CHAPTER 1

Heuristics of noise induced transitions

1.1. Energy balance models of climate dynamics

The simple concept of energy balance models stimulated research not only in the area of conceptual climate models, but was at the cradle of a research direction in physics which subsequently took important examples from various domains of biology, chemistry and neurology: it was one of the first examples for which the phenomenon of *stochastic resonance* was used to explain the transition dynamics between different stable states of physical systems. For a good overview see Gammaitoni et al. [**43**] or Jung [**62**].

In the end of the 70's, Nicolis [**83**] and Benzi et al. [**5**] almost simultaneously tried *stochastic resonance* as a rough and qualitative explanation for the glaciation cycles in earth's history. They were looking for a simple mathematical model appropriate to explain experimental findings from deep sea core measurements according to which the earth has seen ten glacial periods during the last million years, alternating with warm ages rather regularly in periods of about 100 000 years. Mean temperature shifts between warm age and glacial period are reported to be of the order of 10 K, and relaxation times, i.e. transition times between two relatively stable mean temperatures as rather short, of the order of only 100 years. Mathematically, their explanation was based on an equation stating the global energy balance in terms of the average temperature $T(t)$, where the global average is taken meridionally (i.e. over all latitudes), zonally (i.e. over all longitudes), and annually around time t. The global radiative power change at time t is equated to the difference between incoming solar (short wave) radiative power R_{in} and outgoing (long wave) radiative power R_{out}.

The power R_{in} is proportional to the global average of the solar constant $Q(t)$ at time t. To model the periodicity in the glaciation cycles, one assumes that Q undergoes periodic variations due to one of the so-called *Milankovich cycles*, based on periodic perturbations of the earth's orbit around the sun. Two of the most prominent cycles are due to a small periodic variation between 22.1 and 24.5 degrees of the angle of inclination (*obliquity*) of the earth's rotation axis with respect to its plane of rotation, and a very small periodic change of only about 0.1 percent of the *eccentricity*, i.e. the deviation from a circular shape, of the earth's trajectory around the sun. The obliquity cycle has a duration of about 41 000 years, while the eccentricity cycle corresponds to the 100 000 years observed in the temperature proxies from deep sea core measurements mentioned above. They are caused by gravitational influences of other planets of our solar system. In formulas, Q was assumed to be of the form

$$Q(t) = Q_0 + b \sin \omega t,$$

with some constants Q_0, b and a frequency $\omega = 10^{-5}[\frac{1}{y}]$.

The other component determining the absorbed radiative power R_{in} is a rough and difficult to model averaged *surface albedo* of the earth, i.e. the proportion of the solar power absorbed. It is supposed to be just (average) temperature dependent. For temperatures below $\underline{T}$, for which the surface water on earth is supposed to have turned into ice, and the surface is thus constantly bright, the albedo is assumed to be constantly equal to $\overline{a}$, for temperatures above $\overline{T}$, for which all ice has melted, and the surface constantly brown, it is assumed to be given by a constant $\underline{a} < \overline{a}$. For temperatures between $\underline{T}$ and $\overline{T}$, the two constant values $\underline{a}$ and $\overline{a}$ are simply linearly interpolated in the most basic model. The rough albedo function has therefore the *ramp function* shape depicted in Figure 1.1.

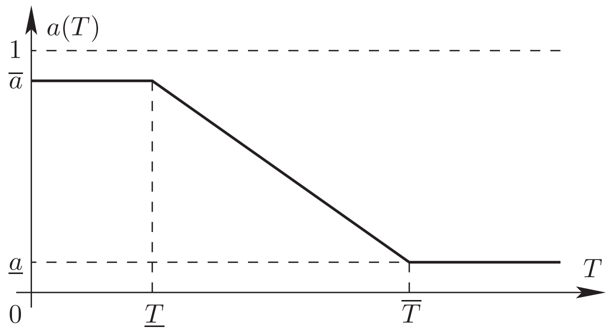

FIGURE 1.1. The albedo function $a = a(T)$.

To have a simple model of R_{out}, the earth is assumed to behave approximately as a *black body radiator*, for which the emitted power is described by the Stefan–Boltzmann law. It is proportional to the fourth power of the body's temperature and is given by $\gamma T^4(t)$, with a constant γ proportional to the Stefan constant.

Hence the simple energy balance equation with periodic input Q on which the model is built is given by

$$c\frac{d}{dt}T(t) = Q(t)\Big(1 - a(T(t))\Big) - \gamma T(t)^4, \tag{1.1}$$

where the constant c describes a global thermal inertia. According to (1.1), (quasi-) stationary states of average temperature should be given by the solutions of the equation $\frac{dT(t)}{dt} = 0$. If the model is good, they should reasonably well interpret glacial period and warm age temperatures. Graphically, they are given by the intersections of the curves of absorbed and emitted radiative power, see Figures 1.2 and 1.3.

