Introduction

Speaking about *noise* we usually mean something that deteriorates the operation of a system. It is understood as a disturbance, a random and persistent one, that obscures or reduces the clarity of a signal.

In nonlinear dynamical systems, however, noise may play a very constructive role. It may enhance a system's sensitivity to a small periodic deterministic signal by amplifying it. The optimal amplification of small periodic signals by noise gives rise to the ubiquitous phenomenon of *stochastic resonance* (SR) well studied in a plethora of papers in particular in the physical and biological sciences. This book presents a mathematical approach to stochastic resonance in a well defined particular mathematical framework. We consider weakly periodic systems with additive noise of small amplitude ε. The systems possess two domains of attraction of stable equilibria separated by a manifold marking a barrier. Both the geometry of the attraction domains as well as the barrier height are not subject to scalings with the amplitude parameter ε. Therefore, as opposed to other approaches, noise induced random transitions in our model happen on time scales of Kramers' law, i.e. they are exponential in the quotient of barrier height and noise amplitude, and are due to large deviations. Our analysis is therefore based on a new large deviations principle of the systems' exit and transition dynamics between different domains of attraction in the limit of small ε. It aims at the description of an optimal interplay between large period length T of the weak periodic motion and noise amplitude ε, where optimization is done with respect to appropriate measures of quality of response of the stochastic system to the periodic input. We will be uniquely concerned with the well founded and self contained presentation of this mathematical approach mainly based on a space-time extension of Freidlin–Wentzell's theory of large deviations of noisy dynamical systems, first on a heuristic and then on a mathematically rigorous level. The two principal messages of the book are these. First we show that — already in space dimension one — the classical physical measures of quality of periodic tuning such as the spectral power amplification, due to the phenomenon of the *small oscillations catastrophe*, are not robust with respect to dimension reduction. Comparing optimal tuning rates for the diffusion processes and the finite state Markov chains retaining the models' essentials one gets essentially different results (Chapter 3, Theorems 3.50, 3.53). We therefore propose — in arbitrary finite space dimension — measures of quality of periodic tuning based uniquely on the transition dynamics and show that these measures are robust and, via a crucial large deviations result, are able to explain stochastic resonance as optimal tuning (Chapter 4, Theorems 4.19, 4.29, 4.31). Concentrating on these more theoretical themes, the book sheds some light on the mathematical shortcomings and strengths of different concepts used in theory and application of stochastic resonance, in a well defined framework. It does not aim at a comprehensive presentation of the many facets of stochastic resonance in various areas of sciences (a sample will be briefly discussed in Chapter 1, Section 1.5). In particular it does not touch computational aspects relevant in particular in high dimensions where analytical methods alone are too complex to be of practical use any more (for an incomplete overview of stochastic resonance from a computational dynamics perspective see also Chapter 1, Section 1.5).

We now explain briefly our motivation and approach. The most prominent and one of the first examples in which phenomena related to stochastic resonance were observed is given by energy balance models of low dimensional conceptual climate dynamics. It was employed for a qualitative explanation of *glacial cycles* in earth's history, i.e. the succession of ice and warm ages observed in paleoclimatic data, by means of stochastic transitions between cold and warm meta-stable climates in a dynamical model. It will be discussed in more detail in Chapter 1. The model proposed by Nicolis [**83**] and Benzi et al. [**6**] is based on the balance between averaged absorbed and emitted radiative energy and leads to a deterministic differential equation for averaged global temperature T of the form

$$\dot{T}(t) = b(t, T(t)).$$

The explicit time dependence of b captures the influence of the solar constant that undergoes periodic fluctuations of a very small amplitude at a very low frequency. The fluctuations are due to periodic changes of the earth's orbital parameters (Milankovich cycles), for instance a small variation of the axial tilt that arises at a frequency of roughly 4×10^{-4} times per year, and coincide roughly with the observed frequencies of cold and warm periods. For frozen t the nonlinear function $b(t, T)$ describes the difference between absorbed radiative energy as a piecewise linear function of the temperature dependent albedo function $a(T)$ and emitted radiative energy proportional to T^4 due to the Stefan–Boltzmann law of black body radiators. In the balance for relevant values of T it can be considered as negative gradient (force) of a double well potential, for which the two well bottoms correspond to stable temperature states of glacial and warm periods. The evolution of temperature in the resulting deterministic dynamical system is analogous to the motion of an overdamped physical particle subject to the weakly periodic force field of the potential. Trajectories of the deterministic system relax to the stable states of the domain of attraction in which they are started. Only the addition of a *stochastic forcing* to the system allows for spontaneous transitions between the different stable states which thus become *meta-stable*.

