imaginary, so $\mathrm{Re}(G_h(z))$ is well-defined.
(2) The value of $\mathrm{Re}(G_h(z))$ is 0 on each segment $[a_j, b_j]$.
(3) $\mathrm{Re}(G_h(z))$ has a singularity of type $-\log(|z-\zeta|)$ at ζ.

Let Γ_j be a loop about $[a_j, b_j]$, traversed counterclockwise. Using Cauchy's theorem, one sees that

$$\int_{\Gamma_j} \frac{h(w)}{q(w)^{1/2}(w-\zeta)}\, dw \;=\; 2\int_{a_j}^{b_j} \frac{h(x)}{q(x)^{1/2}(x-\zeta)}\, dx \;.$$

Thus for the periods of $G_h(z)$ about the intervals $[a_j, b_j]$ to be pure imaginary, we need

$$\mathrm{Re}\,\Big(\int_{a_j}^{b_j} \frac{h(x)}{q(x)^{1/2}(x-\zeta)}\, dx\Big) \;=\; 0 \qquad \text{for } j = 1, \ldots, n \;. \tag{2.40}$$

Let $\varepsilon > 0$ be small enough that the circle $C(\zeta, \varepsilon)$ is disjoint from E. Since the differential $h(w)\,dw/\big(q(w)^{1/2}(w-\zeta)\big)$ is holomorphic at ∞, applying Cauchy's theorem on the domain $\mathbb{P}^1(\mathbb{C})\backslash(E \cup \{\zeta\})$ we obtain

$$\sum_{j=1}^{n} \int_{\Gamma_j} \frac{h(w)}{q(w)^{1/2}(w-\zeta)}\, dw + \int_{C(\zeta,\varepsilon)} \frac{h(w)}{q(w)^{1/2}(w-\zeta)}\, dw \;=\; 0 \;.$$

When (2.40) holds, the period of $G_h(z)$ about ζ (namely $2\pi i h(\zeta)/q(\zeta)^{1/2}$) is pure imaginary as well.

If (2.40) holds, then $\mathrm{Re}(G_h(z))$ is well-defined, harmonic in $\mathbb{P}^1(\mathbb{C})\backslash(E \cup \{\zeta\})$, and constant on each segment $[a_j, b_j]$. Clearly its value on the segment containing z_0 is 0. For it to be identically 0 on E, we need

$$\text{(2.41)} \qquad \mathrm{Re}\,\Big(\int_{b_j}^{a_{j+1}} \frac{h(x)}{q(x)^{1/2}(x-\zeta)}\,dx\Big) \;=\; 0 \qquad \text{for } j = 1, \ldots, n-1 \;.$$

Finally, for $\mathrm{Re}(G_h(z))$ to have a singularity of type $-\log(|z-\zeta|)$ at ζ, we need

$$-1 \;=\; \mathrm{Res}_{w=\zeta}\,\Big(\frac{h(w)}{q(w)^{1/2}(w-\zeta)}\Big) \;=\; \frac{h(\zeta)}{q(\zeta)^{1/2}} \;.$$

Since the period of $G_h(z)$ about ζ is imaginary we have $\mathrm{Im}(h(\zeta)/q(\zeta)^{1/2}) = 0$, and it is enough to require

$$\text{(2.42)} \qquad -1 \;=\; \mathrm{Re}\,\Big(\frac{h(\zeta)}{q(\zeta)^{1/2}}\Big) \;.$$

Writing $h_k = c_k + d_k i$ for $k = 0, \ldots, n-1$, with $c_k, d_k \in \mathbb{R}$, the conditions (2.40), (2.41) and (2.42) represent a system of $2n$ linear equations with real coefficients in $2n$ real unknowns. To show that it has a unique solution, it is enough to show that the only solution to the corresponding homogeneous system is the trivial one.

Suppose that $h(z)$ arises from a solution to the homogeneous system. Then $\mathrm{Re}(G_h(z))$ is harmonic in $\mathbb{P}^1(\mathbb{C})\backslash(E \cup \{\zeta\})$ and extends to a function harmonic at ζ, with boundary values 0 on E. By the Maximum Principle, $\mathrm{Re}(G_h(z)) \equiv 0$. Differentiating, and using the Fundamental Theorem of Calculus on horizontal segments, we see that

$$\mathrm{Re}\,\Big(\frac{h(z)}{q(z)^{1/2}(z-\zeta)}\Big) \;=\; \frac{\partial}{\partial x}\big(\mathrm{Re}(G_h(z))\big) \;\equiv\; 0$$

on $\mathbb{C}\backslash(E \cup \{\zeta\})$. If the real part of an analytic function is identically 0, that function is constant, so we must have

$$\frac{h(z)}{q(z)^{1/2}(z-\zeta)} \;=\; C$$

for some purely imaginary constant C. However, $q(z)^{1/2}(z-\zeta)$ is not a polynomial, so this can hold only if $C = 0$. Thus $h(z) \equiv 0$, which means that $c_1 = d_1 = \cdots = c_n = d_n = 0$.

When $\zeta = \infty$, choosing $z_0 \neq 0$ and noting that $\log(|z|) = \mathrm{Re}\,\big(\int_{z_0}^{\infty} 1/w\,dw\big) - \log(|z_0|)$, Widom gives a formula for the Robin constant equivalent to

$$\begin{aligned} V_\infty(E) \;&=\; \lim_{z\to\infty}\big(G(z,\infty;E) - \log(|z|)\big) \\ &=\; \mathrm{Re}\,\Big(\int_{z_0}^{\infty} \frac{h(w)}{q(w)^{1/2}} - \frac{1}{w}\,dw\Big) \;+\; \log(|z_0|) \;. \end{aligned}$$

Similarly, when $\zeta \in \mathbb{C}\backslash E$,

$$\begin{aligned} V_\zeta(E) \;&=\; \lim_{z\to\zeta}\big(G(z,\zeta;E) + \log(|z-\zeta|)\big) \\ &=\; \mathrm{Re}\,\Big(\int_{z_0}^{\infty} \frac{h(w)}{q(w)^{1/2}(w-\zeta)} + \frac{1}{w-\zeta}\,dw\Big) \;-\; \log(|z_0-\zeta|) \;. \end{aligned}$$

When $\zeta = \infty$, there is a more illuminating formula for $V_\infty(E)$. Put $c = (a_1+b_n)/2$ and let $r = (b_n - a_1)/4$, so $E \subset [a_1, b_n] = [c-2r, c+2r]$. We claim that

$$(2.43) \quad V_\infty(E) \;=\; -\log(\frac{b_n - a_1}{4}) \;+\; \sum_{j=1}^{n-1} \int_{b_j}^{a_{j+1}} G(x,\infty;E)\,\frac{1}{\pi}\frac{dx}{\sqrt{4r^2-(x-c)^2}} \;.$$

Readers familiar with capacity theory will recognize $dx/(\pi\sqrt{4r^2-(x-c)^2})$ as the equilibrium distribution of $[a_1, b_n]$ relative to ∞.

To derive (2.43), assume for simplicity that $c = 0$, so $[a_1, b_n] = [-2r, 2r]$; this can always be arranged by a translation. Note that since $G(z,\infty;[-2r,2r]) \sim \log(|z|) + V_\infty([-2r,2r])$ as $z \to \infty$, and $V_\infty([-2r,2r]) = V_\infty([a_1,b_n]) = -\log((b_n - a_1)/4)$, we have

$$\begin{aligned} V_\infty(E) \;&:=\; \lim_{z\to\infty} G(z,\infty;E) - \log(|z|) \\ (2.44) \qquad &=\; \lim_{z\to\infty} \big(G(z,\infty;E) - G(z,\infty;[-2r,2r])\big) - \log(\frac{b_n-a_1}{4}) \;. \end{aligned}$$

The function $g(z) := G(z,\infty;E) - G(z,\infty;[-2r,2r])$ is harmonic in $\mathbb{C}\backslash[-2r,2r]$ and bounded as $z \to \infty$; hence it extends to a function harmonic in $\mathbb{P}^1(\mathbb{C})\backslash[-2r,2r]$, with

$$(2.45) \qquad g(\infty) \;=\; \lim_{z\to\infty} \big(G(z,\infty;E) - G(z,\infty;[-2r,2r])\big) \;.$$

Let the Joukowski map $z = J_r(w) = w + r^2/w$ be as in (2.11). For each $R > r$, parametrize the ellipse $\mathcal{E}(R + r^2/R, R - r^2/R)$ by $z = J_r(R\cos(\theta) + iR\sin(\theta))$ as in (2.12). Let $\mathcal{D}_R = \mathbb{P}^1(\mathbb{C})\backslash D(0,R)$, and let $\mathcal{E}_R$ be the connected component of $\mathbb{P}^1(\mathbb{C})\backslash\mathcal{E}(R + r^2/R, R - r^2/R)$ containing ∞. The map $J_r(w)$ gives a conformal equivalence from $\mathcal{D}_R$ to $\mathcal{E}_R$, and takes ∞ to ∞. Thus $H(w) := g(J_r(w))$ is harmonic in $\mathcal{D}_R$, and $H(\infty) = g(\infty)$. By the mean value theorem for harmonic functions,

$$\begin{aligned} g(\infty) \;&=\; H(\infty) \;=\; \frac{1}{2\pi}\int_0^{2\pi} H(R\cos(\theta) + iR\sin(\theta))\,d\theta \\ &=\; \frac{1}{2\pi}\int_0^{2\pi} g((R + \frac{r^2}{R})\cos(\theta) + i(R - \frac{r^2}{R})\sin(\theta))\,d\theta \;. \end{aligned}$$

Since $\mathcal{E}(R + r^2/R, R - r^2/R)$ is the level curve $\log(R/r)$ for $G(z,\infty;[-2r,2r])$ (see (2.13)), it follows that

$$g(\infty) \;=\; \frac{1}{2\pi}\int_0^{2\pi} G((R + \frac{r^2}{R})\cos(\theta) + i(R - \frac{r^2}{R})\sin(\theta)),\infty;E)\,d\theta \;-\; \log(R/r) \;.$$

Since $G(z,\infty;E) = G_h(z)$ is continuous on $\mathbb{C}$, letting $R \to r$ we see that

$$g(\infty) \;=\; \frac{1}{2\pi}\int_0^{2\pi} G(2r\cos(\theta),\infty;E)\,d\theta \;.$$

Finally, making the change of variables $x = 2r\cos(\theta)$ yields

$$(2.46) \quad g(\infty) \;=\; \int_{a_1}^{b_n} G(x,\infty;E)\,\frac{1}{\pi}\frac{dx}{\sqrt{4r^2-x^2}} \;=\; \sum_{j=1}^{n-1}\int_{b_j}^{a_{j+1}} G(x,\infty;E)\,\frac{1}{\pi}\frac{dx}{\sqrt{4r^2-x^2}} \;.$$

Combining (2.44), (2.45) and (2.46) gives (2.43).

The Real Projective Line. If $E = \mathbb{P}^1(\mathbb{R})$, the components of its complement in $\mathbb{P}^1(\mathbb{C})$ are the upper and lower half-planes. Fix $\zeta \notin E$. Using the characterization of the Green's function, it is easy to check that if z and ζ belong to the same component of $\mathbb{P}^1(\mathbb{C})\backslash\mathbb{P}^1(\mathbb{R})$, then

$$G(z,\zeta;E) \;=\; -\log\left(\left|\frac{z-\zeta}{z-\overline{\zeta}}\right|\right) \;. \tag{2.47}$$

If z and ζ are not in the same component, then $G(z,\zeta;E)=0$. Taking $g_\zeta(z)=z-\zeta$ and using (2.47) we obtain

$$V_\zeta(E) \;=\; \lim_{z\to\zeta} -\log\left(\left|\frac{z-\zeta}{z-\overline{\zeta}}\right|\right)+\log(|z-\zeta|) \;=\; \log(2|\operatorname{Im}(\zeta)|)\;, \tag{2.48}$$

$$\gamma_\zeta(E) \;=\; \frac{1}{2|\operatorname{Im}(\zeta)|}\;. \tag{2.49}$$

The Disc with Opposite Radial Arms. Take $L_1, L_2 \ge 0$, and let E be the union of $D(0,R)$ with the segment $[-L_1-R, R+L_2]$; thus E is a disc with opposite radial arms of length L_1, L_2. We claim that

$$\gamma_\infty(E) \;=\; \frac{1}{4}\left(2R+\frac{R^2+RL_1+L_1^2}{R+L_1}+\frac{R^2+RL_2+L_2^2}{R+L_2}\right)\;. \tag{2.50}$$

To see this, first take $R=1$. Put $a_1 = 1+L_1$, $a_2 = 1+L_2$; then $E = D(0,1)\cup[-a_1,a_2]$. Let $w=\varphi(z)=(z-1)^2/z$. Then φ is the composite of the maps $z\to t=-1/(z+1)$, $t\to u=t+1/2$, $u\to v=u^2$, and $v\to w=-1/(v-1/4)$. Using standard properties of conformal maps one sees that $\varphi(z)$ maps $\mathbb{C}\backslash E$ conformally onto $\mathbb{C}\backslash[A,B]$, where

$$A=-\frac{(a_1+1)^2}{a_1}\;,\qquad B=\frac{(a_2-1)^2}{a_2}\;.$$

Clearly $\varphi(\infty)=\infty$. Since $\lim_{z\to\infty}\log(|w|/|z|)=0$, it follows that

$$\gamma_\infty(E) \;=\; \gamma_\infty([A,B]) \;=\; \frac{B-A}{4} \;=\; \frac{(a_1a_2+1)(a_1+a_2)}{4a_1a_2}\;. \tag{2.51}$$

In the general case, put $a_1 = 1+L_1/R$, $a_2 = 1+L_2/R$, and scale (2.51) by R; after simplification, one obtains (2.50). The expression (2.51) appears in ([**33**], p.82).

For the set $E=D(0,1)\cup[-a_1,a_2]$ discussed above, and for $z,\zeta\notin E$, one has

$$G(z,\zeta;E) \;=\; G(\varphi(z),\varphi(\zeta);[A,B])$$

where the Green's function of $[A,B]$ is given by (2.8). This can be used to find $\gamma_\zeta(E)$ for any $\zeta\in\mathbb{C}\backslash E$.

Two Concentric Circles. Fix $r>1$, and let E be the union of the circles $|z|=1/r$ and $|z|=r$. The complement of E has three components. If z and ζ belong to different components, then $G(z,\zeta;E)=0$. If they belong to the outer component, then $G(z,\zeta;E)=G(z,\zeta;D(0,r))$, while if they belong to the inner component then $G(z,\zeta;E)=G(1/z,1/\zeta;D(0,r))$.

$G(z,\zeta;E)$ is also known when z and ζ belong to the annular region between the circles. Courant and Hilbert ([**13**], pp. 386–388) derive a formula for it using the Schwarz Reflection Principle: define q by $q^{1/2}=1/r$, and suppose $1/r<|z|,|\zeta|<r$. Courant and Hilbert show that $G(z,\zeta;E)=-\log(|f_\zeta(z)|)$, where

$$f_\zeta(z)=|z|^{-\log(|\zeta|)/\log(q)}\cdot\frac{q^{1/4}\left(\sqrt{\frac{z}{\zeta}}-\sqrt{\frac{\zeta}{z}}\,\right)\prod_{n=1}^\infty(1-q^{2n}\frac{z}{\zeta})(1-q^{2n}\frac{\zeta}{z})}{\prod_{n=1}^\infty(1-q^{2n-1}\overline{\zeta}z)(1-q^{2n-1}\frac{1}{\overline{\zeta}z})}. \tag{2.52}$$

Recalling the product expansions of the theta functions, they note that the second term is a quotient of two theta functions, leading to the following expression: writing $\tau = 2i\log(r)/\pi$, $z = e^{2\pi i u}$ and $\zeta = e^{2\pi i \alpha}$, then for $1/r < |z|, |\zeta| < r$,

$$G(z,\zeta;E) \ = \ -\frac{\log(|z|)\log(|\zeta|)}{2\log(r)} - \log\left(\left|\frac{\theta(u-\alpha,\tau;\frac{1}{2},\frac{1}{2})}{\theta(u-\overline{\alpha},\tau;0,\frac{1}{2})}\right|\right) \ . \tag{2.53}$$

Here we have corrected a minor error in Courant-Hilbert, who state (2.52) for positive real ζ, and omit the conjugate on ζ in generalizing; this changes their $\theta(u+\alpha,\tau;0,\frac{1}{2})$ to $\theta(u-\overline{\alpha},\tau;0,\frac{1}{2})$. Using (2.53), we obtain

$$\begin{aligned} V_\zeta(E) \ &= \ \lim_{z\to\zeta} G(z,\zeta;E) + \log(|z-\zeta|) \\ &= \ -\frac{(\log(|\zeta|))^2}{2\log(r)} + \log(|\theta(\alpha-\overline{\alpha},\tau;0,\frac{1}{2})|) \\ &\qquad - \log(|\frac{d}{du}\theta(0,\tau;\frac{1}{2},\frac{1}{2})| + \log(|\frac{dz}{du}(\alpha)|) \\ &= \ -\frac{(\log(|\zeta|))^2}{2\log(r)} + \log\left(\left|\frac{2\zeta\cdot\theta(\alpha-\overline{\alpha},\tau;0,\frac{1}{2})}{\theta(0,\tau;0,0)\theta(0,\tau;\frac{1}{2},0)\theta(0,\tau;0,\frac{1}{2})}\right|\right) , \end{aligned} \tag{2.54}$$

where we have used Jacobi's identity (2.32) to simplify $\frac{d}{du}\theta(0,\tau;\frac{1}{2},\frac{1}{2})$. When $\zeta = 1$, this becomes

$$V_1(E) \ = \ \log\left(\frac{2}{|\theta(0,\tau;0,0)\theta(0,\tau;\frac{1}{2},0)|}\right), \quad \gamma_1(E) = \frac{|\theta(0,\tau;0,0)\theta(0,\tau;\frac{1}{2},0)|}{2} \ . \tag{2.55}$$

We next consider some sets arising in Polynomial Dynamics:

Julia Sets. Let $\varphi(x) = a_0 + a_1x + \cdots + a_dx^d \in \mathbb{C}[x]$ be a polynomial of degree $d \geq 2$. By definition, the *filled Julia set* $\mathcal{K}_\varphi$ of $\varphi(x)$ is the set of all $z \in \mathbb{C}$ whose forward orbit $z, \varphi(z), \varphi(\varphi(z)), \ldots$ under φ remains bounded; the *Julia set* is its boundary $\mathcal{J}_\varphi = \partial\mathcal{K}_\varphi$. Let $\varphi^{(n)} = \varphi\circ\varphi\circ\cdots\circ\varphi$ be the n-fold iterate. For each sufficiently large R, we have $D(0,R) \supset \varphi^{-1}(D(0,R)) \supset (\varphi^{(2)})^{-1}(D(0,R)) \supset \cdots \supset \mathcal{K}_\varphi$, and

$$\mathcal{K}_\varphi \ = \ \bigcap_{n=1}^{\infty} (\varphi^{(n)})^{-1}(D(0,R)) \ .$$

As in ([**62**], p.147), for each $z \in \mathbb{C}$ we have

$$G(z,\infty;\mathcal{J}_\varphi) \ = \ G(z,\infty;\mathcal{K}_\varphi) \ = \ \lim_{n\to\infty} \frac{1}{d^n}\log^+(|\varphi^{(n)}(z)|) \tag{2.56}$$

(the "escape velocity" of z), and

$$V_\infty(\mathcal{J}_\varphi) = V_\infty(\mathcal{K}_\varphi) = \frac{\log(|a_d|)}{d-1} \ , \qquad \gamma_\infty(\mathcal{J}_\varphi) = \gamma_\infty(\mathcal{K}_\varphi) = |a_d|^{-1/(d-1)} \ . \tag{2.57}$$

The proofs of (2.56) and (2.57) are simple. It is easy to see that $\varphi^{(n)}(z)$ has degree d^n and leading coefficient $a_0^{d^{n-1}+d^{n-2}+\cdots+d+1}$. By the characterization of Green's functions it follows that $G(z,\infty;(\varphi^{(n)})^{-1}(D(0,R))) = d^{-n}\log^+(|\varphi^{(n)}(z)|)$, and that

$$V_\infty((\varphi^{(n)})^{-1}(D(0,R))) \ = \ \frac{d^n-1}{d^n(d-1)}\log(|a_0|) \ = \ \frac{1-1/d^n}{d-1}\log(|a_0|) \ .$$

The functions $G(z,\infty;(\varphi^{(n)})^{-1}(D(0,R))$ decrease monotonically to $G(z,\infty;\mathcal{K}_\varphi)$, and the convergence is uniform outside any neighborhood of $\mathcal{K}_\varphi$, so (2.56) and (2.57) follow.