As we shall more carefully explain below, the lower $(T_1(t))$ and upper $(T_3(t))$ quasi-equilibria are stable, while the middle one $(T_2(t))$ is unstable. The equilibrium $T_1(t)$ should represent an ice age temperature, $T_3(t)$ a warm age, while $T_2(t)$ is not observed over noticeably long periods. In their dependence on t they should describe small fluctuations due to the variations in the solar constant. But here one encounters a serious problem with this purely deterministic model. If the fluctuation amplitude of Q is small, then we will observe the two disjoint branches of stable solutions T_1 and T_3 (Figure 1.4).

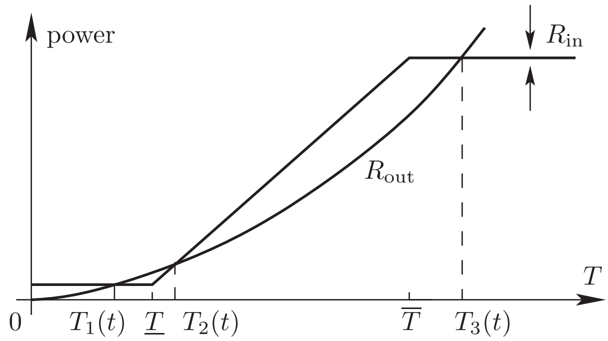

FIGURE 1.2. Incoming vs. outgoing power.

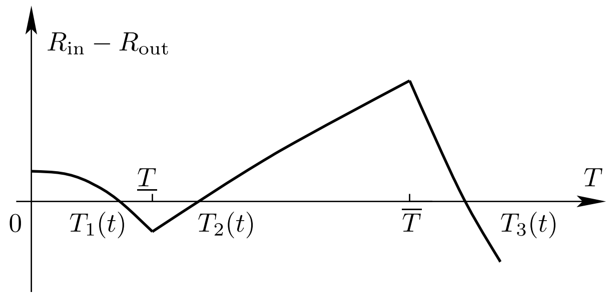

FIGURE 1.3. Difference of the powers of incoming and outgoing radiation.

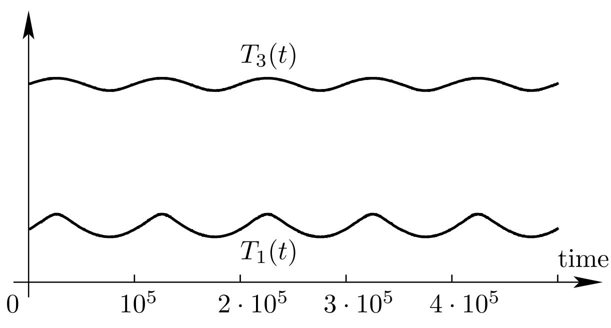

FIGURE 1.4. Equilibrium temperatures $T_1(t)$ and $T_3(t)$ for small fluctuation amplitude b.

However for both branches alone — besides being unrealistically low or high — the difference between minimal and maximal temperature can by no means account for the observed shift of about 10 K, and also the relaxation times are much too long. But the most important shortcoming of the model is the lacking possibility of transitions between the two branches.

If we allow the fluctuation amplitude b to be large, the picture is still very unrealistic: There are intervals during which one of the two branches T_1 or T_3

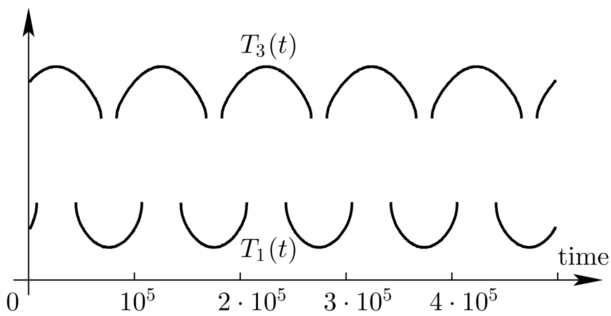

FIGURE 1.5. Unrealistic equilibrium temperatures $T_1(t)$ and $T_3(t)$ for large fluctuation amplitude b.

vanishes completely, and transitions are still impossible, unless one is willing to accept discontinuous behavior (Figure 1.5).

For this reason, Nicolis [**83**] and Benzi et al. [**5**] proposed to add a noise term in (1.1). Despite the fact that then negative temperatures become possible, they worked with the equation

$$c\dot{T}^{\varepsilon}(t) = Q(t)\Big(1 - a(T^{\varepsilon}(t))\Big) - \gamma T^{\varepsilon}(t)^4 + \sqrt{\varepsilon}\,\dot{W}_t, \tag{1.2}$$

$\varepsilon > 0$, where $\dot{W}$ is a white noise. In passing to (1.2), stable equilibria of the deterministic system become — approximately at least — meta-stable states of the stochastic system. And more importantly, the unbounded noise process W makes spontaneous transitions (*tunneling*) between the meta-stable states $T_1(t)$ and $T_3(t)$ possible. In fact, the *random hopping* between the meta-stable states immediately exhibits two features which make the model based on (1.2) much more attractive for a qualitative explanation of glaciation cycles: a) the transitions between T_1 and T_3 allow for far more realistic temperature shifts, b) relaxation times are random, but very short compared to the periods the process solving (1.2) spends in the stable states themselves.