In a more general setting, we study a dynamical system in d-dimensional Euclidean space perturbed by a d-dimensional Brownian motion W, i.e. we consider the solution of the stochastic differential equation

$$dX_t^\varepsilon = b\Big(\frac{t}{T}, X_t^\varepsilon\Big)\, dt + \sqrt{\varepsilon}\, dW_t, \quad t \geq 0. \tag{0.1}$$

One of the system's important features is that its time inhomogeneity is weak in the sense that the drift depends on time only through a re-scaling by the time parameter $T = T(\varepsilon)$ which will be assumed to be exponentially large in ε. This corresponds to the situation in Herrmann and Imkeller [**50**] and is motivated by the well known Kramers–Eyring law which was mathematically underpinned by the Freidlin–Wentzell theory of large deviations [**40**]. The law roughly states that the expected time it takes for a homogeneous diffusion to leave a local attractor e.g.

across a potential wall of height $\frac{v}{2}$ is given to exponential order by $T(\varepsilon) = \exp(\frac{v}{\varepsilon})$. Hence, only in exponentially large scales of the form $T(\varepsilon) = \exp(\frac{\mu}{\varepsilon})$ parametrized by an energy parameter μ we can expect to see effects of transitions between different domains of attraction. We remark at this place that our approach essentially differs from the one by Berglund and Gentz [**13**]. If b represents a negative potential gradient for instance, their approach would typically not only scale time by T, but also the depths of the potential wells by a function of ε. As a consequence, transitions even for the deterministic dynamical system become possible, and their noise induced transitions happen on time scales of intermediate length. In contrast, in our setting transitions between the domains of attraction of the deterministic system are impossible, and noise induced ones are observed on very large time scales of the order of Kramers' time, typically as consequences of large deviations. The function b is assumed to be one-periodic w.r.t. time, and so the system described by (0.1) attains period T by re-scaling time in fractions of T. The deterministic system $\dot{\xi}_t = b(s, \xi_t)$ with *frozen* time parameter s is supposed to have two domains of attraction that do not depend on $s \geq 0$. In the "classical" case of a drift derived from a potential, $b(t, x) = -\nabla_x U(t, x)$ for some potential function U, equation (0.1) is analogous to the overdamped motion of a Brownian particle in a d-dimensional time inhomogeneous double-well potential. In general, trajectories of the solutions of differential equations of this type will exhibit randomly periodic behavior, reacting to the periodic input forcing and eventually amplifying it. The problem of *optimal tuning* at large periods T consists in finding a noise amplitude $\varepsilon(T)$ (the *resonance point*) which supports this amplification effect in a *best possible way.* During the last 20 years, various concepts of measuring the quality of periodic tuning to provide a criterion for optimality have been discussed and proposed in many applications from a variety of branches of natural sciences (see Gammaitoni et al. [**43**] for an overview). Its rigorous mathematical treatment was initiated only relatively late.

The first approach towards a mathematically precise understanding of stochastic resonance was initiated by Freidlin [**39**]. To explain stochastic resonance in the case of diffusions in potential landscapes with finitely many minima (in the more general setting of (0.1), the potential is replaced by a quasi-potential related to the action functional of the system), he goes as far as basic large deviations' theory can take. If noise intensity is ε, in the absence of periodic exterior forcing, the exponential order of times at which successive transitions between meta-stable states occur corresponds to the work to be done against the potential gradient to leave a well (*Kramers' time*). In the presence of periodic forcing with period time scale $e^{\frac{\mu}{\varepsilon}}$, in the limit $\varepsilon \to 0$ transitions between the stable states with critical transition energy close to μ will be periodically observed. Transitions with smaller critical energy may happen, but are negligible in the limit. Those with larger critical energy are forbidden. In case the two local minima of the potential have depths $\frac{V}{2}$ and $\frac{v}{2}$, $v < V$, that switch periodically at time $\frac{1}{2}$ (in scale T accordingly at time $\frac{T}{2}$), for T larger than $e^{\frac{v}{\varepsilon}}$ the diffusion will be close to the deterministic periodic function jumping between the locations of the deepest wells. As T exceeds this exponential order, many short excursions to the wrong well during one period may occur. They will not count on the exponential scale, but trajectories will look less and less periodic. It therefore becomes plausible that physicists' *quality measures for periodic tuning* which always feature some *maximal* tuning quality of the random trajectories to the periodic input signal cannot be captured by this phenomenon of quasi-deterministic periodicity at very large time scales.