The Mandelbrot Set. Each quadratic polynomial $\varphi_c(x) = x^2 + c \in \mathbb{C}[x]$ has 0 as a critical point. The *Mandelbrot set* $\mathcal{M}$ is the set of all $c \in \mathbb{C}$ for which the forward orbit $0, \varphi_c(0), \varphi_c^{(2)}(0), \ldots$ remains bounded; equivalently, $\mathcal{M}$ is the set of all $c \in \mathcal{C}$ for which 0 belongs to the filled Julia set of $\varphi_c(x)$. It is easy to see that $\varphi_c(0) = c$, $\varphi_c^{(2)}(0) = c^2 - c$, and $\varphi_c^{(3)}(0) = c^4 - 2c^3 + c^2 - c$; in general $P_n(c) := \varphi_c^{(n+1)}(0)$ is a monic polynomial of degree 2^n in $\mathbb{Z}[c]$. It can be shown that $D(0,2) \supset P_1^{-1}(D(0,2)) \supset P_2^{-1}(D(0,2)) \supset \cdots \supset \mathcal{M}$, and that

$$\mathcal{M} \ = \ \bigcap_{n=1}^{\infty} P_n^{-1}(D(0,2)) \ ; \tag{2.58}$$

see ([**62**], p.158). By arguments like those for Julia sets, for each $c \in \mathbb{C}$ we have

$$G(c,\infty;\mathcal{M}) \ = \ \lim_{n\to\infty} \frac{1}{2^n} \log^+(|P_n(c)|) \ = \ \lim_{n\to\infty} \frac{1}{2^n} \log^+(\varphi_c^{(n+1)}(0)) \tag{2.59}$$

and

$$V_\infty(\mathcal{M}) = 0 \ , \qquad \gamma_\infty(\mathcal{M}) = 1 \ . \tag{2.60}$$

2. Local Capacities and Green's Functions of Nonarchimedean Sets

In this section, K_v will be a nonarchimedean local field. Identify $\mathbb{P}^1(\mathbb{C}_v)$ with $\mathbb{C}_v \cup \{\infty\}$. There are two methods of determining the Green's function for sets $E_v \subset \mathbb{P}^1(\mathbb{C}_v)$: using the pullback formula for Green's functions, for noncompact sets; or by guessing the equilibrium distribution based on symmetry, for compact sets. We are aided by the fact that the capacity is monotonic under containment of sets.

The pullback formula for Green's functions is as follows. Let $\mathcal{C}_1, \mathcal{C}_2/\mathbb{C}_v$ be smooth, complete curves, and let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a nonconstant rational map. Suppose $E_v \subset \mathcal{C}_2(\mathbb{C}_v)$ is an algebraically capacitable set of positive capacity. Fix $\zeta \in \mathcal{C}_2(\mathbb{C}_v)\backslash E_v$ and write $f^*((\zeta)) = \sum_{j=1}^m m_k(\xi_j)$ as a divisor. Then for each $z \in \mathcal{C}_1(\mathbb{C}_v)$,

$$G(f(z),\zeta;E_v) \ = \ \sum_{j=1}^{m} m_k G(z,\xi_j;f^{-1}(E_v) \ . \tag{2.61}$$

This holds for both nonarchimedean and archimedean sets (see [**51**], Theorems 3.2.9, 4.4.19).

The Closed Disc. If $E_v = D(a,R) = \{z \in \mathbb{C}_v : |z-a|_v \le R\}$ then

$$G(z,\zeta;E_v) \ = \ \begin{cases} \log_v^+(|z-a|_v/R) & \text{if } \zeta = \infty \ , \\ \log_v^+\left(\frac{|\zeta-a|_v}{R} \cdot \left|\frac{z-a}{z-\zeta}\right|_v\right) & \text{if } \zeta \in \mathbb{C}_v\backslash D(a,R) \ . \end{cases} \tag{2.62}$$

The first formula is essentially the definition of the Green's function as given by Cantor ([**16**]); the second follows from the first, by applying the pullback formula (2.61) to the map $f(z) = (z-a)/(z-\zeta)$ which takes $D(a,R)$ to $D(0,R/|\zeta-a|_v)$ and takes ζ to ∞.

Taking $g_\infty(z) = 1/z$, and $g_\zeta(z) = z - \zeta$ if $\zeta \in \mathbb{C}_v \backslash D(a,R)$, we have

$$\begin{aligned} &(2.63) \quad V_\infty(E_v) = -\log_v(R) , \qquad \gamma_\infty(E_v) = q_v^{-V_\infty(E_v)} = R , \\ &(2.64) \quad V_\zeta(E_v) = -\log_v(R/|\zeta - a|_v^2) , \quad \gamma_\zeta(E_v) = R/|\zeta - a|_v^2 . \end{aligned}$$

The Open Disc. If $E_v = D(a,R)^- = \{z \in \mathbb{C}_v : |z|_v < R\}$, formulas (2.62), (2.63), and (2.64) for the Green's function, Robin constant and capacity remain valid.

If $\zeta = \infty$, this is because for $R_1 < R$ we have $D(a,R_1) \subset D(a,R)^- \subset D(a,R)$, and hence

$$\begin{aligned} G(z,\infty;D(a,R_1)) &\le G(z,\infty;D(a,R)^-) \le G(z,\infty;D(a,R)) , \\ \gamma_\infty(D(a,R_1)) &\le \gamma_\infty(D(a,R)^-) \le \gamma_\infty(D(a,R)) . \end{aligned}$$

In the limit as $R_1 \to R$, it follows from (2.62) and (2.63) that $G(z,\infty;D(a,R)^-) = G(z,\infty;D(a,R))$ and $\gamma_\infty(D(a,R)^-) = \gamma_\infty(D(a,R))$.

If $\zeta \in \mathbb{C}_v \backslash D(a,R)^-$, we can reduce to the case where $\zeta = \infty$ by applying the map $f(z) = (z-a)/(z-\zeta)$ and using the pullback formula (2.61). Thus (2.62), (2.63), and (2.64) hold when $E_v = D(a,R)^-$, for any $\zeta \notin E_v$.

The Punctured Disc. Suppose $E_v = D(a,R)\backslash(\bigcup_{i=1}^m D(a_i,R_i)^-)$, where $a_1,\ldots,a_m \in D(0,R)$ and $R_i \le R$ for each i. For each $\zeta \notin D(a,R)$, the Green's function and capacity are still given by (2.62), (2.63), and (2.64). Indeed, for any fixed $a_0 \in E_v$, we have $D(a_0,R)^- \subset E_v \subset D(a_0,R)$, so the result follows from the previous case.

If ζ belongs to one of the "holes" $D(a_i,R_i)^-$, then $D(a_i,R_i)^- = D(\zeta,R_i)^-$ and by applying $f(z) = 1/(z-\zeta)$ and using the pullback formula (2.61), we find that

$$\begin{aligned} &(2.65) \quad G(z,\zeta;E_v) = G(\frac{1}{z-\zeta},\infty;D(0,\frac{1}{R_i})) = \log_v^+\left(\frac{R_i}{|z-\zeta|_v}\right) , \\ &(2.66) \quad V_\zeta(E_v) = \log_v(R_i) , \quad \gamma_\zeta(E_v) = 1/R_i . \end{aligned}$$

The Ring of Integers $\mathcal{O}_w$. We next determine the Green's function of the ring of integers of a finite extension F_w/K_v in $\mathbb{C}_v$.

PROPOSITION 2.1. *Let F_w/K_v be a finite extension in $\mathbb{C}_v$, with ramification index $e = e_{w/v}$ and residue degree $f = f_{w/v}$. Take $E_v = \mathcal{O}_w$, the ring of integers of F_w. Given $z \in \mathbb{C}_v$, put*

$$r = \|z,\mathcal{O}_w\|_v = \min_{x \in \mathcal{O}_w} |z - x|_v .$$

Let $M = \lfloor -e\log_v(r) \rfloor$ be the integer part of $-e\log_v(r)$, and let $\langle -e\log_v(r) \rangle$ be its fractional part. Then

(2.67)

$$G(z,\infty;\mathcal{O}_w) = \begin{cases} 0 & \text{if } z \in \mathcal{O}_w, \\ \frac{1}{e}\frac{1}{q_v^f-1}\frac{1}{q_v^{fM}} - \langle -e\log_v(r)\rangle \frac{1}{e}\frac{1}{q_v^{f(M+1)}} & \text{if } z \notin \mathcal{O}_w,\ |z|_v \le 1, \\ \frac{1}{e}\frac{1}{q_v^f-1} + \log_v(|z|_v) & \text{if } |z|_v > 1 \ , \end{cases}$$

and if capacities are computed relative to the uniformizer $g_\infty(z) = 1/z$ then

$$(2.68) \qquad V_\infty(\mathcal{O}_w) = \frac{1}{e}\frac{1}{q_v^f - 1} , \qquad \gamma_\infty(\mathcal{O}_w) = q_v^{-1/(e(q_v^f-1))} .$$

For any coset $a + b\mathcal{O}_w$ where $a \in \mathbb{C}_v$, $b \in \mathbb{C}_v^\times$,

$$G(z, \infty; a + b\mathcal{O}_w) \;=\; G((z-a)/b, \infty; \mathcal{O}_w) \;,$$

so that $V_\infty(a + b\mathcal{O}_w) = -\log_v(|b|_v) + V_\infty(\mathcal{O}_w)$ and $\gamma_\infty(a + b\mathcal{O}_w) = |b|_v \cdot \gamma_\infty(\mathcal{O}_w)$. In particular, if π_w is a generator for the maximal ideal of $\mathcal{O}_w$, then

(2.69)

$$V_\infty(a + \pi_w^m \mathcal{O}_w) \;=\; \frac{1}{e}\frac{1}{q_v^f - 1} + \frac{m}{e} \;, \qquad \gamma_\infty(a + \pi_w^m \mathcal{O}_w) \;=\; q_v^{-m/e - 1/(e(q_v^f - 1))} \;.$$

Proof. See ([**51**], Example 5.2.17). The equilibrium distribution of $\mathcal{O}_w$ is the additive Haar measure μ for F_w, normalized so that $\mu(\mathcal{O}_w) = 1$ (see [**51**], p.212). It follows that if we write $q_w = q_v^f$, and put $M = \lfloor -e \log_v(r) \rfloor$ if $|z|_v \le 1$, $M = -1$ if $|z|_v > 1$, then the potential function is given by

$$\begin{aligned} u_{\mathcal{O}_w}(z, \infty) &= \int_{\mathcal{O}_w} -\log_v(|z - x|_v) d\mu(x) \\ &= \sum_{k=0}^{M} \frac{k}{e} \cdot \frac{q_w - 1}{q_w^{k+1}} + \sum_{k=M+1}^{\infty} (-\log_v(r)) \cdot \frac{q_w - 1}{q_w^{k+1}} \\ &= \frac{1}{e}\frac{1}{q_w - 1} \cdot [1 - \frac{M+1}{q_w^M} + \frac{1}{q_w^{M+1}}] - \log_v(r) \cdot \frac{1}{q_w^{M+1}} . \end{aligned}$$

The potential function is invariant under translation by $\mathcal{O}_w$, so

$$V_\infty(\mathcal{O}_w) \;=\; u_{\mathcal{O}_w}(0, \infty) \;=\; \frac{1}{e(q_w - 1)} \;.$$

The expression (2.67) is obtained by simplifying

$$G(z, \infty; \mathcal{O}_w) \;=\; \frac{1}{e(q_w - 1)} - u_{\mathcal{O}_w}(z, \infty) \;.$$

(Compare [**51**], Example 4.1.24, p.212). The assertions about cosets follow. □

We now recall a general procedure for computing capacities of finite disjoint unions of nonarchimedean sets (for more details, see Theorem A.13 and Corollary A.14 of Appendix A, or see [**51**], p.354).

Let $\mathcal{C}_v/K_v$ be a curve. Suppose $E_v = \bigcup_{i=1}^N E_{v,i} \subset \mathcal{C}_v(\mathbb{C}_v)$ is a finite disjoint union of compact sets $E_{v,i}$ with positive inner capacity, and that $\zeta \in \mathcal{C}_v(\mathbb{C}_v)$ is such that the canonical distance $[z, w]_\zeta$ (see §3.5) is constant on $E_{v,i} \times E_{v,j}$, for each $i \ne j$. For each i, let $\mu_{\zeta,i}$ be the equilibrium distribution of $E_{v,i}$ (see §3.8). Then each i, the potential function $u_{E_{v,i}}(z, \zeta) = \int_{E_{v,i}} -\log([z, w]_\zeta d\mu_{\zeta,i}(w)$ and Green's function $G(z, \zeta; E_{v,i}) = V_\zeta(E_v) - u_{E_{v,i}}(z, \zeta)$ are constant for $z \in E_{v,j}$, for each $j \ne i$.

We now show that we can compute $G(z, \zeta; E_v)$ and $V_\zeta(E_v)$ in terms of the potential functions $u_{E_{v,i}}(z, \zeta)$. Let capacities be defined in terms of the uniformizer $g_\zeta(z)$. For each $E_{v,i}$, put

$$W_{ii} \;=\; V_\zeta(E_{v,i})$$

and for each $i \neq j$ let W_{ij} be the value that $u_{E_{v,i}}(z,\zeta)$ assumes on $E_{v,j}$. Consider the system of $N+1$ linear equations in the variables V, $s_1,\ldots,s_N$:

$$\begin{aligned} 1 &= 0\cdot V + 1\cdot s_1 + 1\cdot s_2 + \cdots + 1\cdot s_N \ , \\ 0 &= V - W_{i1}s_1 - W_{i2}s_2 - \ldots - W_{iN}s_N, \\ & \qquad \text{for } i = 1,\ldots,N. \end{aligned} \tag{2.70}$$

We claim that this system of equations has a unique solution, given by

$$V_\zeta(E_v) = V \ , \tag{2.71}$$

$$G(z;\zeta;E_v) = \sum s_i G(z,\zeta;E_{v,i}) + \sum s_i W_{ii} - V, \tag{2.72}$$

and that in this solution we have $s_1,\ldots,s_N > 0$.

To see this, let μ be the equilibrium distribution of E_v with respect to ζ, and put $\widehat{s}_i = \mu(E_{v,i})$ for each i. Then $\widehat{s}_i > 0$: otherwise, μ would be supported on $E_v \backslash E_{v,i}$ and then $u_\zeta(z,E_v) = u_\zeta(z,E_v\backslash E_{v,i})$. By ([**51**], Corollary 4.1.12) we would have $u_\zeta(z,E_v\backslash E_{v,i}) < V_\zeta(E_v\backslash E_{v,i}) = V_\zeta(E_v)$ for all $z \in E_{v,i}$, contradicting that $u_\zeta(z;E_v)$ takes the value $V_\zeta(E_v)$ for all $z \in E_v$ except possibly on a set of inner capacity 0 ([**51**], Theorem 4.1.11). Consider the probability measure $\mu_i = \widehat{s}_i^{-1}\mu|_{E_{v,i}}$, and put $u_i(z,\zeta) = \int_{E_{v,i}} -\log_v([z,w]_\zeta)\,d\mu_i(w)$; by our hypothesis on the canonical distance, $u_i(z,\zeta)$ is constant on E_j, for each $j \neq i$. Then

$$\begin{aligned} u_{E_v}(z,\zeta) &= \int_{E_v} -\log_v([z,w]_\zeta)\,d\mu(z) \\ &= \sum_{i=1}^r \int_{E_{v,i}} -\log_v([z,w]_\zeta)\,d\mu(z) \ = \ \sum_{i=1}^r \widehat{s}_i u_i(z,\zeta) \ . \end{aligned}$$

For each i, since $u_{E_v}(z,\zeta)$ and the $u_j(z,\zeta)$ for $j \neq i$ are constant on $E_{v,i}$ except possibly on a set of inner capacity 0 , it follows that $u_i(z,\zeta)$ is constant on $E_{v,i}$ except possibly on a set of inner capacity 0. Since this property characterizes the equilibrium potential, it follows that μ_i must be the equilibrium distribution of $E_{v,i}$ with respect to ζ. Thus there are unique weights $\widehat{s}_1,\ldots,\widehat{s}_N > 0$ with $\sum_{i=1}^N \widehat{s}_i = 1$, for which

$$u_{E_v}(z,\zeta) \ = \ \sum_{i=1}^N \widehat{s}_i u_{E_{v,i}}(z,\zeta) \ . \tag{2.73}$$

Evaluating (2.73) at a generic point of each $E_{v,i}$, we see that $V = V_\zeta(E_v)$ and $\widehat{s}_1,\ldots,\widehat{s}_N$ are a solution to the system (2.70) with each $\widehat{s}_i > 0$. Conversely, any solution to (2.70) gives a system of weights for which $\mu = \sum s_i\mu_i$. The uniqueness of the equilibrium distribution ([**51**], Theorem 4.1.22) shows that $s_1,\ldots,s_N$, and in turn V, are unique. Thus $s_i = \widehat{s}_i$ for each i, and $V_\zeta(E_v) = V$. Since $G(z,\zeta;E_v) = V_\zeta(E_v) - u_{E_v}(z,\zeta)$, formula (2.71) follows.