But now a new problem arises, which actually provided the name *stochastic resonance.*

If, seen on the scale of the period of Q, ε is too small, the solution may be trapped in one of the states T_1 or T_3. By the periodic variation of Q, there are well defined periodically returning time intervals during which $T_1(t)$ is the *more probable* state, while $T_3(t)$ takes this role for the rest of the time. So if ε is small, the process, initially in T_1, may for example fail to leave this state during a whole period while the other one is more probable. The solution trajectory may then look as in Figure 1.6.

If, on the other hand, ε is too large, the big random fluctuation may lead to eventual excursions from the actually more probable equilibrium during its domination period to the other one. The trajectory then typically looks like in Figure 1.7.

In both cases it will be hard to speak of a random periodic curve. Good tuning with the periodic forcing by Q is destroyed by a random mechanism being too slow or too fast to follow. It turned out in numerous simulations in a number of similar systems that there is, however, an *optimal* parameter value ε for which the solution

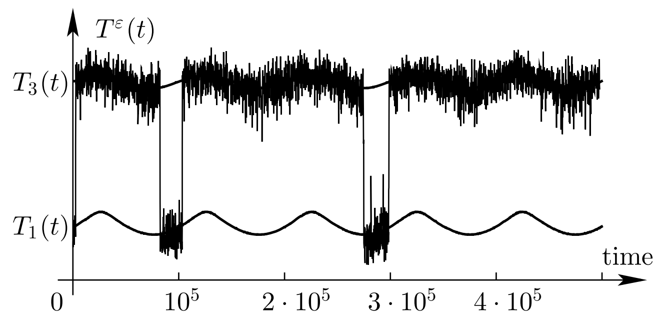

FIGURE 1.6. A typical solution trajectory of equation (1.2) for the small noise amplitude.

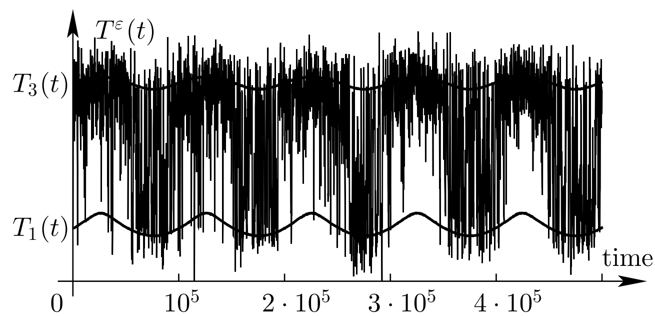

FIGURE 1.7. A typical solution trajectory of equation (1.2) for the large noise amplitude.

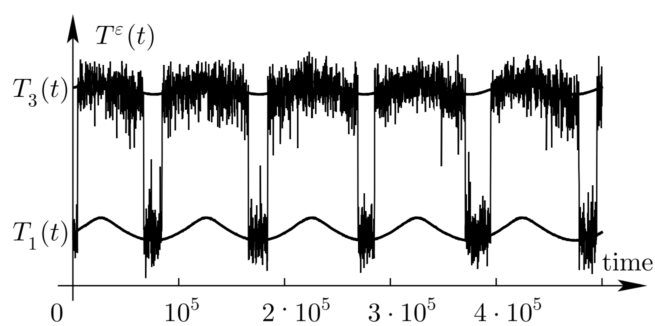

FIGURE 1.8. A typical solution trajectory of equation (1.2), the noise amplitude well tuned.

curves are well tuned with the periodic input. A typical well tuned curve is shown in Figure 1.8.

The optimally tuned system is then said to be in *stochastic resonance.* Nicolis [**83**] and Benzi et al. [**5**], by tuning the noise parameter ε to appropriate values, were able to give qualitative explanations for glaciation cycles based on this phenomenon.

Stochastic resonance proved to be relevant in other elementary climate models than the energy balance models considered so far. In Penland et al. [**87**], Wang et al. [**107, 106**], a two-dimensional stochastic model for a qualitative explanation of the ENSO (El Niño Southern Oscillation) phenomenon also leads to stochastic resonance effects: for certain parameter ranges the model exhibits random tuned transitions between two stable sea surface temperatures. New evidence for the presence of stochastic resonance phenomena in paleo-climatic time series was added by Ganopolski and Rahmstorf [**45**]. Their paper interprets the GRIP ice core record representing temperature proxies from the Greenland glacier that extend over a period of roughly 90 000 years, and showing the fine structure of the temperature record of the last glacial period. The time series shows about 20 intermediate warmings during the last glacial period commonly known under the name of Dansgaard–Oeschger events. These events are clearly marked by rapid spontaneous increases of temperature by about $6K$ followed by slower coolings to return to the initial basic cold age temperature. It was noted in [**45**] that a histogram of the number of Dansgaard–Oeschger events with a duration of $k \cdot 1480$ years, with $k = 1, 2, 3, \dots$ exhibits the typical shape of a stochastic resonance spike train consistent for instance with the results of Herrmann and Imkeller [**49**] for Markov chains describing the effective diffusion dynamics, or Berglund and Gentz [**13**] for diffusion processes with periodic forcing.