These quality measures, studied in Pavlyukevich [**86**] and Imkeller and Pavlyukevich [**59**] assess quality of tuning of the stochastic *output* to the periodic deterministic *input*. The concepts are mostly based on comparisons of trajectories of the noisy system and the deterministic periodic curve describing the location of the relevant meta-stable states, averaged with respect to the equilibrium measure (of the diffusion as a space-time process with time component given by uniform motion in the period interval). Again in the simple one-dimensional situation considered above the system switches between a double well potential state U with two wells of depths $\frac{V}{2}$ and $\frac{v}{2}$, $v < V$, during the first half period, and the spatially opposite one $U(\cdot)$ for the second half period. If as always time is re-scaled by T, the total period length is T, and stochastic perturbation comes from the coupling to a white noise of intensity ε. The most important measures of quality studied are the *spectral power amplification* and the related *signal-to-noise ratio*, both playing an eminent role in the physical literature (see Gammaitoni et al. [**43**], Freund et al. [**41**]). They mainly contain the mean square average in equilibrium of the Fourier component of the solution trajectories corresponding to the input period T, normalized in different ways. These measures of quality are functions of ε and T, and the problem of finding the resonance point consists in optimizing them in ε for fixed (large) T.

Due to the high complexity of original systems, when calculating the resonance point at optimal noise intensity, physicists usually pass to an effective dynamics description. It is given by a simple caricature of the system reducing the diffusion dynamics to the pure *inter well motion* (see e.g. McNamara and Wiesenfeld [**74**]). The reduced dynamics is represented by a continuous time two state Markov chain with transition probabilities corresponding to the inverses of the diffusions' Kramers' times. One then determines the optimal tuning parameters $\varepsilon(T)$ for large T for the approximating Markov chains in equilibrium, a rather simple task. To see that the Markov chain's behavior approaches the diffusion's in the small noise limit, spectral theory for the infinitesimal generator is used. The latter is seen to possess a spectral gap between the second and third eigenvalues, and therefore the closeness of equilibrium measures can be well controlled. Surprisingly, due to the importance of small *intra well fluctuations*, the tuning and resonance pattern of the Markov chain model may differ dramatically from the resonance picture of the diffusion. Subtle dependencies on the geometrical fine structure of the potential function U in the minima beyond the expected curvature properties lead to quite unexpected counterintuitive effects. For example, a subtle drag away from the other well caused by the sign of the third derivative of U in the deep well suffices to make the spectral power amplification curve strictly increasing in the parameter range where the approximating Markov chain has its resonance point.

It was this lack of robustness against model reduction which motivated Herrmann and Imkeller [**50**] to look for different measures of quality of periodic tuning for diffusion trajectories. These notions are designed to depend only on the rough inter well motion of the diffusion. The measure treated in the setting of one-dimensional diffusion processes subject to periodic forcing of small frequency is related to the transition probability during a fixed time window of exponential length $T(\varepsilon) = \exp(\frac{\mu}{\varepsilon})$ parametrized by a free energy parameter μ according to the

Kramers–Eyring formula. The corresponding exit rate is maximized in μ to account for optimal tuning. The methods of investigation of stochastic resonance in [**50**] are heavily based on comparison arguments which are not an appropriate tool from dimension 2 on. Time inhomogeneous diffusion processes such as the ones under consideration are compared to piecewise homogeneous diffusions by freezing the potential's time dependence on small intervals.

In Herrmann et al. [**51**] this approach is extended to the general setting of *finite dimensional diffusion processes* with two meta-stable states. Since the stochastic resonance criterion considered in [**50**] is based on transition times between them, our analysis relies on a suitable notion of transition or exit time parametrized again by the free energy parameter μ from $T(\varepsilon) = \exp(\frac{\mu}{\varepsilon})$ as a natural measure of scale. Assume now that the depths of the two equilibria of the potential in analogy to the scenarios considered before are smooth periodic functions of time of period 1 given for one of them by $\frac{v(t)}{2}$, and for the other one by the same function with some phase delay (for instance by $\frac{1}{2}$). Therefore, at time s the system needs energy $v(s)$ to leave the domain of attraction of the equilibrium. Hence an exit from this set should occur at time

$$a_\mu = \inf\{t \geq 0\colon v(t) \leq \mu\}$$

in the diffusion's natural time scale, in the time re-scaled by $T(\varepsilon)$ thus at time $a_\mu \cdot T(\varepsilon)$. To find a quality measure of periodic tuning depending only on the transition dynamics, we look at the probabilities of transitions to the other domain within a time window $[(a_\mu - h)T(\varepsilon), (a_\mu + h)T(\varepsilon)]$ centered at $a_\mu \cdot T(\varepsilon)$ for small $h > 0$. If τ is the random time at which the diffusion roughly reaches the other domain of attraction (to be precise, one has to look at first entrance times of small neighborhoods of the corresponding equilibrium), we use the quantity (again, to be precise, we use the worst case probability for the diffusion starting in a point of a small neighborhood of the equilibrium of the starting domain)

$$\mathcal{M}^h(\varepsilon, \mu) = \mathbb{P}\Big(\tau \in [(a_\mu - h)T(\varepsilon), (a_\mu + h)T(\varepsilon)]\Big).$$