The Group of Units $\mathcal{O}_w^\times$. Using the machinery above, we will now determine the Green's function and the capacity of the set $\mathcal{O}_w^\times$, relative to the point ∞.

PROPOSITION 2.2. *Let F_w/K_v be a finite extension, with ramification index $e = e_{w/v}$ and residue degree $f = f_{w/v}$. Let $\mathcal{O}_w^\times$ be the group of units of $\mathcal{O}_w$. For $z \in \mathbb{C}_v$, put $r_0 = \min_{x\in\mathcal{O}_w^\times} |z-x|_v$, $M_0 = \lfloor -e\log_v(r_0)\rfloor$; note that $r_0 = |z|_v$ if*

$|z|_v > 1$. *Then*

$$(2.74)\quad G(z,\infty;\mathcal{O}_w^\times) \;=\; \begin{cases} 0 & \text{if } z \in \mathcal{O}_w^\times, \\ \frac{q_v^f}{e(q_v^f-1)^2}\cdot\frac{1}{q_v^{fM_0}} - \langle -e\log_v(r_0)\rangle \frac{1}{e}\frac{1}{q_v^{fM_0}} & \text{if } 0 < r_0 \le 1, \\ \frac{q_v^f}{e(q_v^f-1)^2} + \log_v(|z|_v) & \text{if } |z|_v > 1. \end{cases}$$

If capacities are computed relative to the uniformizer $g_\infty(z) = 1/z$ *then*

$$(2.75)\qquad V_\infty(\mathcal{O}_w^\times) \;=\; \frac{1}{e}\frac{1}{q^f-1}\big(1+\frac{1}{q^f-1}\big) \;=\; \frac{q_v^f}{e(q_v^f-1)^2}\ .$$

Proof. Put $N = q_v^f - 1$ and let $a_1, \ldots, a_N$ be coset representatives for the nonzero classes in $\mathcal{O}_w/\pi_w\mathcal{O}_w$. Then

$$\mathcal{O}_w^\times \;=\; \bigcup_{i=1}^N (a_i + \pi_w\mathcal{O}_w)$$

is a decomposition of the type needed to compute $G(z,\infty;\mathcal{O}_w^\times)$ in terms of the $G(z,\infty;a_i+\pi\mathcal{O}_w)$. Applying the last part of Proposition 2.1, solving the system (2.70) and simplifying (2.71), (2.72) gives the result. Here $W_{ij} = 0$ if $i \ne j$ and each $W_{ii} = q_v^f/(e(q_v^f-1))$, giving $V = q_v^f/(e(q_v^f-1)^2)$ and $s_i = 1/(q_v^f-1)$ for each i. □

Corollary 2.3. *Let K_v be nonarchimedean, and let π_v be a uniformizer for the maximal ideal of $\mathcal{O}_v$. Suppose $a_1, \ldots, a_N$ are representives for distinct cosets of $\mathcal{O}_v/\pi_v\mathcal{O}_v$, and put $E_v = \cup_{i=1}^N(a_i+\pi_v\mathcal{O}_v)$. Then*

$$V_\infty(E_v) \;=\; \frac{q_v}{N(q_v-1)}.$$

Proof. The proof is similar to Proposition 2.2, with $s_i = 1/N$ for $i = 1, \ldots, N$. □

The Punctured $\mathcal{O}_v$-disc. Next we determine the capacity of a union of cosets of $\mathcal{O}_v^\times$, relative to the point $\zeta = \infty$. This computation has important consequences: it is used in the proof Proposition 3.30, which plays a key role in the reduction of Theorem 0.4 to Theorem 4.2.

Proposition 2.4. *Put $E_{v,m} = \bigcup_{k=0}^m \pi_v^k\mathcal{O}_v^\times$, and take $\zeta = \infty$. Then*

$$(2.76)\qquad V_\infty(E_{v,m}) \;=\; \frac{1}{q_v-1} + \frac{1}{(q_v-1)^2(1+q_v^2+q_v^4+\cdots+q_v^{2m})},$$

$$(2.77)\qquad G(0,\infty;E_{v,m}) \;=\; \frac{q_v^{m+1}}{(q_v-1)^2(1+q_v^2+q_v^4+\cdots+q_v^{2m})}\ ,$$

and for each $k = 0, \ldots, m$ the mass of $\pi_v^k\mathcal{O}_v^\times$ under the equilibrium distribution μ_m of $E_{v,m}$ with respect to ∞ is

$$(2.78)\qquad \mu_m(\pi_v^k\mathcal{O}_v^\times) \;=\; \frac{q_v^k + q_v^{2m+1-k}}{1+q_v+q_v^2+q_v^3+\cdots+q_v^{2m+1}}\ .$$

Proof. Write $V_m = V_\infty(E_{v,m})$. By Proposition 2.2, we have

$$(2.79)\qquad V_0 \;=\; \frac{q_v}{(q_v-1)^2} \;=\; \frac{1}{q_v-1} + \frac{1}{(q_v-1)^2}\ .$$

We will prove (2.76) by induction on m. Note that $E_{v,m} = \pi_v E_{v,m-1} \cup \mathcal{O}_v^\times$. For $z \in \pi_v E_{v,m-1}$ and $w \in \mathcal{O}_v^\times$, $-\log_v([z,w]_\infty) = -\log_v(|z-w|_v) = 0$, independent of z, w. Hence $u_{\pi_v E_{v,m-1}}(z,\infty) = 0$ if $z \in \mathcal{O}_w^\times$, and $u_{\mathcal{O}_w^\times}(z,\infty) = 0$ if $z \in \pi_v E_{v,m-1}$. By the scaling property of the capacity, $V_\infty(\pi_v E_{v,m}) = V_\infty(E_{v,m-1})+1 = V_{m-1}+1$. It follows from (2.70) that there are numbers $s_{1,m}, s_{2,m} > 0$ for which

$$\text{(2.80)} \qquad \left\{ \begin{array}{ccc} 1 & = & s_{1,m} + s_{2,m} , \\ V_m & = & (V_{m-1}+1)\cdot s_{1,m} + 0 \cdot s_{2,m} , \\ V_m & = & 0 \cdot s_{1,m} + V_0 \cdot s_{2,m} \end{array} \right. .$$

Solving (2.80) for V_m and inserting (2.79) leads to the recursion

$$V_m \ = \ \frac{q_v(1+V_{m-1})}{q_v + (q_v-1)^2 V_{m-1}}$$

whose solution is easily seen to be (2.76).

Once the V_m are known, one sees that

$$\text{(2.81)} \qquad s_{1,m} \ = \ \frac{q_v(1+q_v+\cdots+q_v^{2m-1})}{1+q_v+\cdots q_v^{2m+1}} \ , \quad s_{2,m} \ = \ \frac{1+q_v^{2m+1}}{1+q_v+\cdots q_v^{2m+1}} \ .$$

To obtain (2.77), note that since $u_{\mathcal{O}_w^\times}(0,\infty) = 0$, we have $u_{E_{v,m}}(0,\infty) = s_{1,m}\cdot(1+u_{E_{v,m-1}}(0,\infty))$. Thus, recursively,

$$\text{(2.82)} \qquad u_{E_{v,m}}(0,\infty) \ = \ s_{1,m} + s_{1,m}s_{1,m-1} + \cdots + s_{1,m}s_{1,m-1}\cdots s_{1,1} \ .$$

One gets (2.77) by inserting (2.76), (2.81) and (2.82) in the formula

$$G(0,\infty;E_{v,m}) \ = \ V_m - u_{E_{v,m}}(0,\infty)$$

and simplifying. Finally, the weights of the cosets $\pi_v^k \mathcal{O}_w^\times$ under the equilibrium distribution μ_m can be found by using

$$\begin{aligned} \mu_m(\pi_v^k \mathcal{O}_w^\times) \ &= \ s_{1,m}\mu_{m-1}(\pi_v^{k-1}\mathcal{O}_w^\times) \ = \ \cdots \\ &= \ s_{1,m}s_{1,m-1}\cdots s_{1,m-k+1}\cdot \mu_{m-k}(\mathcal{O}_w^\times) \end{aligned}$$

where $\mu_{m-k}(\mathcal{O}_w^\times) = s_{2,m-k}$. Using (2.81), and simplifying, yields (2.78). Once the weights $\mu_m(\pi_v^k \mathcal{O}_v^\times)$ are known, $G(z,\infty;E_{v,m})$ can be found for any z. □

The Union of Two Rings of Integers. Let F_w be the unique unramified quadratic extension of K_v, and let F_u be a totally ramified quadratic extension. We will compute the capacity of the set $E_v = \mathcal{O}_w \cup \mathcal{O}_u$ with respect to ∞. This is the only nonarchimedean set known to the author whose Robin constant can be computed explicitly, and is not rational. The importance is not the result itself, but the method, which uses a partial self-similarity of E_v with itself, and can be applied to nondisjoint unions of much more general sets.

PROPOSITION 2.5. *Fix a nonarchimedean local field K_v. Let F_w/K_v be the unique unramified quadratic extension, and let F_u/K_v be a totally ramified quadratic extension. Put $E_v = \mathcal{O}_w \cup \mathcal{O}_u$ and let*

$$\begin{aligned} A \ &= \ 2q_v^4 + 2q_v^3 - 4q_v^2 + 2q_v - 2 \ , \\ B \ &= \ q_v^4 + 2q_v^3 - 2q_v^2 + 2q_v - 1 \ , \\ D \ &= \ q_v^8 + 4q_v^7 + 8q_v^6 + 12q_v^5 + 18q_v^4 + 12q_v^3 + 8q_v^2 + 4q_v + 1 \ . \end{aligned}$$

Then

$$\text{(2.83)} \qquad V_\infty(E_v) \ = \ \frac{-B+\sqrt{D}}{2A} \ .$$

Below are some numerical examples when $K_v = \mathbb{Q}_p$, for small primes p. We give the values of $V_\infty(\mathcal{O}_w)$ and $V_\infty(\mathcal{O}_u)$ for comparison.

	$q_v = 2$	$q_v = 3$	$q_v = 5$	$q_v = 7$	$q_v = 11$
$V_\infty(E_v)$	.2750820518	.1060035774	.0366954968	.0188065868	.0077456591
$V_\infty(\mathcal{O}_w)$	.3333333333	.1250000000	.0416666666	.0208333333	.0083333333
$V_\infty(\mathcal{O}_u)$	.5000000000	.2500000000	.1250000000	.0833333333	.0500000000

It can be shown that as $q_v \to \infty$, then $V_\infty(E_v) = 1/q_v^2 - 1/q_v^3 + O(1/q_v^4)$.

Proof of Proposition 2.5. Let $\pi = \pi_v$ be a generator for the maximal ideal of $\mathcal{O}_v$, and write $q = q_v$. Then $\#(\mathcal{O}_v/\pi\mathcal{O}_v) = q$; let $\gamma_1, \ldots, \gamma_q$ be coset representatives for $\mathcal{O}_v/\pi\mathcal{O}_v$. Put $E_{0,i} = \gamma_i + \pi E_v = \gamma_i + \pi(\mathcal{O}_w \cup \mathcal{O}_u)$, for $i = 1, \ldots, q$. There are $q^2 - q$ cosets of $\mathcal{O}_w/\pi\mathcal{O}_w$ which do not contain elements of $\mathcal{O}_v$; let these be $E_{1,j} = \alpha_j + \pi\mathcal{O}_w$, for $j = 1, \ldots, q^2 - q$. Similarly, there are $q^2 - q$ cosets of $\mathcal{O}_u/\pi\mathcal{O}_u$ which do not contain elements of $\mathcal{O}_v$; let these be $E_{2,k} = \beta_k + \pi\mathcal{O}_u$, for $k = 1, \ldots, q^2 - q$. Then the sets $E_{0,i}$, $E_{1,j}$ and $E_{2,k}$ are pairwise disjoint (in fact, they are contained in pairwise disjoint cosets $a + \pi\widehat{\mathcal{O}}_v$, where $\widehat{\mathcal{O}}_v = D(0,1)$ is the ring of integers of $\mathbb{C}_v$), and we can write

$$E_v \ = \ \Big(\bigcup_{i=1}^{q} E_{1,i}\Big) \cup \Big(\bigcup_{j=1}^{q^2-q} E_{2,j}\Big) \cup \Big(\bigcup_{k=1}^{q^2-q} E_{3,k}\Big) \ .$$

Let μ be the equilibrium distribution of E_v with respect to ∞, and put $w_{0,i} = \mu(E_{0,i})$, $w_{1,j} = \mu(E_{1,j})$, $w_{2,k} = \mu(E_{2,k})$ for all i, j, k. Then

$$u_\infty(z, E_v) \ = \ \sum_{i=1}^{q} w_{1,i} u_\infty(z, E_{0,i}) + \sum_{j=1}^{q^2-q} w_{0,j} u_\infty(z, E_{1,j}) + \sum_{k=1}^{q^2-q} w_{2,k} u_\infty(z, E_{2,k}) \ . \tag{2.84}$$

Let $V = V_\infty(E_v)$ be the (as yet unknown) Robin constant of $E_v = \mathcal{O}_w \cup \mathcal{O}_u$, and let $V_1 = V_\infty(\mathcal{O}_w)$, $V_2 = V_\infty(\mathcal{O}_u)$. Since $E_v \subset D(0,1)$, we must have $V \geq 0$. By Proposition 2.1,

$$V_1 = \frac{1}{q^2 - 1} \ , \qquad V_2 = \frac{1}{2(q-1)} \ . \tag{2.85}$$

In general, for any compact set $\widetilde{E} \subset \mathbb{C}_v$ of positive capacity, we have $V_\infty(a + \pi\widetilde{E}) = V_\infty(\widetilde{E}) + 1$ for each $a \in \mathbb{C}_v$. If $\widetilde{E} \subset D(a,r)$, then $u_\infty(z, \widetilde{E}) = -\log_v(|z - a|_v)$ for all $z \notin D(a,r)$. It follows that for each $E_{0,i}$, one has $u_\infty(z, E_{0,i}) = V + 1$ on $E_{0,i}$. On the $q - 1$ cosets $E_{2,k}$ contained in $\gamma_i + \sqrt{\pi}\widehat{\mathcal{O}}_v$, one has $u_\infty(z, E_{0,i}) = 1/2$. On the other $q^2 - 2q + 1$ cosets $E_{2,k}$ and the other $q - 1$ cosets $E_{0,i'}$, as well as all the cosets $E_{1,j}$, one has $u_\infty(z, E_{0,i}) = 0$. For each $E_{1,j}$, one has $u_\infty(z, E_{1,j}) = V_1 + 1$ on $E_{1,j}$, and $u_\infty(z, E_{1,j}) = 0$ on all the $E_{0,i}$, all the $E_{2,j}$ and all the $E_{1,j'}$ distinct from j. For each $E_{2,k}$, one has $u_\infty(z, E_{2,k}) = V_2 + 1$ on $E_{2,k}$. There are $q - 2$ other cosets $E_{2,k'}$ and one coset $E_{1,j}$ contained in $\beta_k + \sqrt{\pi}\widehat{\mathcal{O}}_v$. On those cosets we have $u_\infty(z, E_{2,k}) = 1/2$. On the remaining $q^2 - 2q + 1$ cosets $E_{2,k'}$ and on all the cosets $E_{1,j}$, one has $u_\infty(z, E_{2,k}) = 0$.

Evaluating $u_\infty(z, E_v)$ on each of the sets $E_{r,s}$ in turn yields a system of $2q^2 - q$ equations satisfied by V and the $w_{r,s}$. Since μ and $V = V_\infty(E_v)$ are unique, these equations uniquely determine the $w_{r,s}$. Hence for any permutation σ of the sets $E_{r,s}$

which takes sets of type $r = 0, 1, 2$ to sets of the same type, and which preserves distances between corresponding pairs of sets, we must have $w_{r,\sigma(s)} = w_{r,s}$ for all r, s. It is easy to see that there are enough permutations satisfying these conditions to assure that there are w_0, w_1, w_2 such that for all i, j, k

$$w_{0,i} = w_0 \ , \qquad w_{1,j} = w_1 \ , \qquad w_{2,k} = w_2 \ .$$

We can now determine V. From $\mu(E_v) = 1$, we obtain the mass equation

$$1 \ = \ (q) \cdot w_0 + (q^2 - q) \cdot w_1 + (q^2 - q) \cdot w_2 \ .$$

Evaluating $u_\infty(z, E_v)$ on the sets $E_{0,i}$, $E_{1,j}$ and $E_{2,k}$ gives the equations

$$\begin{aligned} V &= w_0 \cdot (V+1) + w_2 \cdot (q-1) \cdot (1/2) \ , \\ V &= w_1 \cdot (V_1 + 1) \ , \\ V &= w_0 \cdot (1/2) + w_2 \cdot ((V_2+1) + (q-2) \cdot (1/2)) \ . \end{aligned}$$

Treating this as a linear system in w_0, w_1, w_2, solving it in terms of V, V_1, V_2, and inserting the resulting values in the mass equation leads to

$$1 \ = \ (q) \frac{V(\frac{1}{2} + V_2)}{(1+V)(V_2 + \frac{q}{2}) - \frac{q-1}{4}} + (q^2 - q) \frac{V}{1 + V_1} + (q^2 - q) \frac{V^2 + \frac{1}{2} V}{(1+V)(V_2 + \frac{q}{2}) - \frac{q-1}{4}} \ .$$

Clearing denominators and using the values for V_1, V_2 from (2.85) yields a quadratic in V. Its unique nonnegative root (simplified using Maple) is the one in (2.83). □

3. Global Examples on $\mathbb{P}^1$

As will be seen, capacity theory gives a "Calculus" for answering certain types of questions about algebraic integers and units. Note that $\alpha \in \widetilde{\mathbb{Q}}$ is an algebraic integer if and only if its conjugates all satisfy $|\sigma(\alpha)|_v \le 1$ for all nonarchimedean v, and it is a unit if and only if $|\sigma(\alpha)|_v = 1$ for all nonarchimedean v.