1.2. Heuristics of our mathematical approach

The rigorous mathematical elaboration of the concept of *stochastic resonance* is the main objective of this book. We start its mathematically sound treatment by giving a heuristical outline of the main stream of ideas and arguments based on the methods of *large deviations* for random dynamical systems in the framework of the Freidlin–Wentzell theory. Freidlin [**39**] is able to formulate Kramers' [**65**] very old seminal approach mathematically rigorously in a very general setting, and this way provides a lower estimate for the good tuning (see also the numerical results by Milstein and Tretyakov [**77**]). To obtain an upper estimate, we finally argue by embedding time discrete Markov chains into the diffusion processes that describe the effective dynamics of noise induced transitions. Optimal tuning results obtained for the Markov chains will then be transferred to the original diffusion processes.

To describe the idea of our approach, let us briefly return to our favorite example explained in the preceding section. Recall that the function

$$f(t,T) = Q(t)\Big(1 - a(T)\Big) - \gamma T^4, \quad T, t \geq 0,$$

describes a multiple of $R_{\text{in}} - R_{\text{out}}$, and its very slow periodicity in t is initiated by the assumption on the solar constant $Q(t) = Q_0 + b\sin(\omega t)$. Let us compare this quantity, sketched in Figure 1.9 schematically for two times, say t_1, t_2 such that Q takes its minimum at t_1 and its maximum at t_2. The graph of f moves periodically between the two extreme positions. Note that in the one-dimensional situation considered, $f(t, \cdot)$ can be seen as the negative gradient of a potential function $U(t, \cdot)$ which depends periodically on time t.

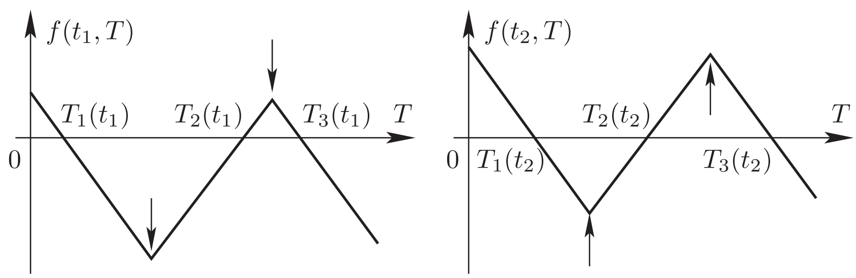

FIGURE 1.9. Schematical form of radiation power difference at times t_1 and t_2

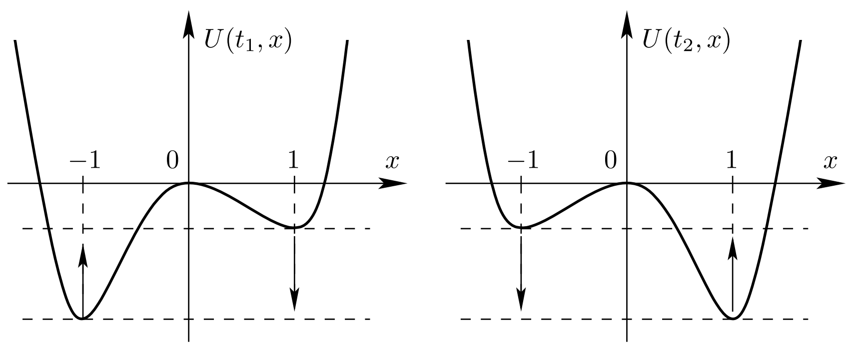

FIGURE 1.10. Potential function U at time instants t_1 and t_2.

We now turn to a more general context. For simplicity of the heuristical exposition still sticking to a one-dimensional scenario, we start by considering a temporally varying *potential function* U and set

$$f(t,\cdot) = -\frac{\partial}{\partial x}U(t,\cdot), \quad t \geq 0.$$

We assume that U oscillates in time between the two extreme positions depicted schematically in Figure 1.10.

In Figure 1.10 (l.), the potential well on left hand side is deeper than on the right hand side, in Figure 1.10 (r.) the role of the deeper well has changed. As t varies, we will observe a smoothly time dependent potential with two wells of periodically and smoothly fluctuating relative depth. Just the function describing the position of the deepest well will in general be discontinuous. It will play a crucial role in the analysis now sketched.