To symmetrize this quality measure with respect to switching of the equilibria, we refine it by taking its minimum with the analogous expression for interchanged equilibria. In order to exclude trivial or chaotic transition behavior, the scale parameter μ has to be restricted to an interval I_R of reasonable values which we call *resonance interval*. With this measure of quality, the stochastic resonance point may be determined as follows. We first fix ε and the window width parameter $h > 0$, and maximize $\mathcal{M}^h(\varepsilon, \mu)$ in μ, eventually reached for the time scale $\mu_0(h)$. Then the eventually existing limit $\lim_{h\to 0} \mu_0(h)$ will be the *resonance point*.

To calculate $\mu_0(h)$ for fixed positive h we use large deviations techniques. In fact, our main result consists in an extension of the Freidlin–Wentzell large deviations result to weakly time inhomogeneous dynamical systems perturbed by small Gaussian noise which states that

$$\lim_{\varepsilon\to 0} \varepsilon \ln\Big(1 - \mathcal{M}^h(\varepsilon, \mu)\Big) = \mu - v(a_\mu - h),$$

again in a form which is symmetric for switched equilibria. We show that this asymptotic relation holds uniformly w.r.t. μ on compact subsets of I_R, a fact which enables us to perform a maximization and find $\mu_0(h)$. The resulting notion of stochastic resonance is strongly related to the notions of periodic tuning based on *interspike intervals* (see [**49**]), which describe the probability distribution for transitions as functions of time with exponentially decaying spikes near the integer multiples of the forcing periods. It has the big advantage of being robust for model reduction, i.e. the passage from the diffusion to the two state Markov chain describing its reduced dynamics.

The techniques needed to prove this main result feature non-trivial extensions and refinements of the fundamental large deviations theory for time homogeneous diffusions by Freidlin–Wentzell [**40**]. We prove a large deviations principle for the inhomogeneous diffusion (0.1) and further strengthen this result to get uniformity in system parameters. Similarly to the time homogeneous case, where large deviations theory is applied to the problem of *diffusion exit* culminating in a mathematically rigorous proof of the Kramers–Eyring law, we study the problem of diffusion exit from a domain which is carefully chosen in order to allow for a detailed analysis of transition times. The main idea behind our analysis is that the natural time scale is so large that re-scaling in these units essentially leads to an asymptotic freezing of the time inhomogeneity, which has to be carefully controlled, to hook up to the theory of large deviations of time homogeneous diffusions.

The material in the book is organized as follows. In Chapter 1 we give a detailed treatment of the heuristics behind our mathematical approach, mostly in space dimension 1. We start by giving a fairly thorough account of the paradigm of glacial cycles which was the historical root of physical models exhibiting stochastic resonance. It gives rise to the model equation of a weakly periodically forced dynamical system with noise that can be interpreted as the motion of an overdamped physical particle in a weakly periodically forced potential landscape subject to noise. The heuristics of exit and transition behavior between domains of attraction (potential wells) of such systems based on the classical large deviations theory is explained in two steps: first for time independent potential landscapes, then for potentials switching discontinuously between two anti-symmetric states every half period. Freidlin's quasi-deterministic motion is seen to not cover the concept of optimal periodic tuning between weak periodic input and randomly amplified output. They determine stochastic resonance through measures of quality of periodic tuning such as the spectral power amplification or the signal-to-noise ratio. The latter concepts are studied first for finite state Markov chains capturing the dynamics of the underlying diffusions reduced to the meta-stable states, and then for the diffusions with time continuous periodic potential functions. The robustness defect of the classical notions of resonance in passing from Markov chain to diffusion is pointed out. Then alternative notions of resonance are proposed which are based purely on the asymptotic behavior of transition times. Finally, examples of systems exhibiting stochastic resonance features from different areas of science are presented and briefly discussed. They document the ubiquity of the phenomenon of stochastic resonance.