Algebraic Integers. The following is a trivial application of the Fekete-Szegö theorem with local rationality, but appears hard to prove without it.

Example 2.6. Let $\mathcal{M}$ be the Mandelbrot set. Then

(A) There are infinitely many algebraic integers with all their conjugates in $\mathcal{M}$.

(B) For each $B > 0$ there are only finitely many algebraic integers α whose conjugates all belong to $\mathcal{M}$, and some prime $p \le B$ splits completely in $\mathbb{Q}(\alpha)$. Indeed, there is a neighborhood $U = U(B)$ of $\mathcal{M}$ with this property.

(C) On the other hand, for each neighborhood U of $\mathcal{M}$ in $\mathbb{C}$, there is a $C = C(U)$ such that for each prime $p > C$, there are infinitely many algebraic integers α such that all the conjugates of α belong to U, and p splits completely in $\mathbb{Q}(\alpha)$.

Proof. Take $K = \mathbb{Q}$, $\mathcal{C} = \mathbb{P}^1$, and $\mathfrak{X} = \{\infty\}$.

Part (A) is well known. Indeed, put $\varphi_c(z) = z^2 + c$ and for each integer $n \ge 1$ put $P_n(c) = \varphi_c^{(n+1)}(0)$, as in the discussion preceding (2.58). Then $P_n(c)$ is a monic polynomial in $\mathbb{Z}[c]$ of degree 2^n. If α is a root of $P_n(c) = 0$, then $z = 0$ is periodic for $\varphi_\alpha(z)$ (with period dividing $n+1$) since $\varphi_\alpha^{(n+1)}(0) = 0$. The same is true for all the $\mathrm{Gal}(\widetilde{\mathbb{Q}}/\mathbb{Q})$-conjugates of α, so α is an algebraic integer whose conjugates all belong to $\mathcal{M}$.

There are many ways to see that as a collection, the $P_n(c)$ have infinitely many distinct roots. For example, note that $c = 0$ is the only number such that 0 is periodic for $\varphi_c(z)$ with period 1. Taking $n = p-1$ where p is prime, we obtain $2^n - 1$ values of c such that 0 is periodic for $\varphi_c(z)$ with exact period p; thus there are infinitely many algebraic integers whose conjugates all belong to $\mathcal{M}$.

For part (B), fix a prime p, let $E_\infty = \mathcal{M}$, $E_p = \mathbb{Z}_p$, and let $E_q = D(0,1) \subset \mathbb{C}_q$ for each prime $q \neq p$. Put $\mathbb{E} = \prod_v E_v$. Then $\mathbb{E}$ is algebraically capacitable, and

$$\gamma(\mathbb{E}, \mathfrak{X}) \;=\; \gamma_\infty(\mathcal{M}) \cdot \gamma_\infty(E_p) \;=\; p^{-1/(p-1)} \;<\; 1 \;.$$

By Theorem 1.5 there is an adelic neigbhorhood $\mathbb{U} = \mathbb{U}_p = \prod_v U_{p,v}$ of $\mathbb{E}$ such that there are only finitely many $\alpha \in \widetilde{\mathbb{Q}}$ which have all their conjugates in $\mathbb{U}_p$. Each algebraic integer α such that p splits completely in $\mathbb{Q}(\alpha)$ is such a number.

Given $B > 0$, put $U = U(B) = \cap_{p \le B} U_{p,\infty} \subset \mathbb{C}$. Then $U(B)$ has the desired properties.

For part (C), let $U \subset \mathbb{C}$ be any neighborhood of $\mathcal{M}$. By enlarging $\mathcal{M}$ within U (for example by choosing a point $a \in (U \cap \mathbb{R}) \backslash \mathcal{M}$ and adjoining a suitably small disc $D(a,r)$) we can obtain a set $\mathcal{M}_U \subset U$ which has $\gamma_\infty(\mathcal{M}_U) > 1$ and is stable under complex conjugation.

Fix a prime p, and take $E_\infty = \mathcal{M}_U$, $E_p = \mathbb{Z}_p$, and $E_q = D(0,1) \subset \mathbb{C}_p$ for each prime $q \neq p$. Put $\mathbb{E} = \prod_v E_v$. By (2.63) and (2.68), the capacity of $\mathbb{E}$ with respect to $\mathfrak{X}$ is

$$\gamma(\mathbb{E}, \mathfrak{X}) \;=\; \gamma_\infty(\mathcal{M}_U) \cdot p^{-1/(p-1)} \;.$$

It follows that if p is sufficiently large, then $\gamma(\mathbb{E}, \mathfrak{X}) > 1$. Put $U_\infty = U$, $U_p = \mathbb{Z}_p$, and $U_q = D(0,1)$ for each prime $q \neq p$. By Theorem 1.2, there are infinitely many $\alpha \in \widetilde{\mathbb{Q}}$ whose conjugates belong to U_v for each v. Each such α is an algebraic integer whose archimedean conjugates belong to U and whose conjugates in $\mathbb{C}_p$ belong to $\mathbb{Z}_p$, so p splits completely in $\mathbb{Q}(\alpha)$. □

We remark that Example 2.6 holds with $\mathcal{M}$ replaced by the Julia set of any monic polynomial $g(x) \in \mathbb{Z}[x]$ with degree $d > 1$ (see (2.57)). In this case, each repelling periodic point for $g(x)$ is an algebraic integer whose conjugates belong to the Julia set.

Our next result, originally formulated by Cantor ([**16**]) and proved in ([**52**]), generalizes Robinson's theorem on totally real algebraic integers ([**48**]) stated in the introduction.

Example 2.7. Let $\mathcal{Q}$ be a finite set of primes of $\mathbb{Q}$, and let $[a,b] \subset \mathbb{R}$. If

$$b - a \;>\; 4 \cdot \prod_{q \in \mathcal{Q}} q^{1/(q-1)} \;,$$

there are infinitely many algebraic integers α whose conjugates all belong to $[a,b]$, such that the primes in $\mathcal{Q}$ split completely in $\mathbb{Q}(\alpha)$; if $b - a < 4 \cdot \prod_{q \in \mathcal{Q}} q^{1/(q-1)}$ there are only finitely many.

Proof. Take $K = \mathbb{Q}$, $\mathcal{C} = \mathbb{P}^1$, and $\mathfrak{X} = \{\infty\}$. Put $E_\infty = [a,b]$, $E_q = \mathbb{Z}_q$ for $q \in \mathcal{Q}$, and $E_p = \widehat{\mathcal{O}}_p$ for finite $p \notin \mathcal{Q}$. Then for $\mathbb{E} = E_\infty \times \prod_p E_p$, using $g_\infty = 1/z$ to compute the local capacities, formulas (2.9) and (2.68) give

$$\gamma(\mathbb{E}, \mathfrak{X}) \;=\; \prod_{p,\infty} \gamma_\infty(E_p) \;=\; \frac{b-a}{4} \cdot \prod_{q \in \mathcal{Q}} q^{-1/(q-1)} \;.$$

Thus the result follows from Theorem 1.5. □

Over an arbitrary number field, we have the following generalization of Example 2.7, motivated by a result of Moret-Bailly ([**39**], Théorème 1.3, p.182).

EXAMPLE 2.8. Let K be a number field, with r_1 real places and r_2 complex places. Write $n = [K : \mathbb{Q}]$, and let $\mathcal{Q}$ be a finite set of nonarchimedean places of K. For each $v \in \mathcal{Q}$, let F_w/K_v be a finite Galois extension, with ramification index $e_{w/v}$ and residue degree $f_{w/v}$. If

$$R^n \ > \ 2^{r_1} \prod_{v \in \mathcal{Q}} q_v^{1/(e_{w/v}(q_v^{f_{w/v}}-1))} \tag{2.86}$$

then there are infinitely many algebraic integers α whose archimedean conjugates belong to $D(0,R)$ at each v where $K_v \cong \mathbb{C}$, to $[-R,R]$ at each v where $K_v \cong \mathbb{R}$, and are such that for each $v \in \mathcal{Q}$ all the conjugates in $\mathbb{C}_v$ belong to $\mathcal{O}_{F_w}$. If R^n is less than the bound in (2.86), there are only finitely many such algebraic integers.

PROOF. For each complex archimedean v, put $E_v = D(0,R)$; then $\gamma_\infty(E_v) = R$. For each real archimedean v, put $E_v = [-R,R] \subset \mathbb{R}$; then $\gamma_\infty(E_v) = R/2$. For each nonarchimedean $v \in \mathcal{Q}$, put $E_v = \mathcal{O}_w$, and write $e = e_{w/v}$, $f = f_{w/v}$; then $\gamma_\infty(E_v) = q_v^{-1/e(q_v^f-1)}$ by (2.68). For all other nonarchimedean v, put $E_v = D(0,1)$, and put $\mathbb{E} = \prod_v E_v$. By our convention about weights and absolute values in the complex archimedean case,

$$\begin{aligned} \gamma(\mathbb{E},\{\infty\}) &= \prod_{\text{real } v} \gamma_\infty(E_v) \cdot \prod_{\text{complex } v} \gamma_\infty(E_v)^2 \cdot \prod_{\text{finite } v} \gamma_\infty(E_v) \\ &= R^n \cdot 2^{-r_1} \cdot \prod_{v \in \mathcal{Q}} q_v^{-1/(q_v-1)} \ . \end{aligned}$$

Again the result follows from Theorem 1.5. □

In general, not much is known about the case when $R^n = 2^{r_1} \prod_{v \in S} q_v^{1/(q_v-1)}$. When K is totally real, and S is empty, we have $R = 2$, and the roots of Chebyshev polynomials belong to $[-2,2]$. When K is totally complex, $R = 1$ and the roots of unity belong to $D(0,1)$. Thus in these two cases there are infinitely many algebraic integers whose conjugates satisfy the required conditions. There are no known examples where there are only finitely many.

Algebraic numbers satisfing prescribed arithmetic conditions, with controlled archimedean conjugates, can often be constructed by imposing appropriate geometric conditions on the sets E_v. The following (admittedly contrived) example illustrates some of the possibilities.

EXAMPLE 2.9. For any $\varepsilon > 0$, there are infinitely many algebraic integers α such that

(1) each archimedean conjugate $\sigma(\alpha)$ is real and satisfies $0 < \sigma(\alpha) < 12\sqrt{5} + \varepsilon$;
(2) the primes above 2 in $\mathbb{Q}(\alpha)$ have residue degree 1, and $|\alpha|_v = 1$ at each $v|2$.
(3) 3 is unramified in $\mathbb{Q}(\alpha)$, and $\mathrm{ord}_v(\alpha) = 1$ at all v above 3;
(4) 5 splits completely in $\mathbb{Q}(\alpha)$, and α is a quadratic nonresidue at each $v|5$;
(5) for all primes $\mathfrak{p}_v$ of $\mathbb{Q}(\alpha)$ above 7, we have $\alpha \equiv -1 \pmod{\mathfrak{p}_v}$.

PROOF. Take $K = \mathbb{Q}$ and let $L > 0$ be a parameter. Put $E_\infty = [0,L] \subset \mathbb{R}$, $E_2 = D(1,1)^-$, $E_3 = D(2/3,1/3)^-$, $E_5 = \mathbb{Z}_5 \cap (D(2,1)^- \cup D(3,1)^-)$, $E_7 = D(-1,1)^-$. Put $E_p = D(0,1)$ for all other primes p. Then $\gamma_\infty(E_\infty) = L/4$.

As seen in the discussion of capacities of nonarchimedean open discs, $\gamma_\infty(E_2) = \gamma_\infty(D(1,1)) = \gamma_\infty(D(0,1)) = 1$ and $\gamma_\infty(E_3) = \gamma_\infty(D(2/3,1/3)) = \gamma_\infty(D(0,1/3)) = 1/3$. Corollary 2.3 shows that $\gamma_\infty(E_5) = 5^{-2/4}$. At $p = 7$, we have $E_7 \subset D(0,1)$ so $\gamma_\infty(E_7) \le 1$; on the other hand for each totally ramified finite extension $F_w/\mathbb{Q}_7$ we have $-1 + \pi_w \mathcal{O}_w \subset E_7$, and Proposition 2.1 shows that $\gamma_\infty(-1 + \pi_w \mathcal{O}_w) = 7^{-1/e_w} \cdot 7^{-1/6e_w}$. Letting $e_w \to \infty$, we see that $\gamma_\infty(E_7) = 1$. As noted in Theorem 1.4, the condition that the primes above 2 have residue degree 1, and the primes above 3 be unramified, can be imposed "for free". For Theorem 1.4 to be applicable, we need $L/4 \cdot 1/3 \cdot 5^{-1/2} > 1$. □

The following example illustrates a case in which some of the E_v are unions of "different types" of sets, with overlaps.

EXAMPLE 2.10. Take $K = \mathbb{Q}$; let $\mathfrak{X} = \{\infty\}$, put $E_\infty = D(0,1) \cup [1, 1+L]$ where $L \ge 0$, and put $E_3 = \mathcal{O}_{v_1} \cup \mathcal{O}_{v_2}$ where $\mathcal{O}_{v_1}$ is the ring of integers of the unramified extension $L_{v_1} = \mathbb{Q}_3(\sqrt{-1})$, and $\mathcal{O}_{v_2}$ is the ring of integers of the totally ramified extension $L_{v_2} = \mathbb{Q}_3(\sqrt{-3})$. For each finite prime $p \ne 2$, let $E_p = \widehat{\mathcal{O}}_v$ be the $\mathfrak{X}$-trivial set.

By formula (2.50) with $L_1 = 0$ and $L_2 = L$, and formula (2.83) with $q_v = 3$, we have

$$\begin{aligned} \gamma(\mathbb{E}, \{\infty\}) &= 3^{-(-61+\sqrt{6481})/184} \cdot \Big(1 + \frac{L^2}{4(1+L)}\Big) \\ &\cong 0.89000685 + 0.22251713 L^2/(1+L) \ . \end{aligned}$$

By Theorems 0.4 and 1.5, if $L > 0.99240793$ then there are infinitely many algebraic integers whose archimedean conjugates all lie in $D(0,1) \cup [1, 1+L]$ and whose $\mathbb{C}_3$-conjugates all lie in $\mathcal{O}_{v_1} \cup \mathcal{O}_{v_2}$, while if $L < 0.99240792$ there are only finitely many.

Our last result in this section is a continuation of an example of Cantor ([**16**], p.167). Suppose that instead of constructing algebraic integers, one has a rational function $f(x)$ and is interested in constructing numbers α for which $f(\alpha)$ is an algebraic integer.

For instance, let $f(x) = 1/(1+x^2)$. Using Fekete's Theorem, Cantor showed that there are only finitely many totally real α for which $f(\alpha)$ is an algebraic integer; indeed, $\alpha = 0$ and $\alpha = \infty$ are the only such points.

Suppose, however, that we were willing to accept numbers α, all of whose conjugates had a small imaginary part. How large would the imaginary parts have to be to guarantee the existence of infinitely many solutions?

EXAMPLE 2.11. Take $f(x) = 1/(1+x^2)$. Suppose $T > 3/4$. Then there are infinitely many $\alpha \in \widetilde{\mathbb{Q}}$, all of whose conjugates satisfy $|\operatorname{Im}(\sigma(\alpha))| < T$, for which $f(\alpha)$ is an algebraic integer. However, if $T < 3/4$, there are only finitely many.

PROOF. Take $K = \mathbb{Q}$, and let $\mathfrak{X} = \{i, -i\}$, the set of poles of $f(x)$. Put $g_i(z) = z - i$, $g_{-i}(z) = z + i$, and let $L = \mathbb{Q}(i)$. Fix $T > 0$. At the archimedean place of $\mathbb{Q}$, take

$$E_\infty \ = \ \{z \in \mathbb{C} : -T \le \operatorname{Im}(z) \le T\} \cup \{\infty\} \ .$$

At each finite prime p, put $E_p = f^{-1}(\widehat{\mathcal{O}}_p)$. One sees that $E_2 = \mathbb{P}^1(\mathbb{C}_2)\backslash B(1,1)^-$, while for each $p \geq 3$, $E_p = \mathbb{P}^1(\mathbb{C}_p)\backslash(B(i,1)^- \cup B(-i,1)^-)$ where the two balls are disjoint. Put $\mathbb{E} = E_\infty \times \prod_p E_p$.

