Preface

Stochastic resonance is a phenomenon arising in many systems in the sciences in a wide spectrum extending from physics through neuroscience to chemistry and biology. It has attracted an overwhelming attendance in the science literature for the last two decades, more recently also in the mathematics literature. It is generally understood as the optimal amplification of a weak periodic signal in a dynamical system by random noise.

This book presents a mathematical approach of stochastic resonance in a well defined framework. We consider weakly periodic systems in arbitrary finite dimension with additive noise of small amplitude ε. They possess two domains of attraction of stable equilibria separated by a manifold marking a barrier. Both the geometry of the attraction domains as well as the barrier height are not scaled with the amplitude parameter ε. Therefore, in contrast to other approaches, noise induced random transitions in our model happen on time scales given by the exponential of the quotient of barrier height and noise amplitude (Kramers' times), and are due to *large deviations*. Our analysis is therefore based on a new space-time large deviations principle for the system's exit and transition dynamics between different domains of attraction in the limit of small ε. It aims at the description of an optimal interplay between large period length T of the weak periodic motion and noise amplitude ε. Optimization is done with respect to appropriate measures of quality of tuning of the stochastic system to the periodic input.

The two principal messages of the book are these. First we show that—already in space dimension one—the classical physical measures of quality of periodic tuning such as the *spectral power amplification* or *signal-to-noise ratio*, due to the impact of small random oscillations near the equilibria, are not robust with respect to dimension reduction. Comparing optimal tuning rates for the unreduced (diffusion) model and the associated reduced (finite state Markov chain) model one gets essentially different tuning scenarios. We therefore propose—in arbitrary finite space dimension—measures of quality of periodic tuning based uniquely on the transition dynamics and show that these measures are robust. Via our central space-time large deviations result they are able to explain stochastic resonance as optimal tuning.

Concentrating on these more theoretical themes, the book sheds some light on the mathematical shortcomings and strengths of different concepts used in theory and application of stochastic resonance. It does not aim at a comprehensive presentation of the many facets of stochastic resonance in various areas of sciences. In particular it does not touch computational aspects relevant in particular in high dimensions where analytical methods alone are too complex to be of practical use any more.

With this scope the book addresses researchers and graduate students in mathematics and the sciences interested in stochastic dynamics, in a quite broad sense, and wishing to understand the conceptual background of stochastic resonance, on the basis of large deviations theory for weakly periodic dynamical systems with small noise. Chapter 1 explains our approach on a heuristic basis on the background of paradigmatic examples from climate dynamics. It is accessible to a readership without a particular mathematical training. Chapter 2 provides a self-contained treatment of the classical Freidlin–Wentzell theory of diffusion exit from domains of attraction of dynamical systems in the simpler additive noise setting starting from a wavelet expansion of Brownian motion. It should be accessible to readers with basic knowledge of stochastic processes. In Chapter 3 based on an approach from the perspective of semi-classical analysis, i.e. spectral theory of infinitesimal generators of diffusion processes, the conceptual shortcomings of the classical physical concepts of stochastic resonance are presented. In Chapter 4 the Freidlin–Wentzell theory is extended to the non-trivial setting of weakly time-periodic dynamical systems with noise, and concepts of optimal tuning discussed which avoid the defects of the classical notions. Both Chapters are accessible on the basis of the background knowledge presented in Chapter 2.