Our approach is based on concepts of large deviations. Therefore Chapter 2 is devoted to a self-contained treatment of the theory of large deviations for randomly perturbed dynamical systems in finite dimensions. Following a direct and elegant approach of Baldi and Roynette [**3**], we describe Brownian motion in its Schauder decomposition. It not only allows a direct approach to its regularity properties in terms of Hölder norms on spaces of continuous functions. It also allows a derivation of Schilder's large deviation principle (LDP) for Brownian motion from the elementary LDP for one-dimensional Gaussian random variables. The key to this elegant and direct approach is Ciesielski's isomorphism of normed spaces of continuous functions with sequence spaces via Fourier representation. The proof of the LDP for Brownian motion using these arguments is given after recalling general notions and basic concepts about large deviations, especially addressing their construction from exponential decay rates of probabilities of basis sets of topologies, and their transport between different topological spaces via continuous mappings (contraction principle). Since we only consider diffusion processes with additive noise for which Itô's map is continuous, an appeal to the contraction principle provides the LDP for the homogeneous diffusion processes we study. Finally, we follow Dembo and Zeitouni [**25**] to derive the exit time laws due to Freidlin and Wentzell [**40**] for time homogeneous diffusions from domains of attraction of underlying dynamical systems in the small noise limit.

Chapter 3 deals with an approach to stochastic resonance for diffusions with weakly time periodic drift and additive noise in the spirit of the associated Markovian semigroups and their spectral theory. This approach, presented in space dimension 1, is clearly motivated by the physical notions of periodic tuning, in particular the spectral power amplification coefficient. It describes the average spectral component of the diffusion trajectories corresponding to the frequency of the periodic input signal given by the drift term. We first give a rigorous account of Freidlin's quasi-deterministic limiting motion for potential double well diffusions of this type. We then follow the paradigm of the physics literature, in particular NcNamara and Wiesenfeld [**74**], and introduce the effective dynamics of our weakly periodically forced double well diffusions given by reduced continuous time Markov chains jumping between their two meta-stable equilibria. In this setting, different notions of periodic tuning can easily be investigated. We not only consider the physicists' favorites, spectral power amplification and signal-to-noise ratio, but also other reasonable concepts in which the energy carried by the Markov chain trajectories or the entropy of their invariant measures are used. Turning to diffusions with weakly time periodic double well potentials and additive noise again, we then develop an asymptotic analysis of their spectral power amplification coefficient based on the spectral theory of their infinitesimal generators. It is based on the crucial observation that in the case of double well potentials its spectrum has a gap between the second and third eigenvalue. Therefore we have to give the corresponding eigenvalues and eigenfunctions a more detailed study, in particular with respect to their asymptotic behavior in the small noise limit. Its results then enable us to give a related small noise asymptotic expansion both of the densities of the associated invariant measures as for the spectral power amplification coefficients. We finally compare spectral power amplification coefficients of the Markov chains describing the reduced dynamics and the associated diffusions, to find that in the small noise limit they may be essentially different, caused by the *small oscillations catastrophe* near the potential wells' bottoms.

This motivates us in Chapter 4 to look for notions of periodic tuning for the solution trajectories of diffusions in spaces of arbitrary finite dimension with weakly periodic drifts and additive small noise which do not exhibit this robustness defect. We aim at notions related to the maximal probabilities that the random exit or transition times between different domains of attraction of the underlying dynamical systems happen in time windows parametrized by free energy parameters on an exponential scale. For the two-state Markov chains describing the effective dynamics of the diffusions with slow and weak time inhomogeneity this optimal transition rate is readily calculated. This concept moreover has the advantage that their related transition times, as well as the corresponding ones for diffusions with a weak noise dependent time inhomogeneity, allow a treatment by methods of large deviations in the small noise limit. We therefore start with a careful extension of large deviations theory to diffusions with slow time inhomogeneity. The central result for the subsequent analysis of their exit times is contained in a large deviations principle, uniform with respect to the energy parameter. It allows us in the sequel to derive upper and lower bounds for the asymptotic exponential exit rate from domains of attraction for slowly time dependent diffusions. They combine to the main large deviations result describing the exact asymptotic exponential exit rates for slowly and weakly time inhomogeneous diffusions in the small noise limit. This central result is tailor made for providing the optimal tuning rate related to maximal probability of transition during an exponential time window. We finally compare the resulting stochastic resonance point to the ones obtained for the Markov chains of the reduced dynamics, and conclude that they agree in the small noise limit, thus establishing robustness.

In two appendices — for easy reference in the text — we collect some standard results about Gronwall's lemma and Laplace's method for integrals with exponential singularities of the integrand.