We assume in the sequel for simplicity that $U(t,x)$, $t \geq 0$, $x \in \mathbb{R}$, is a smooth function such that for all $t \geq 0$, $U(t,\cdot)$ has exactly two minima, one at -1, the other at 1, and that the two wells at -1 and 1 are separated by the saddle 0, where $U(t,0)$ is assumed to take the value 0. Two moment pictures of the potential may look as in Figure 1.10.

We further assume time periodicity for U, more formally that

$$U(t,\cdot) = U(t+1,\cdot).$$

The variable period of the input will be denoted by some positive number T. We therefore consider the stochastic differential equation

$$\frac{d}{dt}X_t^\varepsilon = f\Big(\frac{t}{T}, X_t^\varepsilon\Big) + \sqrt{\varepsilon}\,\dot{W}_t, \tag{1.3}$$

with a one-dimensional Wiener process W (white noise $\dot{W}$). We may circumscribe a more mathematical concept of stochastic resonance like this: given T ($\omega = \frac{1}{T}$), find the parameter $\varepsilon = \varepsilon(T)$ such that X^ε is *optimally tuned* with the periodic input $f(\frac{t}{T}, \cdot)$. We pose the problem in the following (almost equivalent) way: given $\varepsilon > 0$, find the good scale $T = T(\varepsilon)$ such that optimal tuning of X^ε with the periodic input is given, at least in the limit $\varepsilon \to 0$.

1.2.1. Random motion of a strongly damped Brownian particle. The analogy with the motion of a physical particle in a periodically changing double well potential landscape alluded to in (1.3) (see also Mazo [**72**] and Schweitzer [**97**]) motivates us to pause for a moment and give it a little more thought. As in the previous section, let us concentrate on a one-dimensional setting, remarking that our treatment easily generalizes to a finite-dimensional setting. Due to Newton's law, the motion of a particle is governed by the impact of all forces acting on it. Let us denote F the sum of these forces, m the mass, x the space coordinate and v the velocity of the particle. Then

$$m\dot{v} = F.$$

Let us first assume the potential to be turned off. In their pioneering work at the turn of the twentieth century, Marian Smoluchowski and Paul Langevin introduced stochastic concepts to describe the Brownian particle motion by claiming that at time t

$$F(t) = -\gamma v(t) + \sqrt{2kT\gamma}\dot{W}_t.$$

The first term results from friction γ and is velocity dependent. An additional stochastic force represents random interactions between Brownian particles and their simple molecular random environment. The white noise $\dot{W}$ (the formal derivative of a Wiener process) plays the crucial role. The diffusion coefficient (standard deviation of the random impact) is composed of Boltzmann's constant k, friction and environmental temperature T. It satisfies the condition of the fluctuation-dissipation theorem expressing the balance of energy loss due to friction and energy gain resulting from noise. The equation of motion becomes

$$\begin{cases} \dot{x}(t) = v(t), \\ \dot{v}(t) = -\dfrac{\gamma}{m}\,v(t) + \dfrac{\sqrt{2kT\gamma}}{m}\dot{W}_t. \end{cases}$$

In equilibrium, the stationary Ornstein–Uhlenbeck process provides its solution:

$$v(t) = v(0)\,e^{-\frac{\gamma}{m}t} + \frac{\sqrt{2kT\gamma}}{m}\int_0^t e^{-\frac{\gamma}{m}(t-s)}\,dW_s.$$

The ratio $\beta := \frac{\gamma}{m}$ determines the dynamic behavior. Let us focus on the over-damped situation with large friction and very small mass. Then for $t \gg \frac{1}{\beta} = \tau$ (relaxation time), the first term in the expression for velocity can be neglected, while the stochastic integral represents a Gaussian process. By integrating, we obtain in the over-damped limit ($\beta \to \infty$) that v and thus x is Gaussian with almost constant mean

$$m(t) = x(0) + \frac{1 - e^{-\beta t}}{\beta}\, v(0) \approx x(0)$$

and covariance close to the covariance of white noise, see Nelson [**82**]:

$$\begin{aligned} K(s,t) &= \frac{2kT}{\gamma}\min(s,t) + \frac{kT}{\gamma\beta}\Big(-2 + 2e^{-\beta t} + 2e^{-\beta s} - e^{-\beta|t-s|} - e^{-\beta(t+s)}\Big) \\ &\approx \frac{2kT}{\gamma}\min(s,t), \quad s,t \geq 0. \end{aligned}$$

Hence the time-dependent change of the velocity of the Brownian particle can be neglected, the velocity rapidly converges to thermal equilibrium ($\dot{v} \approx 0$), while the spatial coordinate remains far from it. In the so-called adiabatic transformation, the evolution of the particle's position is thus given by the transformed Langevin equation

$$\dot{x}(t) = \frac{\sqrt{2kT}}{\gamma}\, \dot{W}_t.$$

Let us next suppose that we face a Brownian particle in an external field of force, associated with a potential $U(t,x)$, $t \geq 0$, $x \in \mathbb{R}$. This then leads to the Langevin equation

$$\begin{cases} \dot{x}(t) = v(t), \\ m\dot{v}(t) = -\gamma\, v(t) - \dfrac{\partial U}{\partial x}(t, x(t)) + \sqrt{2kT\gamma}\, \dot{W}_t. \end{cases}$$