To compute the Green's matrices we must make a base change to L. Recall that $\Gamma(\mathbb{E},\mathfrak{X}) = [L:K]^{-1}\Gamma(\mathbb{E}_L,\mathfrak{X})$. There is one archimedean place of L, which we will denote w_∞. By (2.48), we have $V_i(E_{w_\infty}) = V_{-i}(E_{w_\infty}) = \log(2(1-T))$, while $G(-i,i;E_{w_\infty}) = G(i,-i;E_{w_\infty}) = 0$ since i and $-i$ belong to distinct components of $\mathbb{P}^1(\mathbb{C})\backslash E_\infty$. Since $L_{w_\infty} \cong \mathbb{C}$, we have $\log(q_{w_\infty}) = \log(e^2) = 2$. There is one place w_2 of L above 2; $L_{w_2}/\mathbb{Q}_2$ is totally ramified. Fixing an isomorphism $\mathbb{C}_{w_2} \cong \mathbb{C}_2$, identify E_{w_2} with E_2. Then $V_i(E_{w_2}) = V_{-i}(E_{w_2}) = 0$, while $G(-i,i;E_{w_2}) = G(i,-i;E_{w_2}) = -\log_2(|i-(-i)|_{w_2}) = 2$. We have $\log(q_{w_2}) = \log(2)$. For all other places v of L the Green's matrices are trivial. Thus

$$\Gamma(\mathbb{E},\mathfrak{X}) \;=\; \begin{pmatrix} \log(2(1-T)) & \log(2) \\ \log(2) & \log(2(1-T)) \end{pmatrix} . \tag{2.87}$$

By definition $\gamma(\mathbb{E},\mathfrak{X}) = \exp(-\operatorname{val}(\Gamma(\mathbb{E},\mathfrak{X})))$, where

$$\operatorname{val}(\Gamma(\mathbb{E},\mathfrak{X})) \;=\; \min_{\vec{r}\in\mathfrak{p}}\max_{\vec{s}\in\mathfrak{p}} {}^t\vec{r}\,\Gamma(\mathbb{E},\mathfrak{X})\vec{s} \;=\; \max_{\vec{r}\in\mathfrak{p}}\min_{\vec{s}\in\mathfrak{p}} {}^t\vec{r}\,\Gamma(\mathbb{E},\mathfrak{X})\vec{s}$$

is the value of $\Gamma(\mathbb{E},\mathfrak{X})$ as a matrix game; here $\mathfrak{p}$ is the set of probability vectors in $\mathbb{R}^2$. If we take $\vec{s} = {}^t(\frac{1}{2},\frac{1}{2})$ then both entries of $\Gamma(\mathbb{E},\mathfrak{X})\vec{s}$ are equal to $\frac{1}{2}\log(4(1-T))$; combining the "mini-max" and "maxi-min" expressions for $\operatorname{val}(\Gamma(\mathbb{E},\mathfrak{X}))$ shows that $\operatorname{val}(\Gamma(\mathbb{E},\mathfrak{X})) = \frac{1}{2}\log(4(1-T))$.

Thus $\gamma(\mathbb{E},\mathfrak{X}) > 1$ iff $T > 3/4$, while $\gamma(\mathbb{E},\mathfrak{X}) < 1$ iff $T < 3/4$, and the result follows from the Fekete-Szegö Theorem 1.5. □

Algebraic Units. As in Example 2.9, the condition that an algebraic integer α be a unit at finitely many specified primes can be imposed "for free". However, if we want global algebraic units, the construction must assure that they avoid 0 (and ∞) at all nonarchimedean v. This can be accomplished by using the capacity relative to two points $\mathfrak{X} = \{0,\infty\}$.

Below is Robinson's unit theorem ([**49**]) cited in the Introduction, which was originally proved without using capacity theory. Cantor ([**16**]) was the first to recognize that Robinson's conditions arise naturally in the context of capacities.

Example 2.12 (**Robinson**). Suppose $0 < a < b \in \mathbb{R}$ satisfy the conditions

$$\log(\frac{b-a}{4}) \;>\; 0 \;, \tag{2.88}$$

$$\log(\frac{b-a}{4})\cdot\log(\frac{b-a}{4ab}) - \Big(\log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}})\Big)^2 \;>\; 0 \;. \tag{2.89}$$

Then there are infinitely many totally real units α whose conjugates belong to $[a,b]$.

Proof. We follow Cantor ([**16**], p.166). Take $K = \mathbb{Q}$, $\mathcal{C} = \mathbb{P}^1$, and $\mathfrak{X} = \{0,\infty\}$. Put $E_\infty = [a,b]$, and put $E_p = D(0,1)\backslash D(0,1)^-$ for each finite prime p. Each nonarchimedean E_p is the "$\mathfrak{X}$-trivial" set in $\mathbb{P}^1(\mathbb{C}_p)$, so we can take $\mathbb{E} = E_\infty \times \prod_p E_p$.

Let the uniformizing parameters used to compute capacities be $g_0(z) = z$, $g_\infty(z) = 1/z$. By formulas (2.9), (2.7), at the archimedean place $V_\infty([a,b]) = \log(4/(b-a))$ and $G(0,\infty;[a,b]) = \log((\sqrt{b}+\sqrt{a})/(\sqrt{b}-\sqrt{a}))$. Pulling back by $1/z$, we have $G(z,0;[a,b]) = G(1/z,\infty;[1/b,1/a])$. In view of our choices of the uniformizing parameters, this yields $V_0([a,b]) = V_\infty([1/b,1/a]) = \log(4ab/(b-a))$.

At each finite prime p, one sees easily that $V_0(E_p) = V_\infty(E_p) = G(0,\infty;E_p) = 0$. Thus

$$\Gamma(\mathbb{E},\mathfrak{X}) \;=\; \Gamma(E_\infty,\mathfrak{X}) \;=\; \begin{pmatrix} \log(\frac{4ab}{b-a}) & \log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}) \\ \log(\frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}-\sqrt{a}}) & \log(\frac{4}{b-a}) \end{pmatrix} . \tag{2.90}$$

The conditions (2.88), (2.89) in Robinson's theorem are simply determinant inequalities on the minors of $\Gamma(\mathbb{E},\mathfrak{X})$, necessary and sufficient for it to be negative definite. Hence the result follows from the Fekete-Szegö Theorem 1.5. □

In the next result, we bound the size of the units and their reciprocals, as well as imposing conditions at nonarchimedean places.

Example 2.13. There are infinitely many totally real algebraic units α whose archimedean conjugates belong to $[-r,-1/r] \cup [1/r,r]$, if

$$r \;>\; 1+\sqrt{2}\ .$$

More generally, let $\mathcal{Q}$ be a finite set of primes, and put $A = \prod_{q\in\mathcal{Q}} q^{q/(q-1)^2}$. Then there are infinitely many totally real algebraic units α for which the primes $q \in \mathcal{Q}$ split completely in $\mathbb{Q}(\alpha)$, and whose archimedean conjugates belong to $[-r,-1/r]\cup [1/r,r]$ if

$$r \;>\; A^2+\sqrt{A^4+1}\ . \tag{2.91}$$

If the opposite inequality holds, there are only finitely many.

Proof. Take $K = \mathbb{Q}$, $\mathcal{C} = \mathbb{P}^1$, and $\mathfrak{X} = \{0,\infty\}$. Let the uniformizers be $g_0(z) = z$, $g_\infty(z) = 1/z$ as before. Take $r \ge 1$ and put $E_\infty = [-r,-1/r] \cup [1/r,r] \subset \mathbb{R}$. For each $q \in \mathcal{Q}$, put $E_q = \mathbb{Z}_q^\times$. For all other primes p, put $E_p = D(0,1)\backslash D(0,1)^- \subset \mathbb{C}_p$, then let $\mathbb{E} = E_\infty \times \prod_p E_p$.

By formulas (2.15), (2.16) and (2.18), we have

$$\Gamma(E_\infty,\mathfrak{X}) \;=\; \begin{pmatrix} \frac{1}{2}\log(\frac{4r^2}{r^4-1}) & \frac{1}{2}\log(\frac{r^2+1}{r^2-1}) \\ \frac{1}{2}\log(\frac{r^2+1}{r^2-1}) & \frac{1}{2}\log(\frac{4r^2}{r^4-1}) \end{pmatrix} .$$

For primes $q \in \mathcal{Q}$, formulas (2.74) and (2.75) give $V_\infty(E_q) = G(0,\infty;E_q) = q/(q-1)^2$. Pulling back by $1/z$ and using that $\mathbb{Z}_p^\times$ is stable under taking reciprocals, we have $G(z,0;E_q) = G(1/z,\infty;E_q)$ and hence $V_0(E_q) = G(\infty,0;E_q) = q/(q-1)^2$ as well. Thus

$$\Gamma(E_q,\mathfrak{X}) \;=\; \begin{pmatrix} q/(q-1)^2 & q/(q-1)^2 \\ q/(q-1)^2 & q/(q-1)^2 \end{pmatrix} . \tag{2.92}$$

For all other p, $\Gamma(E_p,\mathfrak{X})$ is the 0 matrix. Hence

$$\begin{aligned} \Gamma(\mathbb{E},\mathfrak{X}) \;&=\; \Gamma(E_\infty,\mathfrak{X}) + \sum_p \Gamma(E_p,\mathfrak{X})\log(p) \\ &=\; \frac{1}{2}\begin{pmatrix} \log(\frac{4A^2r^2}{r^4-1}) & \log(A^2\frac{r^2+1}{r^2-1}) \\ \log(A^2\frac{r^2+1}{r^2-1}) & \log(\frac{4A^2r^2}{r^4-1}) \end{pmatrix} . \end{aligned}$$

Take $\vec{s} = {}^t(\frac{1}{2},\frac{1}{2})$. Then $\Gamma(\mathbb{E},\mathfrak{X})\vec{s}$ has equal entries

$$V \;=\; \frac{1}{2}\log\left(\frac{4A^4r^2}{(r^2-1)^2}\right) .$$

By the definition of the value of a matrix game, $V(\mathbb{E},\mathfrak{X}) := \mathrm{val}(\Gamma(\mathbb{E},\mathfrak{X})) = V$. Since $r \ge 1$, $\gamma(\mathbb{E},\mathfrak{X}) = e^{-V(\mathbb{E},\mathfrak{X})} = (r^2-1)/(2A^2r)$. It is easy to see that $\gamma(\mathbb{E},\mathfrak{X}) > 1$ if and only if condition (2.91) holds, and that $\gamma(\mathbb{E},\mathfrak{X}) < 1$ if and only if the opposite inequality holds. Hence the result follows from Theorem 1.5. □

If $r = 1+\sqrt{2}$ in the first part of Example 2.13, then there are infinitely many units whose conjugates lie in $[-r,-1/r] \cup [1/r,r]$. Note that this set is the pullback of $[-2,2]$ by $f(z) = z - 1/z$. For each $n \ge 1$, let $T_n(x)$ denote the Chebyshev polynomial of degree n. It is well known that $T_n(x)$ is a monic polynomial with integer coefficients, whose roots are simple and belong to the interval $[-2,2]$. Put $P_n(z) = z^n T_n(z-1/z)$. Then $P_n(z)$ is monic with integer coefficients, and has constant coefficient $(-1)^n$. Thus the roots of the $P_n(z)$ are the units we need.

Next we give an S-unit analogue of Example 2.13. By a trick, we are able to require that the S-units constructed be totally p-adic, while their archimedean conjugates all have absolute value 1:

Example 2.14. Let $k = \mathbb{Q}$ and fix a (nonarchimedean) prime p. Let $\mathcal{Q}$ be a finite set of nonarchimedean primes of $\mathbb{Q}$, disjoint from $\{p\}$, and put $A = \prod_{q\in\mathcal{Q}} q^{q/(q-1)^2}$ as in Example 2.13. Suppose $0 < m \in \mathbb{Z}$ is such that

$$m\log(p) \ > \ 2\log(A) + \frac{p\,(p^{2m}+1)}{(p-1)(p^{2m+1}-1)}\log(p)\ . \tag{2.93}$$

Then there are infinitely many numbers $\alpha \in \widetilde{\mathbb{Q}}$ for which the primes $q \in \mathcal{Q}$ split completely in $\mathbb{Q}(\alpha)$, which are units at all nonarchimedean places v of $\mathbb{Q}(\alpha)$ not above p, whose archimedean conjugates all satisfy $|\sigma(\alpha)| = 1$, and whose conjugates in $\mathbb{C}_p$ all belong to $\mathbb{Q}_p$ and satisfy $|\,\mathrm{ord}_p(\sigma(\alpha))| \le m$.

If the opposite inequality to (2.93) holds, there are only finitely many.

Proof. Take $K = \mathbb{Q}$, and let $\mathfrak{X} = \{0,\infty\}$. Choose the uniformizing parameters $g_0(z) = z$, $g_\infty(z) = 1/z$ as usual.

The proof makes use of two $\mathbb{Q}$-rational adelic sets, which we will denote $\mathbb{E}$ and $\mathbb{E}'$. To construct $\mathbb{E}$, let $E_\infty = C(0,1)$, the unit circle. For each $q \in \mathcal{Q}$, put $E_q = \mathbb{Z}_q^\times$, and put

$$E_p \ = \ \{x \in \mathbb{Q}_p : -m \le \mathrm{ord}_p(x) \le m\} \ = \ p^{-m}E_{p,2m}$$

where $E_{p,2m} = \bigcup_{k=0}^{2m} p^k\mathbb{Z}_p^\times$ is as in Proposition 2.4. For all other finite primes q take $E_q = \widehat{\mathcal{O}}_q^\times = D(0,1)\backslash D(0,1)^-$, the $\mathfrak{X}$-trivial set in $\mathbb{P}^1(\mathbb{C}_q)$. Set $\mathbb{E} = E_\infty \times \prod_{q\ne\infty} E_q$.

To construct $\mathbb{E}'$, first choose a square-free integer $d < 0$ which satisfies $d \equiv 1 \pmod 8$ and $d \equiv 1 \pmod q$ for each $q \in \mathcal{Q}\cup\{p\}$. Thus the primes in $\mathcal{Q}\cup\{p\}$ split completely in the quadratic imaginary field $F = \mathbb{Q}(\sqrt{d})$. Let

$$f(x) \ = \ \frac{x-\sqrt{d}}{x+\sqrt{d}}\ ,$$

and for each prime q (archimedean or nonarchimedean) put $E'_q = f^{-1}(E_q)$. Then $E'_\infty = \mathbb{P}^1(\mathbb{R})$, while for each $q \in \mathcal{Q}\cup\{p\}$ we have $E'_q \subset \mathbb{Q}_q$. For all other primes q, E'_q is the RL-domain in $\mathbb{P}^1(\mathbb{C}_q)$ gotten by omitting two open discs centered on $\pm\sqrt{d}$; for all but finitely many q these discs are disjoint and have radius 1. Note that for each q, the set E'_q is stable under $\mathrm{Aut}_c(\mathbb{C}_q/\mathbb{Q}_q)$. If $q \in \mathcal{Q}\cup\{p,\infty\}$ this is trivial; for all other q, note that for each $\sigma \in \mathrm{Aut}_c(\mathbb{C}_q/\mathbb{Q}_q)$, either $\sigma(f)(x) = f(x)$

or $\sigma(f)(x) = 1/f(x)$. Since $E_q = \widehat{\mathcal{O}}_q^\times$ is stable under inversion and $\mathrm{Aut}_c(\mathbb{C}_q/\mathbb{Q}_q)$, it follows that $x \in E_q'$ if and only if $\sigma(x) \in E_q'$.

Set $\mathbb{E}' = E_\infty' \times \prod_{q \neq \infty} E_q'$, and take $\mathfrak{X}' = \{\sqrt{d}, -\sqrt{d}\}$. We claim that

$$\Gamma(\mathbb{E}', \mathfrak{X}') \ = \ \Gamma(\mathbb{E}, \mathfrak{X}) \ .$$

This follows by pulling back using $f(x)$: let $\mathbb{E}_F$, $\mathbb{E}_F'$ be the F-rational adelic sets obtained from $\mathbb{E}$, $\mathbb{E}'$ by base change (see [**51**], §5.1). Then $\Gamma(\mathbb{E}_F, \mathfrak{X}) = [F : \mathbb{Q}] \cdot \Gamma(\mathbb{E}, \mathfrak{X})$ and $\Gamma(\mathbb{E}_F', \mathfrak{X}') = [F : \mathbb{Q}] \cdot \Gamma(\mathbb{E}', \mathfrak{X}')$ ([**51**], p.326, formula (9)). On the other hand $f(x)$ is rational over F, so by ([**51**], p.335, formula (16)) and the fact that $\deg(f) = 1$,

$$\Gamma(\mathbb{E}_F', \mathfrak{X}') \ = \ \Gamma(\mathbb{E}_F, \mathfrak{X}) \ .$$

This establishes the claim.

By the Fekete-Szegö Theorem 0.4, if $\Gamma(\mathbb{E}', \mathfrak{X}')$ is negative definite, there are infinitely many algebraic numbers whose archimdean conjugates belong to $E_\infty' = \mathbb{P}^1(\mathbb{R})$ and whose q-adic conjugates belong to E_q' for all nonarchimedean q. The images of these numbers under $f(x)$ will be the numbers α in the example.

Hence it suffices to show that $\Gamma(\mathbb{E}', \mathfrak{X}') = \Gamma(\mathbb{E}, \mathfrak{X})$ is negative definite under condition (2.93). We have

$$\Gamma(\mathbb{E}, \mathfrak{X}) \ = \ \Gamma(E_\infty, \mathfrak{X}) + \sum_{q \neq \infty} \Gamma(E_q, \mathfrak{X}) \log(q) \ .$$

Here $\Gamma(E_\infty, \mathfrak{X})$ is the 0 matrix. For each $q \in \mathcal{Q}$, $\Gamma(E_q, \mathfrak{X})$ is the same as in (2.92) in the proof of Example 2.13. To express $\Gamma(E_p, \mathfrak{X})$, write

$$\begin{aligned} B &= \frac{1}{p-1} + \frac{1}{(p-1)^2(1+p^2+p^4+\cdots+p^{4m})} \ , \\ C &= \frac{p^{2m+1}}{(p-1)^2(1+p^2+p^4+\cdots+p^{4m})} \ . \end{aligned}$$

By Proposition 2.4 and the scaling property of the capacity,

$$V_\infty(E_p) \ = \ V_\infty(p^{-m} E_{p,2m}) \ = \ -m + B \ .$$

The map $f(z) = 1/z$ stabilizes E_p but takes 0 to ∞, so our choice of uniformizing parameters gives $V_0(E_p) = V_\infty(E_p)$. Finally $f(z) = p^m z$ takes $0 \mapsto 0$, $\infty \mapsto \infty$, and E_p to $E_{p,2m}$, so by the pullback formula (2.61) and Proposition (2.4), $G(0, \infty; E_p) = G(0, \infty; E_{p,2m}) = C$. Thus

$$\Gamma(E_p, \mathfrak{X}) \ = \ \begin{pmatrix} -m+B & C \\ C & -m+B \end{pmatrix} \ .$$

For all other primes q, $\Gamma(E_q, \mathfrak{X})$ is the 0 matrix, so

$$\Gamma(\mathbb{E}, \mathfrak{X}) \ = \ \begin{pmatrix} \log(A) + (-m+B)\log(p) & \log(A) + C\log(p) \\ \log(A) + C\log(p) & \log(A) + (-m+B)\log(p) \end{pmatrix} \ .$$

Since $\Gamma(\mathbb{E}, \mathfrak{X})$ has equal row sums, as in the proof of Example 2.13 it follows that $\mathrm{val}(\Gamma(\mathbb{E}, \mathfrak{X})) = \frac{1}{2}\big(-m\log(p) + 2\log(A) + (B+C)\log(p)\big)$. Simplifying, we have $\mathrm{val}(\Gamma(\mathbb{E}, \mathfrak{X})) < 0$, and hence $\gamma(\mathbb{E}, \mathfrak{X}) > 1$, if and only if (2.93) holds; similarly, $\gamma(\mathbb{E}, \mathfrak{X}) < 1$ if and only if the opposite inequality holds. Thus the result follows from the Fekete-Szego Theorem 1.5. $\square$

For instance, if $\mathcal{Q} = \{2\}$ in Example 2.14, then for $p = 3$ we need $m \geq 4$; for $5 \leq p \leq 17$ we can take $m = 2$, and for $p \geq 19$ we can take $m = 1$. If $\mathcal{Q} = \{2, 3\}$ in Example 2.14, then for $p = 5$ we need $m \geq 7$; for $7 \leq p \leq 11$ we can take $m = 5$; for $13 \leq p \leq 23$ we can take $m = 4$; for $29 \leq p \leq 109$ we can take $m = 3$; for $113 \leq p \leq 11673$ we can take $m = 2$; and for $p \geq 11677$ we can take $m = 1$.