In the over-damped limit, after relaxation time, the adiabatic elimination of the fast variables (see Gardiner [**46**]) then leads to an equation similar to the one encountered in the previous section, namely

$$\dot{x}(t) = -\frac{1}{\gamma}\frac{\partial U}{\partial x}(t, x(t)) + \frac{\sqrt{2kT}}{\gamma}\dot{W}_t.$$

1.2.2. Time independent potential. We now continue discussing the heuristics of stochastic resonance for systems described by equations of the type encountered in the previous two sections. To motivate the link to the theory of large deviations, we first study the case in which $U(t,\cdot)$ is given by some time independent potential function U for all t. Following Freidlin and Wentzell [**40**], the description of the asymptotics contained in the *large deviations principle* requires the crucial notion of *action functional*. It is defined for $T > 0$ and absolutely continuous functions $\varphi \colon [0,T] \to \mathbb{R}$ with derivative $\dot{\varphi}$ by

$$S_{0T}(\varphi) = \frac{1}{2}\int_0^T \Big[\dot{\varphi}_s - \Big(-\frac{\partial}{\partial x}U\Big)(\varphi_s)\Big]^2\, ds.$$

By means of the action functional we can define the *quasipotential function*

$$V(x,y) = \inf\{S_{0T}(\varphi) \colon \varphi_0 = x, \varphi_T = y, T > 0\},$$

for x, $y \in \mathbb{R}$. Intuitively, $V(x,y)$ describes the minimal work to be done in the potential landscape given by U to pass from x to y. Keeping this in mind, the relationship between U and V is easy to understand (for a more formal argument see Chapter 3). If x and y are in the same potential well, we have

$$V(x,y) = 2(U(y) - U(x))^+, \tag{1.4}$$

where $a^+ = a \vee 0 = \max\{a, 0\}$ denotes the positive part of a real number a. In particular, if $U(y) < U(x)$, then $V(x, y) = 0$, i.e. going downhill in the landscape does not require work. If, however, x and y are in different potential wells, we have (recall $U(0) = 0$)

$$V(x, y) = -2U(x). \tag{1.5}$$

This equation reflects the fact that the minimal work to do to pass to y consists in reaching the saddle 0, since then one can just go downhill.

Rudiments of the following arguments can also be found in the explanation of stochastic resonance by McNamara and Wiesenfeld [**74**]. The main ingredient is the *exit time law* by Freidlin and Wentzell [**40**] (see also Eyring [**37**], Kramers [**65**] and Bovier et al. [**14**]). For $y \in \mathbb{R}$, $\varepsilon > 0$ the first time y is visited is defined by

$$\tau_y^\varepsilon = \inf\{t \geq 0\colon X_t^\varepsilon = y\}.$$

If $\mathbb{P}_x$ denotes the law of the diffusion $(X_t^\varepsilon)_{t \geq 0}$ started at $x \in \mathbb{R}$, the exit time law states that for any $\delta > 0$, $x \in \mathbb{R}$ we have

$$\mathbb{P}_x\Big(e^{\frac{V(x,y)-\delta}{\varepsilon}} \leq \tau_y^\varepsilon \leq e^{\frac{V(x,y)+\delta}{\varepsilon}}\Big) \to 1 \tag{1.6}$$

as $\varepsilon \to 0$.

In other words, in the limit $\varepsilon \to 0$, the process started at x takes approximately time $\exp(\frac{V(x,y)}{\varepsilon})$ to reach y, or more roughly

$$\varepsilon \ln \tau_y^\varepsilon \cong V(x, y)$$

as $\varepsilon \to 0$. As a consequence, one finds that as $\varepsilon \to 0$, on time scales $T(\varepsilon)$ at least as long as $\exp(\frac{V(x,y)}{\varepsilon})$ or such that

$$\varepsilon \ln T(\varepsilon) > V(x, y),$$

we may expect with $\mathbb{P}_x$-probability close to 1 that the process $X_{tT(\varepsilon)}^\varepsilon$ has reached y by time 1. Remembering (1.4) and (1.5) one obtains the following statement formulated much more generally by Freidlin. Suppose

$$\lim_{\varepsilon \to 0} \varepsilon \ln T(\varepsilon) > 2 \max\{-U(-1), -U(1)\}, \tag{1.7}$$

and $U(-1) < U(1)$. Then the Lebesgue measure of the set

$$\Big\{t \in [0, 1]\colon |X_{tT(\varepsilon)}^\varepsilon - (-1)| > \delta\Big\} \tag{1.8}$$

tends to 0 in $\mathbb{P}_x$-probability as $\varepsilon \to 0$, for any $\delta > 0$.

In other words, the process X^ε, run in a time scale $T(\varepsilon)$ large enough, will spend most of the time in the deeper potential well. Excursions to the other well are *exponentially negligible* on this scale, as $\varepsilon \to 0$. The picture is roughly as deployed in Figure 1.11.

1.2.3. Periodic step potentials and quasi-deterministic motion. As a rough approximation of temporally continuously varying potential functions we may consider periodic *step function* potentials such as

$$U(t, \cdot) = \begin{cases} U_1(\cdot), & t \in [k, k + \frac{1}{2}), \\ U_2(\cdot), & t \in [k + \frac{1}{2}, k + 1), \quad k \in \mathbb{N}_0. \end{cases} \tag{1.9}$$

We assume that both U_1 and U_2 are of the type described above, that $U_1(x) = U_2(-x)$, $x \in \mathbb{R}$, and that U_1 has a well of depth $\frac{V}{2}$ at -1, and a well of depth $\frac{v}{2}$ at

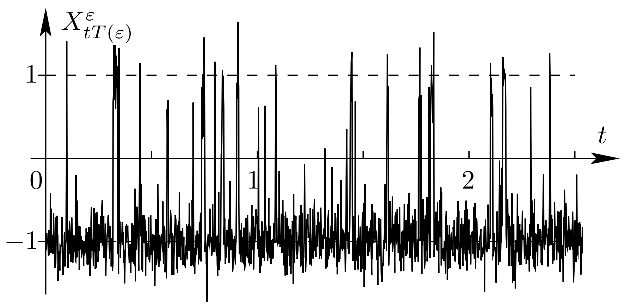

FIGURE 1.11. Solution trajectory of the diffusion $X^\varepsilon_{tT(\varepsilon)}$ in the time independent double-well potential U.

1, with $V > v$ (and U_2 wells with respectively opposite roles). Let us briefly point out the main features of the transition times for periodic step potentials described in (1.6). According to (1.6) the exponential rate of the transition time from -1 to 1 in U_1 in the small noise limit is asymptotically given by $\exp(\frac{V}{\varepsilon})$, as long as the time scale T of the diffusion allows no switching of the potential states before, i.e. as long as $T = T(\varepsilon) > \exp(\frac{V}{\varepsilon})$. Accordingly, the transition time from 1 to -1 in U_1 is given by $\exp(\frac{v}{\varepsilon})$, as long as $T = T(\varepsilon) > \exp(\frac{v}{\varepsilon})$. Similar statements hold for transitions between states of U_2. It is therefore also plausible that (1.8) generalizes to the following statement due to Freidlin [**39**, Theorem 2].

Suppose

$$\lim_{\varepsilon\to 0} \varepsilon \ln T(\varepsilon) > V. \tag{1.10}$$

Define

$$\phi(t) = \begin{cases} -1, & t \in [k, k+\frac{1}{2}), \\ 1, & t \in [k+\frac{1}{2}, k+1), \quad k \in \mathbb{N}_0. \end{cases}$$

Then the Lebesgue measure of the set

$$\Big\{ t \in [0,1] \colon |X^\varepsilon_{t\,T(\varepsilon)} - \phi(t)| > \delta \Big\} \tag{1.11}$$

tends to 0 as $\varepsilon \to 0$ in $\mathbb{P}_x$-probability, for any $\delta > 0$, $x \in \mathbb{R}$.

Again, this just means that the process X^ε, run in a time scale $T(\varepsilon)$ large enough, will spend most of the time in the minimum of the deepest potential well which is given by the time periodic function ϕ. Excursions to the other well are *exponentially negligible* on this scale, as $\varepsilon \to 0$. The picture is typically the one depicted in Figure 1.12.

1.2.4. Periodic potentials and quasi-deterministic motion. Since the function ϕ appearing in the previous theorem is already discontinuous, it is plausible that the step function potential is in fact a reasonable approximation of the general case of continuously and (slowly) periodically changing potential functions. It is intuitively clear how the result has to be generalized to this situation. We just have to replace the periodic step potentials by potentials *frozen* along a partition of the period interval on the potential state taken at its starting point, and finally let the mesh of the partition tend to 0. To continue the discussion in the spirit of the previous section and with the idea of instantaneously frozen potential states, we

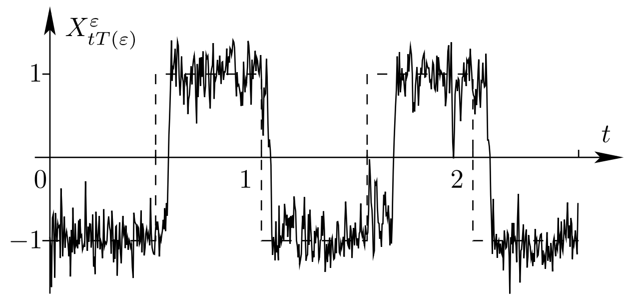

FIGURE 1.12. Solution trajectory of the diffusion $X^\varepsilon_{tT(\varepsilon)}$ in the double-well periodic step potential.