Note that the S-units constructed in Example 2.14 are not roots of unity, because there are only finitely many roots of unity ζ_n for which p splits completely in $\mathbb{Q}(\zeta_n)$.

In the next example, a limit argument allows us deal with a situation where one of the points in $\mathfrak{X}$ belongs to E_∞:

EXAMPLE 2.15. Let $A > 0$. If $A \geq 4$, then there are infinitely many units whose conjugates all lie in $[0, 1] \cup [A, A + 1] \subset \mathbb{R}$. If $A < 4$, there are only finitely many.

PROOF. Write $E = [0, 1] \cup [A, A+1]$. If $A \leq 1$, then $E \subset [0, 2]$, so $\gamma_\infty(E) < 1/2$. Otherwise, E is a translate of $[-(A + 1)/2, -(A - 1)/2] \cup [(A - 1)/2, (A + 1)/2]$ and so $\gamma_\infty(E) = \sqrt{A}/2$ by formula (2.15). Thus if $A < 4$, we have $\gamma_\infty(E) < 1$, and Fekete's Theorem 1.5(B) shows there are only finitely many *algebraic integers*, and in particular finitely many units, whose conjugates lie in E.

Next suppose $A = 4$. We will explicitly construct infinitely many units whose conjugates lie in $[0, 1] \cup [4, 5]$. To do so, note that $[0, 1] \cup [4, 5]$ is the pullback of $[-2, 2]$ by $f(z) = z^2 - 5z + 2$. Let $T_n(x)$ denote the Chebyshev polynomial of degree n. As before, $T_n(x)$ is a monic polynomial with integer coefficients whose roots are simple and belong to $[-2, 2]$. It oscillates n times between ± 2 on $[-2, 2]$; in particular, $T_n(2) = 2$. Furthermore, $T_n(x)$ is an even function if n is even, and is an odd function if n is odd. Consider the polynomials $Q_n(z) = T_n(f(z))$. They are monic with integer coefficients, and have all their roots in $[0, 1] \cup [4, 5]$. Unfortunately, $Q_n(z)$ has constant coefficient $Q_n(0) = T_n(2) = 2$. However, if n is odd, then $Q_n(z)$ has $f(z)$ as a factor, so $P_n(z) := Q_n(z)/f(z)$ has constant coefficient 1. Thus the roots of the $P_n(z)$ for odd n are the required units.

Finally, suppose $A > 4$, and let $0 < \varepsilon < 1$. Consider the set $E_{\varepsilon,A} = [\varepsilon, 1] \cup [A, A + 1]$. Take $K = \mathbb{Q}$, $\mathcal{C} = \mathbb{P}^1$, $\mathfrak{X} = \{0, \infty\}$. Let $E_\infty = E_{\varepsilon,A}$, and for each finite prime p let $E_p = D(0, 1)\backslash D(0, 1)^- \subset \mathbb{C}_p$ be the $\mathfrak{X}$-trivial set. Put $\mathbb{E}_{\varepsilon,A} = E_{\varepsilon,A} \times \prod_p E_p$, and take $g_0(z) = z$, $g_\infty(z) = 1/z$ as before. Then

$$\Gamma(\mathbb{E}_{\varepsilon,A}, \mathfrak{X}) = \begin{pmatrix} V_0(E_{\varepsilon,A}) & G(\infty, 0; E_{\varepsilon,A}) \\ G(0, \infty; E_{\varepsilon,A}) & V_\infty(E_{\varepsilon,A}) \end{pmatrix} .$$

Formula (2.27) expresses the Green's function of two intervals in terms of a quotient of two theta functions. These theta-functions and their parameters vary continuously with ε and A, hence the Green's function varies continuously as well. Letting $\varepsilon \to 0$, formulas (2.15), (2.27), and (2.34) show that

$$\begin{aligned} \lim_{\varepsilon \to 0} V_\infty(E_{\varepsilon,A}) &= V_\infty(E) = -\frac{1}{2}\log(A/4) < 0 , \\ \lim_{\varepsilon \to 0} G(\infty, 0; E_{\varepsilon,A}) &= \lim_{\varepsilon \to 0} G(0, \infty; E_{\varepsilon,A}) = G(0, \infty; E) = 0 , \\ \lim_{\varepsilon \to 0} V_0(E_{\varepsilon,A}) &= -\infty . \end{aligned}$$

Thus for all sufficiently small $\varepsilon > 0$, $\Gamma(\mathbb{E}_{\varepsilon,A}, \mathfrak{X})$ is negative definite, and the Fekete-Szegö Theorems 0.4 and 1.5 yield the result. $\square$

As a whimsical side note, we remark that an argument similar to the one in Example 2.15 shows that $E = [0,1] \cup [A, A+.001]$ contains infinitely many conjugate sets of units if $A \geq 30.19249489$, but only finitely many if $0 < A < 30.19249488$. This is obtained by using Maple to evaluate $V_\infty(E)$ in formula (2.33).

It is also possible to use the Fekete-Szegö theorem to construct units whose conjugates *globally omit* residue classes. View $\mathbb{P}^1/\mathbb{Q}$ as the generic fibre of $\mathbb{P}^1_{\mathbb{Z}}/\operatorname{Spec}(\mathbb{Z})$. Given $\alpha, \beta \in \mathbb{P}^1(\widetilde{\mathbb{Q}})$, we will say that α is integral with respect to β if the Zariski closures of α and β in $\mathbb{P}^1_{\mathbb{Z}}$ do not meet. If $\beta_1, \ldots, \beta_N$ are the conjugates of β, this is equivalent to requiring that for every prime p, all the conjugates $\sigma(\alpha)$ in $\mathbb{P}^1(\mathbb{C}_p)$ belong to $\mathbb{P}^1(\mathbb{C}_p)\backslash(\bigcup_{i=1}^N B(\beta_i, 1)^-)$.

Recall that the (absolute, logarithmic) Weil height of a number $\alpha \in \widetilde{\mathbb{Q}}$ is

$$h(\alpha) \ = \ \frac{1}{[F:\mathbb{Q}]} \sum_{w \text{ of } F} \log_w^+(|\alpha|_w) \log(q_w) \ ,$$

where F is any finite extension $\mathbb{Q}(\alpha)$. The height is independent of the field used to compute it. By definition, $h(\infty) = 0$. The points of $\mathbb{P}^1(\widetilde{\mathbb{Q}})$ with $h(\alpha) = 0$ are precisely 0, ∞, and the roots of unity.

To put the following result in context, we remark that in ([**8**]) the authors show that if $h(\beta) \neq 0$, there are only finitely many roots of unity which are integral with respect to β.

EXAMPLE 2.16. Take $K = \mathbb{Q}$, and let $\beta \in \widetilde{\mathbb{Q}}$. Then there are infinitely many algebraic units $\eta \in \widetilde{\mathbb{Q}}$ which are integral with respect to β. For any $\varepsilon > 0$, these units can be required to have Weil height $h(\eta) < \varepsilon$.

PROOF. Put $L = \mathbb{Q}(\beta_1, \ldots, \beta_N)$, where $\beta_1, \ldots, \beta_N$ are the conjugates of β over $\mathbb{Q}$, and take $\mathfrak{X} = \{\infty, 0, \beta_1, \ldots, \beta_N\}$. Let $g_\infty(z) = 1/z$, $g_0(z) = z$, and $g_{\beta_i}(z) = z - \beta_i$ for each i. In the discussion below, we will assume that $\beta \neq 0$. If $\beta = 0$, then we are merely asking for units with height $h(\eta) < \varepsilon$, and the argument carries through in a simplified form.

Viewing the β_i as embedded in $\mathbb{C}$, let $r > 1$ be any number small enough that $r < \min_{|\beta_i|>1}(|\beta_i|)$, and then let $\rho > 0$ be any number small enough that the discs $D(\beta_i, \rho)$ for $i = 1, \ldots, N$ and $D(0, \delta)$ are pairwise disjoint and do not meet the circles $\{|z| = r\}$. (Note that r and ρ are independent of choice of the embedding of the β_i.). Eventually we will let $\rho \to 0$, and then let $r \to 1$. Put

$$E_\infty \ = \ \Big(D(0,r) \cup \big(\bigcup_{|\beta_i|>r} C(\beta_i, \rho)\big)\Big) \backslash \Big(D(0,\rho)^- \cup \bigcup_{|\beta_i|<r} D(\beta_i, \rho)^-\Big) \ .$$

Thus, E_∞ consists of a disc $D(0,r)$ with tiny holes deleted around 0 and the $\beta_i \in D(0,r)$, together with tiny circles adjoined around the $\beta_i \notin D(0,r)$. By construction E_∞ is stable under complex conjugation.

For each finite prime p, regarding the β_i as embedded in $\mathbb{C}_p$, put

$$E_p \ = \ \mathbb{P}^1(\mathbb{C}_p)\backslash\big(B(\infty,1)^- \cup B(0,1)^- \cup \bigcup_{i=1}^N B(\beta_i,1)^-\big) \ .$$

Then E_p is an RL-domain, stable under $\mathrm{Aut}_c(\mathbb{C}_p/\mathbb{Q}_p)$, and for all but finitely many p it is $\mathfrak{X}$-trivial.

Put $\mathbb{E} = E_\infty \times \prod_p E_p$. To compute $\Gamma(\mathbb{E}, \mathfrak{X})$, we must first make a base change to the field L, over which the β_i are rational. By definition

$$\Gamma(\mathbb{E}, \mathfrak{X}) \;=\; \frac{1}{[L:K]}\Gamma(\mathbb{E}_L, \mathfrak{X}) \;=\; \frac{1}{[L:K]} \sum_{\text{places } w \text{ of } L} \Gamma(E_w, \mathfrak{X}) \log(q_w) \tag{2.94}$$

where $\mathbb{E}_L = \prod_{w \text{ of } L} E_w$. Here, for each place w of L, if w lies over p, then E_w is gotten by choosing an embedding $\sigma : L \hookrightarrow \mathbb{C}_p$ which induces v, extending σ to an isomorphism $\overline{\sigma} : \mathbb{C}_w \to \mathbb{C}_p$, and setting $E_w = \overline{\sigma}^{-1}(E_p)$. Basically, E_w is the same as E_p, but the way the β_i are embedded depends w.

We now compute the matrices $\Gamma(\mathbb{E}_w, \mathfrak{X})$. First suppose $w|\infty$. By construction, each point of $\mathfrak{X}$ belongs to a different connected component of $\mathbb{P}^1(\mathbb{C}_w)\backslash E_w$, so $\Gamma(E_w, \mathfrak{X})$ is a diagonal matrix. We have $V_\infty(E_w) = -\log(r) - \delta(\rho)$ where $\delta(\rho) > 0$ and $\delta(\rho) \to 0$ as $\rho \to 0$, while $V_{\beta_i}(E_w) = -\log(\rho)$ for each i.

Next suppose w is nonarchimedean. Since E_w is obtained by deleting a finite number of open discs of radius 1 from $\mathbb{P}^1(\mathbb{C}_w)$, one of which is $B(\infty, 1)^-$, we have $V_\infty(E_w) = V_\infty(D(0,1)) = 0$. The other entries of $\Gamma(E_w, \mathfrak{X})$ will not matter to us: $\Gamma(E_w, \mathfrak{X})$ is an $(N+2) \times (N+2)$ matrix whose $V_\infty(E_w)$ entry is 0. For all but finitely many w, E_w is $\mathfrak{X}$-trivial and $\Gamma(E_w, \mathfrak{X})$ is the 0 matrix.

By definition $\Gamma(\mathbb{E}_F, \mathfrak{X}) = \sum_w \Gamma(E_w, \mathfrak{X}) \log(q_w)$; for archimedean w, $q_w = e$ if $L_w \cong \mathbb{R}$, while $q_v = e^2$ if $L_w \cong \mathbb{C}$. By (2.94)

$$\Gamma(\mathbb{E}, \mathfrak{X}) \;=\; \begin{pmatrix} -\log(r) - \delta(\rho) & A_{12} & \cdots & A_{1,N+2} \\ A_{21} & -\log(\rho) + A_{22} & \cdots & A_{2,N+2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N+2,1} & A_{N+1,2} & \cdots & -\log(\rho) + A_{N+2,N+2} \end{pmatrix}$$

where the A_{ij} do not depend on r or ρ, and $A_{ij} = A_{ji}$ for all i, j. By the determinant criterion for negative definiteness from linear algebra (see for example [**51**], Proposition 5.1.8, p.331), for each fixed r if ρ is sufficiently small then $\Gamma(\mathbb{E}, \mathfrak{X})$ is negative definite. Thus for any neighborhood U of E_∞ the Fekete-Szegö Theorem 1.5 produces infinitely many units η whose archimedean conjugates all lie in U, and whose nonarchimedean conjugates avoid the balls $B(\beta_i, 1)^-$ at all places w of L.

To see why the numbers η can be assumed to have arbitrarily small height requires some understanding of the proof of the Fekete-Szegö theorem (either Theorem 6.3.2 of [**51**], or Theorem 4.2 in this work). We will now sketch the argument, assuming the reader is coarsely familiar with the proof.

Fix $r > 1$, and let ρ be small enough that $\Gamma(\mathbb{E}, \mathfrak{X})$ is negative definite. Then there is a probability vector $\vec{s} = {}^t(s_1, \ldots, s_{N+2})$ for which $\Gamma(\mathbb{E}, \mathfrak{X})\vec{s}$ has all its entries equal to $V(\mathbb{E}, \mathfrak{X})$. These s_i are essentially the relative orders of the poles of the initial patching functions $G_v^{(0)}(z)$ at the points $x_i \in \mathfrak{X}$. As $\rho \to 0$, we will have $s_1 \to 1$ and $s_2, \ldots, s_{N+2} \to 0$ since the first row of $\Gamma(\mathbb{E}, \mathfrak{X})$ (and hence $V(\mathbb{E}, \mathfrak{X})$) remains bounded but the diagonal entries in the other rows approach ∞.

The archimedean initial local patching function $G_\infty^{(0)}(z)$ is chosen so that the discrete probability density of its zeros approximates $\sum_{i=1}^{N+2} s_i \mu_i$, where μ_i is the equilibrium distribution of E_v with respect to x_i. Here each μ_i is a probability measure supported on the boundary of the component of $\mathbb{P}^1(\mathbb{C}_v)\backslash E_v$ containing x_i. As $\rho \to 0$, the amount of mass which μ_1 (corresponding to $x_1 = \infty$) places on the circles $C(\beta_i, \rho)$ goes to 0. Thus the proportion of the zeros of $G_\infty^{(0)}(z)$ which lie near near $C(0, r)$ goes to 1. The remaining zeros all lie near the circles $C(\beta_i, \rho)$. If U is chosen small enough that each $C(\beta_i, \rho)$ outside $D(0, r)$ lies in a separate component of U, then the patching process preserves the number of zeros which lie in each component. Thus the final patched function $G^{(n)}(z)$, whose zeros are numbers constructed by the Fekete-Szegö theorem, has the same number of zeros in each component of U as the initial function $G_\infty^{(0)}(z)$.

Since the zeros of $G^{(n)}(z)$ (in our instance) are algebraic units, the only contribution to their height is from archimedean places. By the discussion above, that contribution approaches $\log(r)$ as $\rho \to 0$. So, if we first let $\rho \to 0$, and then let $r \to 1$, the Fekete-Szegö theorem produces numbers whose heights approach 0. □

Our final example constructs units which avoid the residue class of 1 at every prime, and whose archimedean conjugates all lie very close to the circle $C(0, r)$ or the circle $C(0, 1/r)$ (so $|\log(|\sigma(\alpha)|)| \approx \log(r)$), for suitable r.

EXAMPLE 2.17. Let r satisfy $1 < r < 2.96605206$. Then for any $\varepsilon > 0$, there are infinitely many units α whose conjugates all satisfy

$$||\sigma(\alpha)| - r| \ < \ \varepsilon \quad \text{or} \quad ||\sigma(\alpha)| - 1/r| \ < \ \varepsilon \ ,$$

and are such that $\sigma(\alpha) \not\equiv 1 \pmod{\mathfrak{p}}$ for each prime $\mathfrak{p}$ of $\mathcal{O}_{\mathbb{Q}(\sigma(\alpha))}$. If $r > 2.96605207$, there are only finitely many.

PROOF. Take $K = \mathbb{Q}$, $\mathcal{C} = \mathbb{P}^1$, and $\mathfrak{X} = \{0, 1, \infty\}$. Let $E_\infty = C(0, r) \cup C(0, 1/r)$, and for each finite prime let E_p be the $\mathfrak{X}$-trivial set

$$E_p \ = \ \mathbb{P}^1(\mathbb{C}_p) \backslash (B(0, 1)^- \cup B(1, 1)^- \cup B(\infty, 1)^-) \ .$$

Put $\mathbb{E} = E_\infty \times \prod_p E_p$, and take $g_0(z) = z$, $g_1(z) = z - 1$, $g_\infty(z) = 1/z$.

Note that 0, 1 and ∞ belong to different components of $\mathbb{P}^1(\mathbb{C}) \backslash E_\infty$. Then $V_\infty(E_\infty) = V_0(E_\infty) = -\log(r)$ by formula (2.3), while $V_1(E_\infty)$ is given by (2.55) with $\tau = 2i \log(r)/\pi$. At each nonarchimedean place, $\Gamma(E_p, \mathfrak{X})$ is the 0 matrix. Hence $\Gamma(\mathbb{E}, \mathfrak{X}) = \Gamma(E_\infty, \mathfrak{X})$ is the diagonal matrix

$$\Gamma(\mathbb{E}, \mathfrak{X}) \ = \ \begin{pmatrix} -\log(r) & 0 & 0 \\ 0 & -\log(\frac{|\theta(0,\tau;0,0)\theta(0,\tau;\frac{1}{2},0)|}{2}) & 0 \\ 0 & 0 & -\log(r) \end{pmatrix} . \tag{2.95}$$

Clearly $\Gamma(\mathbb{E}, \mathfrak{X})$ is negative definite if and only if the middle term is negative. A computation with Maple yields the result. □

4. Function Field Examples concerning Separability

In this section, we will take $K = \mathbb{F}_p(t)$ where p is a prime, $\mathbb{F}_p$ is the finite field with p elements, and t is transcendental over $\mathbb{F}_p$. We give three examples showing the need for the separability hypothesis in Theorem 0.4.C.2, and in Theorems 1.2, 1.3, and 1.5. This was discovered by Daeshik Park in his doctoral thesis ([**45**]).

Let v_0, v_1 and v_∞ be the valuations of $\mathbb{F}_p(t)$ for which $v_0(t) = 1$, $v_1(t-1) = 1$, and $v_\infty(1/t) = 1$, respectively. For each of the corresponding places, the residue field is $\mathbb{F}_p$, and we have $\mathcal{O}_{v_0} \cong \mathbb{F}_p[[t]]$, $\mathcal{O}_{v_1} \cong \mathbb{F}_p[[t-1]]$, and $\mathcal{O}_{v_\infty} \cong \mathbb{F}_p[[\frac{1}{t}]]$.

Our first example, which is due to Park, concerns a set where all the hypotheses of the Fekete-Szegö Theorem 0.4 are satisfied except for separability of the extension F_{w_0}/K_{v_0}, yet the conclusion of the theorem fails for r in a certain range.

Take $E_2 = \mathcal{E}(\mathbb{Z}_2)$. By formula (2.118), $V_{\overline{o}}(E_2) = 26/35$. Similarly, take $E_3 = \mathcal{E}(\mathbb{Z}_3)$. By formula (2.107) with $n = 4$, we have $V_{\overline{o}}(E_3) = 123/238$. For $p > 3$, take $E_p = \mathcal{E}(\widehat{\mathcal{O}}_p)$. Since the given model of $\mathcal{E}$ and the parameter z have good reduction at p, $V_{\overline{o}}(E_p) = 0$.

Let $\mathbb{E} = \prod_{p,\infty} E_p$, and take $\mathfrak{X} = \{\overline{o}\}$. Then

$$V(\mathbb{E}, \mathfrak{X}) \;=\; f_\infty(T) + \frac{26}{35}\ln(2) + \frac{123}{238}\ln(3)\ .$$

If we had taken E_∞ to be the real loop $x^{-1}([-6, 2])$, then by the Fekete-Szegö theorem there would be only finitely many $\alpha \in \mathcal{E}(\mathbb{Q})$ whose conjugates meet the given conditions. Maple shows that the value of T for which $V(\mathbb{E}, \mathfrak{X}) = 0$ satisfies

$$28.890384201 \;<\; T \;<\; 28.890384202\ ,$$

and the Fekete-Szegö Theorems 0.4 and 1.5 yield the result. □

Example 2.28 (N = 360). Let $\mathcal{E}/\mathbb{Q}$ be the elliptic curve defined by the Weierstrass equation $y^2 = x^3 + 117x + 918$, curve $360(E4)$ in Cremona's tables. Then for any $R \geq 142.388571238$ there are infinitely many $\alpha \in \mathcal{E}(\widetilde{\mathbb{Q}})$ whose archimedean conjugates satisfy $|y(\alpha)| \leq R$, whose conjugates in $\mathcal{E}(\mathbb{C}_2)$ all belong to $\mathcal{E}(\mathbb{Z}_2)$, whose conjugates in $\mathcal{E}(\mathbb{C}_3)$ all belong to $\mathcal{E}(\mathbb{Z}_3)$, whose conjugates in $\mathcal{E}(\mathbb{C}_5)$ all belong to $\mathcal{E}(\mathbb{Z}_5)$, and whose conjugates in $\mathcal{E}(\mathbb{C}_p)$ belong to $\mathcal{E}(\widehat{\mathcal{O}}_p)$, for all primes $p > 5$.

If $R \leq 142.388571237$, there are only finitely many such α.

Proof. The given Weierstrass equation is minimal. At $p = 2$ it has reduction type III^*, with 2 rational components; at $p = 3$ it has reduction type I_0^*, with 2 rational components; and at $p = 5$ it has nonsplit multiplicative reduction ($n = 4$) with 2 rational components (see Cremona [**21**], p.133). We compute capacities with respect to the uniformizing parameter $z = x/y$.

Take $E_\infty = y^{-1}(D(0, R))$, where $R > 0$. By formula (2.99), $V_{\overline{o}}(E_\infty) = -\frac{1}{3}\ln(R)$. At $p = 2$, take $E_2 = \mathcal{E}(\mathbb{Z}_2)$. By formula (2.121), $V_{\overline{o}}(E_2) = 10/9$. At $p = 3$, take $E_3 = \mathcal{E}(\mathbb{Z}_3)$. By formula (2.104) $V_{\overline{o}}(E_3) = 1/2$. At $p = 5$, take $E_5 = \mathcal{E}(\mathbb{Z}_5)$. By formula (2.109), $V_{\overline{o}}(E_5) = 29/140$. For $p > 5$, take $E_p = \mathcal{E}(\widehat{\mathcal{O}}_p)$. Since the given model of $\mathcal{E}$ and the parameter z have good reduction at p, $V_{\overline{o}}(E_p) = 0$.

Let $\mathbb{E} = \prod_{p,\infty} E_p$, and take $\mathfrak{X} = \{\overline{o}\}$. Then

$$V(\mathbb{E}, \mathfrak{X}) \;=\; -\frac{1}{3}\ln(R) + \frac{10}{9}\ln(2) + \frac{1}{2}\ln(3) + \frac{29}{140}\ln(5)\ .$$

The value of R for which $V(\mathbb{E}, \mathfrak{X}) = 0$ satisfies

$$142.388571237 \;<\; R \;<\; 142.388571238\ ,$$

and the Fekete-Szegö Theorems 0.4 and 1.5 yield the result. □

6. The Fermat Curve

Let $p \geq 3$ be an odd prime, and let $\zeta = e^{2\pi i/p}$. In this section we will apply the Fekete-Szegö theorem with local rationality to the Fermat curve

$$\mathcal{F}^p : X^p + Y^p \;=\; Z^p\ , \tag{2.150}$$

taking the ground field to be $K = \mathbb{Q}$. Let $\mathfrak{F}^p/\operatorname{Spec}(\mathbb{Z})$ be the corresponding scheme. To obtain a nontrivial set $\mathbb{E}$, we make use of McCallum's description ([**42**]) of a regular model for $\mathfrak{F}^p_{v_p} := \mathfrak{F}^p \times \mathrm{Spec}(\mathcal{O}_{L,v_p})$, where $L = \mathbb{Q}(\zeta)$ and v_p is the unique place of L over p. The author thanks Dino Lorenzini for suggesting this example.

Writing $x = X/Z$, $y = Y/Z$, let the part of $\mathcal{F}^p$ in the coordinate patch $Z \neq 0$ be the affine curve

$$\mathcal{F}^{p,0} : \ x^p + y^p \ = \ 1 \ . \tag{2.151}$$

Let $\mathfrak{X} = \{\xi_1, \ldots, \xi_p\}$ be the set of points at infinity, where $\xi_k = (1 : -\zeta^k : 0)$, and take $\mathbb{E} = \prod_v E_v$ where the sets E_v are as follows: for the archimedean place, let

$$E_\infty \ = \ x^{-1}(D(0,R)) \ = \ \{z \in \mathcal{F}^p(\mathbb{C}) : |x(z)| \le R\} \ .$$

At the place p, take $E_p = \mathcal{F}^{p,0}(\mathcal{O}_{L,v_p})$, and for all the other nonarchimedean places q, take E_q to be the $\mathfrak{X}$-trivial set $E_q = \mathcal{F}^{p,0}(\widehat{\mathcal{O}}_q)$.

Let z vary over $\mathcal{F}^p(\mathbb{C})$. Writing (2.151) in the form

$$\prod_{k=1}^{p} (\frac{y}{x} + \zeta^k) \ = \ (\frac{1}{x})^p$$

we see that as $z \to \xi_k$, then $(y/x) + \zeta^k$ vanishes to order p; at each $\xi_k \in \mathfrak{X}$ we will take the local uniformizing parameter to be

$$g_{\xi_k}(z) = \frac{1}{x(z)} \ .$$

The Green's Matrix at the Archimedean Place. Since E_∞ and $|1/x(z)|$ are invariant under the automorphisms of $\mathcal{F}^p$ given by

$$(X : Y : Z) \ \mapsto \ (\zeta^k X : \zeta^\ell Y : Z) \ ,$$

while the ξ_k are permuted by those automorphisms, there are numbers A, B such that $G(\xi_k, \xi_\ell; E_\infty) = A$ and $V_{\xi_k}(E_\infty) = B$ for all $k \neq \ell$. Thus the archimedean local Green's matrix is

$$\Gamma(E_\infty, \mathfrak{X}) \ = \ \begin{pmatrix} B & A & \cdots & A \\ A & B & \cdots & A \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \cdots & B \end{pmatrix} . \tag{2.152}$$

Although we are unable to determine the numbers A, B explicitly, we will see below that

$$(p-1)A + B \ = \ -\log(R) \ . \tag{2.153}$$

This relation will enable us to determine the capacities we need.

For the divisor (∞) on $\mathbb{P}^1$, we have $x^{-1}((\infty)) = (\xi_1) + \cdots + (\xi_p)$, so the pullback formula (2.61) shows that for each $z \in \mathcal{F}^p(\mathbb{C})$,

$$G(x(z), \infty; D(0,R)) \ = \ \sum_{\ell=1}^{p} G(z, \xi_\ell, E_\infty) \ .$$

Since $G(w, \infty; D(0,R)) = \log^+(|w/R|)$ in $\mathbb{P}^1$, for each ξ_k we have

$$\begin{aligned} -\log(R) &= \lim_{z \to \xi_k} G(x(z), \infty; D(0,R)) + \log(|1/x(z)|) \\ &= V_{\xi_k}(E_\infty) + \sum_{\ell \neq k} G(\xi_k, \xi_\ell; E_\infty) \, , \end{aligned} \tag{2.154}$$

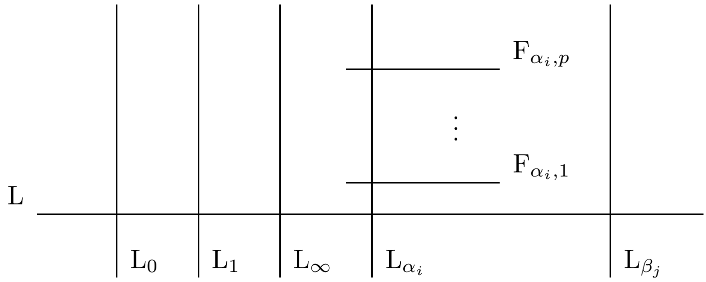

FIGURE 1. Fermat curve special fiber

and (2.153) follows.

The Green's Matrix at the Place p. Put $L = \mathbb{Q}(\zeta)$, and let v_p be the unique place of L above p; thus $\mathcal{O}_{L,v_p} \cong \mathbb{Z}_p[\zeta]$. Put $\pi_{v_p} = 1 - \zeta$. The residue field $k_v = \mathcal{O}_{L,v_p}/\pi_{v_p}\mathcal{O}_{L,v_p}$ is isomorphic to $\mathbb{F}_p$. Write $\mathcal{F}^p_{v_p} = \mathcal{F}^p \times_{\mathbb{Q}} \operatorname{Spec}(L_{v_p})$ and $\mathfrak{F}^p_{v_p} = \mathfrak{F}^p \times_{\mathbb{Z}} \operatorname{Spec}(\mathcal{O}_{L,v_p})$.

McCallum ([**42**], see Theorem 3, p.59; Diagram 3, p.69) has determined a regular model for $\mathcal{F}^p_{v_p}$. Put

$$\phi(x,y) \;=\; \frac{(x+y)^p - x^p - y^p}{p} \,.$$

Then $\phi(x,y)$ is a polynomial with integer coefficients, divisible by $xy(x+y)$. Let $\widetilde{\mathbb{F}}_p$ be the algebraic closure of $\mathbb{F}_p$; McCallum notes that $\phi(x,-y) \pmod p$ has a factorization over $\widetilde{\mathbb{F}}_p$ of the form

$$xy(x-y)\cdot\prod_i (x-\alpha_i y)^2 \cdot \prod_j (x-\beta_j y)$$

in which the $\alpha_i, \beta_j \in \widetilde{\mathbb{F}}_p$ are distinct, the α_i belong to $\mathbb{F}_p\backslash\{0,1\}$, and the β_j belong to $\widetilde{\mathbb{F}}_p\backslash\mathbb{F}_p$.

McCallum shows that there is a regular model $\mathfrak{G}^p_{v_p}/\operatorname{Spec}(\mathcal{O}_{L,v_p})$, gotten by blowing up $\mathfrak{F}^p_{v_p}$, whose geometric special fibre has the configuration shown in Figure 1. The components L_0, L_1, L_∞, L_{α_i} and L_{β_j} meeting L are indexed by the irreducible factors of $\phi(x,-y) \pmod p$, and for each α_i there are p components $\mathrm{F}_{\alpha_i,k}$ meeting L_{α_i}. All components are nonsingular and isomorphic to $\mathbb{P}^1/\widetilde{\mathbb{F}}_p$, and all intersections are transverse. The components L, L_0, L_1, L_∞, L_{α_i}, and $\mathrm{F}_{\alpha_i,j}$ are rational over $k_v = \mathbb{F}_p$; each L_{β_j} is rational over $\mathbb{F}_p(\beta_j)$. Furthermore L has multiplicity p and self-intersection -1; L_0, L_1, and L_∞ have multiplicity 1 and self-intersection $-p$; the L_{α_i} have multiplicity 2 and self-intersection $-p$; the L_{β_j} have multiplicity 1 and self-intersection $-p$; and the $\mathrm{F}_{\alpha_i,k}$ have multiplicity 1 and self-intersection -2.

The points of $\mathcal{F}^p_{v_p}(L_{v_p})$ specialize to the k_v-rational closed points of the k_v-rational multiplicity 1 components, which are not intersection points of components. There are p such points on each of L_0, L_1, and the components $\mathrm{F}_{\alpha_i,k}$. Each such point lifts to a subset of E_p isomorphic to $\pi_{v_p}\mathcal{O}_{L,v_p}$, and $E_p = \mathcal{F}^{p,0}(\mathcal{O}_{L,v_p})$ is the union of those subsets. On the other hand, $\xi_1, \ldots, \xi_p$ specialize to distinct k_v-rational closed points of L_∞.

Using Proposition 2.22 and the above description of E_p, we can determine $G(z, \xi_k; E_p)$ for each k. Since the computations are tedious, and the methods are the same as those in the proof of Theorem 2.21, we only give the final result: if n_p is the number of components L_{α_i}, and if

$$V \;=\; \frac{1}{p} + \frac{2p-1}{(2n_p+2)p - n_p}\ , \tag{2.155}$$

then for points $z \in \mathcal{F}^p(L_{v_p})$ specializing to L_∞,

$$G(z;\xi_k;E_p) \;=\; \frac{1}{p-1}\cdot\Big(V + \log_{v_p}\big(((z)\cdot(\xi_k))_{\mathfrak{G}^p_{v_p}}\big)\Big)$$

where $((z)\cdot(\xi_k))_{\mathfrak{G}^p_{v_p}}$ is the intersection number of the closures of z and ξ_k in the model $\mathfrak{G}^p_{v_p}$. The factor $1/(p-1)$ appears because the ramification index of $L_{v_p}/\mathbb{Q}_p$ is $p-1$. By analyzing the blowups in the construction of $\mathfrak{G}^p_{v_p}$, and writing $z \equiv_{\mathrm{L}_\infty} \xi_k$ if z and ξ_k specialize to the same closed point of L_∞, one further sees that

$$\log_{v_p}\big(((z)\cdot(\xi_k))_{\mathfrak{G}^p_{v_p}}\big) = \begin{cases} 0 & \text{if } z \not\equiv_{\mathrm{L}_\infty} \xi_k, \\ \log_{v_p}(|x(z)|_{v_p})) - 1 & \text{if } z \equiv_{\mathrm{L}_\infty} \xi_k\,. \end{cases}$$

Since $g_{\xi_k}(z) = 1/x(z)$ for each k, the local Green's matrix at p is

$$\Gamma(E_p, \mathfrak{X}) \;=\; \frac{1}{p-1}\begin{pmatrix} V-1 & V & \cdots & V \\ V & V-1 & \cdots & V \\ \vdots & \vdots & \ddots & \vdots \\ V & V & \cdots & V-1 \end{pmatrix}. \tag{2.156}$$

The Global Green's Matrix. For each prime q of $\mathbb{Q}$ with $q \neq p$, the model $\mathfrak{F}^p$ has good reduction at q, the points ξ_k specialize to distinct points of the special fibre, and the function $1/x(z)$ specializes to a nonconstant function $\pmod{q}$. Since E_q is $\mathfrak{X}$-trivial, $\Gamma(E_q, \mathfrak{X})$ is the zero matrix.

Thus the Global Green's matrix is

$$\Gamma(\mathbb{E}, \mathfrak{X}) = \Gamma(E_\infty, \mathfrak{X}) + \Gamma(E_p, \mathfrak{X}) \log(p)\ .$$

When $\vec{s} = {}^t(\frac{1}{p}, \cdots, \frac{1}{p}) \in \mathcal{P}^p(\mathbb{R})$, entries of $\Gamma(\mathbb{E}, \mathfrak{X})\vec{s}$ are all equal, so using (2.153) we conclude that

$$\begin{aligned} V(\mathbb{E}, \mathfrak{X}) \;&=\; \frac{1}{p}\Big(\big(B + (p-1)A\big) + \frac{1}{p-1}\big(pV - 1\big)\Big) \\ &=\; \frac{1}{p}\big(-\log(R) + \frac{pV-1}{p-1}\big) \end{aligned}$$

Thus, by (2.155) and the Fekete-Szegö Theorems 0.4 and 1.5 we obtain:

THEOREM 2.29. *Let p be an odd prime. Then on the affine Fermat curve $x^p + y^p = 1$, if*

$$R \;>\; p^{\frac{p(2p-1)}{(p-1)^2((2n_p+2)p-n_p)}}\ , \tag{2.157}$$

there are infinitely many integral points α whose p-adic conjugates are all rational over L_{v_p} and whose archimedean conjugates satisfy $|x(\sigma(\alpha))| < R$.

If the inequality (2.157) *is reversed, there are only finitely many.*

For small primes, n_p can be computed using Maple. For $p = 2$ and $p = 5$, we have $n_p = 0$; for all primes with $5 < p < 75$ except $p = 59$, we have $n_p = 2$; for $p = 59$ we have $n_p = 13$. Below are some examples for the critical value of R:

p	n_p	critical R	
3	0	$3^{5/8}$	$\cong 1.987013346$
5	0	$5^{9/32}$	$\cong 1.572480664$
7	2	$7^{91/1440}$	$\cong 1.130851299$
53	2	$53^{5565/854464}$	$\cong 1.026195152$
59	13	$59^{6903/5513596}$	$\cong 1.005118113$
61	2	$61^{7381/1310400}$	$\cong 1.023425196$
73	2	$73^{10585/2260224}$	$\cong 1.020296147$

As McCallum remarks, n_p is the number of "tame curves" C_s for which $\mathrm{Jac}(C_s)$ is isogenous to a factor of $\mathrm{Jac}(\mathcal{F}_p)$. For $p = 59$ the abnormally large number of tame curves means the critical value of R is unusually small. It would be interesting to know if there are other phenomena related to this.

7. The Modular Curve $X_0(p)$

In this section we will give an example applying the Fekete-Szegö theorem with local rationality to the modular curve $X_0(p)/\mathbb{Q}$, where $p \geq 5$ is prime. The author thanks Pete Clark for suggesting this, and for help with properties of orders.

As is well known, $X_0(p)$ is the compactification of the moduli space for pairs (E, C) consisting of an elliptic curve and a cyclic subgroup of order p. As a Riemann surface, $X_0(p)(\mathbb{C})$ is gotten from $\Gamma_0(p)\backslash\mathfrak{H}$ by adjoining the "cusps" c_0 and c_∞; here $\mathfrak{H}$ is the complex upper half-plane and $\Gamma_0(p)$ is the congruence subgroup

$$\Gamma_0(p) \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{p} \right\} .$$

The function field of $X_0(p)/\mathbb{Q}$ is $\mathbb{Q}(j(z), j(pz))$ where $j(z)$ is the modular function

$$j(z) \;=\; \frac{1728 g_2^3}{g_2^3 - 27 g_3^2} \;=\; \frac{1}{q} + 744 + 196884q + \cdots .$$

Here $X = j(z)$ and $Y = j(pz)$ satisfy the "Modular Equation" $\Phi(X, Y) = 0$, where

$$\Phi(X, Y) \;=\; -(X^p - Y)(Y^p - X) + \sum_{\max(i,j) \leq p} a_{ij} X^i Y^j \;\in\; \mathbb{Z}[X, Y]$$

and each a_{ij} is divisible by p. The genus of $X_0(p)$ is

$$g_p \;=\; \begin{cases} (p-13)/12 & \text{if } p \equiv 1 \pmod{12}, \\ (p-5)/12 & \text{if } p \equiv 5 \pmod{12}, \\ (p-7)/12 & \text{if } p \equiv 7 \pmod{12}, \\ (p+1)/12 & \text{if } p \equiv 11 \pmod{12}, \end{cases}$$

Deligne and Rapoport determined a regular model $\mathfrak{M}_0(p)/\operatorname{Spec}(\mathbb{Z})$ for $X_0(p)$. It can be described as follows (see [**41**], Theorem 1.1, p.175). First, consider the projective normalization $M_0(p)$ of $\operatorname{Spec}(\mathbb{Z}[X, Y]/(\Phi(X, Y))$. It is smooth outside the points corresponding to supersingular elliptic curves in characteristic p with $j \neq 0, 1728$; its special fibre at p has two components, each isomorphic to $\mathbb{P}^1$, which meet transversely at the supersingular points. These components will be denoted

Z_0 and Z_∞; the reduction of j (that is, X) is a coordinate function on Z_∞. If $p \equiv 2 \pmod 3$ then $j = 0$ is supersingular in the fibre at p, and $M_0(p)$ has a singularity of type A_3 at the corresponding point; if $p \equiv 3 \pmod 4$ then $j = 1728$ is supersingular in the fibre at p and $M_0(p)$ has a singularity of type A_2 at the corresponding point.

The model $\mathfrak{M}_0(p)$ is obtained by resolving these singularities, introducing a chain of two components F_1, F_2 in the first case, and a single component G in the second. The special fibre of $\mathfrak{M}_0(p)$ is reduced, and all its components are rational over $\mathbb{F}_p$. There are $m = g_p + 1$ supersingular points, each of which is rational over $\mathbb{F}_{p^2}$. The components Z_0 and Z_∞ have self-intersection $-m$; the components F_1, F_2, and G (if present) have self-intersection -2. The cusps are rational over $\mathbb{Q}$; c_0 specializes to Z_0, and c_∞ specializes to Z_∞. Their images are not supersingular, and are the points "at infinity" on those components.

We will take $\mathfrak{X} = \{c_\infty, c_0\}$ to be the set of cusps, and we will take $\mathbb{E} = \prod_v E_v$, where

$$E_\infty \;=\; j^{-1}(D(0,R)) \;=\; \{z \in X_0(p)(\mathbb{C}) : |j(z)| \le R\} \ .$$

and where E_p is the set of points of $X_0(p)(\mathbb{Q}_p)$ specializing to the "ordinary" (i.e., nonsupersingular and noncuspidal) points of Z_∞. For all the other nonarchimedean places q, we will take E_q to be the $\mathfrak{X}$-trivial set

$$E_q \;= \mathfrak{M}_0(p)(\mathbb{C}_p)\backslash(B(c_0,1)^- \bigcup B(c_\infty,1)^-) \ .$$

We will take the uniformizing parameters to be $g_{c_\infty}(z) = 1/j(z)$, $g_{c_0}(z) = 1/j(pz)$.

This set $\mathbb{E}$ is chosen mainly because we can do explicit computations with it. However, it illustrates nicely how arithmetic and geometric information about a curve enter into capacities.

The Green's Matrix at the Archimedean Place. Let $\mathcal{D} = \{z \in \mathfrak{H} : -1/2 \le \mathrm{Re}(z) \le 1/2, |z| \ge 1\}$ be the standard closed fundamental domain for $SL_2(\mathbb{Z})$. As a function from $\mathfrak{H}$ to $\mathbb{C}$, $j(z)$ maps this region conformally onto $\mathbb{C}$, taking the ray from i to ∞ along the imaginary axis to the real interval $[1728, \infty)$, with $j(i) = 1728$; the circular arc at the bottom of $\mathcal{D}$ to the real interval $[0, 1728]$ (covering it twice), with $j(e^{\pi i/3}) = j(e^{2\pi i/3}) = 0$; and the vertical sides of $\mathcal{D}$ to the real interval $[-\infty, 0]$. It also takes the part of the imaginary axis from 0 to i to $[1728, \infty)$. A fundamental domain for $\Gamma_0(p)$ is given by

$$\mathcal{D}(p) \;=\; \mathcal{D} \cup \Big(\bigcup_{k=-(p-1)/2}^{(p-1)/2} f_k(\mathcal{D})\Big)$$

where $f_k(z) = -1/(z+k)$. Under the quotient $\Gamma_0(p)\backslash\mathfrak{H}$, the image of the circular arc at the bottom of $\mathcal{D}$ separates $X_0(p)(\mathbb{C})$ into two components, one containing c_∞ and the other containing c_0. On the other hand, the image of the imaginary axis joins the cusps c_0, c_∞.

By our choice of E_∞ and discussion above, it follows that when $j(z)$ is viewed as a map from $X_0(p)(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{C})$, if $R \ge 1728$ then $X_0(p)(\mathbb{C})\backslash E_\infty$ has two connected components, while if $R < 1728$ it has one component.

As a divisor $j^{-1}((\infty)) = p(c_0) + (c_\infty)$, so the pullback formula (2.61) gives

$$G(j(z), \infty; D(0,R)) \;=\; pG(z, c_0; E_\infty) + G(z, c_\infty; E_\infty) \ .$$

Since $1/j(z)$ is the uniformizing parameter at c_∞, it follows that

$$-\log(R) \;=\; pG(c_\infty, c_0; E_\infty) + V_{c_\infty}(E_\infty) \ .$$

Similarly, since $\lim_{z\to c_0} j(z)^p/j(pz) = 1$, and since $1/j(pz)$ is the uniformizing parameter at c_0,

$$-\log(R) \;=\; G(c_0,c_\infty;E_\infty)+pV_{c_0}(E_\infty)\ .$$

Hence, writing $B(R) = G(c_\infty,c_0;E_\infty) = G(c_0,c_\infty;E_\infty)$, the archimedean local Green's matrix is

$$\Gamma(E_\infty,\mathfrak{X}) \;=\; \begin{pmatrix} -\log(R) & 0 \\ 0 & -\frac{1}{p}\log(R)\end{pmatrix} + B(R)\begin{pmatrix} -p & 1 \\ 1 & -1\end{pmatrix}\ . \tag{2.158}$$

Here $B(R) = 0$ if $R \geq 1728$, while $B(R) > 0$ if $R < 1728$. It will turn out that $R > 1728$ in the situation of interest to us; however, note that in any case the second matrix in (2.158) is negative semi-definite.

The Green's Matrix at the Place p. Using Proposition 2.22 and the definition of E_p as the set of points of $X_0(\mathbb{Q}_p)$ specializing to ordinary points on the component Z_∞, we can determine $G(z,c_\infty;E_p)$ and $G(z,c_0;E_p)$.

Let $\mathcal{N}_p$ be the number of $\mathbb{F}_p$-rational ordinary points on Z_∞. For $z \in X_0(p)(\mathbb{C}_p)$, write $z \equiv_{Z_0} c_0$ if z specializes to same point of Z_0 as c_0, and write $z \equiv_{Z_\infty} c_\infty$ specializes to the same point of Z_∞ as c_∞. Using Proposition 2.22 and the methods in the proof of Theorem 2.21, we find that

$$G(z,c_\infty;E_p) \;=\; \begin{cases} \frac{1}{\mathcal{N}_p}\frac{p}{p-1} & \text{if } z\equiv_{Z_0} c_0, \\ \frac{1}{\mathcal{N}_p}\frac{p}{p-1} + \log_p(|j(z)|_p) & \text{if } z\equiv_{Z_\infty} c_\infty, \end{cases}$$

$$G(z,c_0;E_p) \;=\; \begin{cases} \frac{1}{\mathcal{N}_p}\frac{p}{p-1} & \text{if } z\equiv_{Z_\infty} c_\infty, \\ \frac{1}{\mathcal{N}_p}\frac{p}{p-1} + \frac{12}{p-1} + \log_p(|j(pz)|_p) & \text{if } z\equiv_{Z_0} c_0. \end{cases}$$

Here the number $12/(p-1)$ is actually the quantity $j_{c_0}(Z_\infty,Z_\infty)$ in the notation of (2.131), obtained by solving the equations (2.128) relating components. Since the special fibre of $\mathfrak{M}_0(p)$ at p has different configurations according as $p \equiv 1,5,7,11 \pmod{12}$, it is somewhat surprising that the same value arises in all cases.

It follows that the local Green's matrix at p is

$$\Gamma(E_p,\mathfrak{X}) \;=\; \begin{pmatrix} \frac{1}{\mathcal{N}_p}\frac{p}{p-1} & \frac{1}{\mathcal{N}_p}\frac{p}{p-1} \\ \frac{1}{\mathcal{N}_p}\frac{p}{p-1} & \frac{1}{\mathcal{N}_p}\frac{p}{p-1}+\frac{12}{p-1}\end{pmatrix}\ . \tag{2.159}$$

The number $\mathcal{N}_p$ can be expressed in terms of the class number $h(-p)$ of the ring of integers of $\mathbb{Q}(\sqrt{-p})$. Indeed, $\mathcal{N}_p = p - n_{ss}(\mathbb{F}_p)$, where $n_{ss}(\mathbb{F}_p)$ is the number of $\mathbb{F}_p$-rational supersingular points on Z_∞. It is known (see for example [**20**], pp.75-76) that

$$n_{ss}(\mathbb{F}_p) \;=\; \frac{h'(-p)+h'(-4p)}{2}$$

where $h'(D)$ is the class number of the quadratic order of discriminant D if there is such an order, and is 0 otherwise. Using the formula relating class numbers of orders in quadratic fields to those of the maximal orders (see [**36**], Theorem 7, p.95), this simplifies to $n_{ss}(\mathbb{F}_p) = c_p h(-p)$, where

$$c_p \;=\; \begin{cases} 1/2 & \text{if } p\equiv 1 \pmod 8, \\ 2 & \text{if } p\equiv 3 \pmod 8, \\ 1/2 & \text{if } p\equiv 5 \pmod 8, \\ 1 & \text{if } p\equiv 7 \pmod 8. \end{cases} \tag{2.160}$$

Thus $\mathcal{N}_p = p - c_p h(-p)$. It is known that $\mathcal{N}_p$ is always positive, so E_p is nonempty.

The Global Green's Matrix. For each prime q of $\mathbb{Q}$ with $q \neq p$, the model $\mathfrak{M}_0(p)$ has good reduction at q, the cusps c_∞, c_0 specialize to distinct points of the special fibre, and the uniformizing parameters $g_{c_\infty}(z)$ and $g_{c_0}(z)$ specialize to nonconstant functions $\pmod q$. Since E_q is $\mathfrak{X}$-trivial, $\Gamma(E_q, \mathfrak{X})$ is the zero matrix.

Suppose for the moment that $R \geq 1728$; this assumption will be justified below. Then $B(R) = 0$, and the global Green's matrix $\Gamma(\mathbb{E}, \mathfrak{X})$ is

$$\begin{pmatrix} -\log(R) + \frac{1}{\mathcal{N}_p}\frac{p}{p-1}\log(p) & \frac{1}{\mathcal{N}_p}\frac{p}{p-1}\log(p) \\ \frac{1}{\mathcal{N}_p}\frac{p}{p-1}\log(p) & -\frac{1}{p}\log(R) + (\frac{1}{\mathcal{N}_p}\frac{p}{p-1} + \frac{12}{p-1})\log(p) \end{pmatrix} .$$

By the minimax definition of $V(\mathbb{E}, \mathfrak{X})$ (see formula (3.50) below)

$$V(\mathbb{E}, \mathfrak{X}) \ = \ \min_{\vec{s} \in \mathcal{P}^2(\mathbb{R})} \max_i \big(\Gamma(\mathbb{E}, \mathfrak{X})\vec{s}\big)_i \ .$$

Thus $V(\mathbb{E}, \mathfrak{X}) < 0$ if and only if for some $\vec{s} = {}^t(s_1, s_2) \in \mathcal{P}^2(\mathbb{R})$,

$$\begin{cases} s_1\big(-\log(R) + \frac{1}{\mathcal{N}_p}\frac{p}{p-1}\log(p)\big) + s_2\big(\frac{1}{\mathcal{N}_p}\frac{p}{p-1}\log(p)\big) \ < \ 0 \ , \\ s_1\big(\frac{1}{\mathcal{N}_p}\frac{p}{p-1}\log(p)\big) + s_2\big(-\frac{1}{p}\log(R) + \frac{1}{\mathcal{N}_p}\frac{p}{p-1} + \frac{12}{p-1}\log(p)\big) \ < \ 0 \ . \end{cases}$$

Equivalently, $V(\mathbb{E}, \mathfrak{X}) < 0$ if and only if for some $s \in \mathbb{R}$ with $0 < s < 1$,

$$\begin{cases} \frac{1}{s} \cdot \big(\frac{1}{\mathcal{N}_p}\frac{p}{p-1}\big) \ < \log_p(R) \ , \\ \frac{1}{1-s} \cdot \big(\frac{1}{\mathcal{N}_p}\frac{p^2}{p-1}\big) + \frac{12p}{p-1} \ < \ \log_p(R) \ . \end{cases} \tag{2.161}$$

The left side of the first inequality in (2.161) is decreasing with s, while that in the second inequality is increasing, so the extremal value of R is obtained when they are equal. Solving, and using the Fekete-Szegö Theorems 0.4 and 1.5, one obtains

THEOREM 2.30. *Let $p \geq 5$ be a prime, and consider the Deligne-Rapoport model $\mathfrak{M}_0(p)$ for modular curve $X_0(p)/\mathbb{Q}$. Put $\mathcal{N}_p = p - c_p h(-p)$, where c_p is as in* (2.160) *and $h(-p)$ is the class number of $\mathbb{Q}(\sqrt{-p})$. Then if*

$$R \ > \ p^{\frac{1+p+12\mathcal{N}_p + \sqrt{(1+p+12\mathcal{N}_p)^2 - 48\mathcal{N}_p}}{2\mathcal{N}_p} \cdot \frac{p}{p-1}} \ , \tag{2.162}$$

there are infinitely many $\alpha \in X_0(p)(\widetilde{\mathbb{Q}})$ with archimedean conjugates which satisfy $|j(\sigma(\alpha))| < R$, whose p-adic conjugates all belong to $X_0(p)(\mathbb{Q}_p)$ and specialize $\pmod p$ *to ordinary points in Z_∞, and whose conjugates in $X_0(\mathbb{C}_q)$ specialize* mod q *to noncuspidal points of $\mathfrak{M}_0(p)$, for all $q \neq p$.*

If the inequality (2.162) *is reversed, there are only finitely many.*

Note that the right side of (2.162) is greater than p^6, and for $p \geq 5$ this is at least 15625. By the second part of the theorem and the monotonicity of the sets $j^{-1}(D(0, R))$, the first part cannot hold for any $R < 15625$. This validates our assumption that $R > 1728$.