have to explain the asymptotics of the minimal time a Brownian particle needs to exit from the (frozen) starting well, say the left one. Freezing the potential at some time s, the asymptotics of its exit time is derived from the classical large deviation theory of randomly perturbed dynamical systems, see Freidlin and Wentzell [**40**]. Let us assume that U is locally Lipschitz continuous. We recall that for any $t \geq 0$ the potential $U(t, \cdot)$ has its minima at -1 and 1, separated by the saddle point 0. The law of the first exit time $\tau_1^\varepsilon = \inf\{t \geq 0\colon X_t^\varepsilon > 0\}$ is described by some particular functional related to large deviation. For $t > 0$, we introduce the *action functional* on the space of real valued continuous functions $C([0,t], \mathbb{R})$ on $[0,t]$ by

$$S_t^s(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^t \Big(\dot{\varphi}_u + \frac{\partial}{\partial x}U(s, \varphi_u)\Big)^2 du & \text{if } \varphi \text{ is absolutely continuous,} \\ +\infty & \text{otherwise,} \end{cases}$$

which is non-negative and vanishes on the set of solutions of the ordinary differential equation $\dot{\varphi} = -\frac{\partial}{\partial x}U(s, \varphi)$. Let x and y be real numbers. With respect to the (frozen) action functional, we define the (frozen) quasipotential

$$V_s(x, y) = \inf\{S_t^s(\varphi)\colon \varphi \in C([0,t], \mathbb{R}),\ \varphi_0 = x,\ \varphi_t = y,\ t \geq 0\}$$

which represents the minimal work the diffusion with a potential frozen at time s and starting in x has to do in order to reach y. To switch wells, the Brownian particle starting in the left well's bottom -1 has to overcome the barrier. So we let

$$\overline{V}_s = V_s(-1, 0).$$

This minimal work needed to exit from the left well can be computed explicitly, and is equal to twice its depth at time s. The asymptotic behavior of the exit time is expressed by

$$\lim_{\varepsilon \to 0} \varepsilon \ln \mathbb{E}_x \tau_1^\varepsilon = \overline{V}_s$$

or in generalization of (1.6)

$$\lim_{\varepsilon \to 0} \mathbb{P}_x\Big(e^{\frac{\overline{V}_s - \delta}{\varepsilon}} < \tau_1^\varepsilon < e^{\frac{\overline{V}_s + \delta}{\varepsilon}}\Big) = 1 \text{ for any } \delta > 0 \text{ and } x < 0.$$

Let us now assume that the left well is the deeper one at time s. If the Brownian particle has enough time to cross the barrier, i.e. if $T(\varepsilon) > e^{\frac{\overline{V}_s}{\varepsilon}}$, then, generalizing

(1.8), Freidlin in [**39**, Theorem 1] proves that independently of the starting point x it should stay near -1 in the following sense. The Lebesgue measure of the set

$$\Big\{t \in [0,1]\colon |X^\varepsilon_{tT(\varepsilon)} - (-1)| > \delta\Big\}$$

converges to 0 in probability as $\varepsilon \to 0$. If $T(\varepsilon) < e^{\frac{\overline{V}_s}{\varepsilon}}$, the time left is not long enough for any crossing: the particle, starting at x, stays in the starting well, near the stable equilibrium point. In other words, the Lebesgue measure of the set

$$\Big\{t \in [0,1]\colon |X^\varepsilon_{tT(\varepsilon)} - (-\mathbb{I}_{(-\infty,0)}(X^\varepsilon_{tT(\varepsilon)}) + \mathbb{I}_{[0,\infty)}(X^\varepsilon_{tT(\varepsilon)}))| > \delta\Big\}$$

converges to 0 in the small noise limit. This observation is at the basis of Freidlin's law of quasi-deterministic periodic motion discussed in the subsequent section. The lesson it teaches is this: to observe switching of the position to the energetically most favorable well, $T(\varepsilon)$ should be larger than some critical level $e^{\frac{\lambda}{\varepsilon}}$, where $\lambda = \inf_{s\geq 0}\overline{V}_s$. Measuring time in exponential scales by μ through the equation $T(\varepsilon) = e^{\frac{\mu}{\varepsilon}}$, the condition translates into $\mu > \lambda$. Continuing the reasoning of the preceding subsection, if this condition is satisfied, we may define a periodic function ϕ denoting the deepest well position in dependence on t. Then, in generalization of (1.11), the process X^ε will spend most of the time, measured by Lebesgue's measure, near ϕ for small ε.

1.2.5. Quality of periodic tuning and reduced motion. Do the manifestations of quasi-deterministic motion in instantaneously frozen potentials just discussed explain *stochastic resonance*? The problem is obvious. They just give lower bounds for the scale $T(\varepsilon) = e^{\frac{\mu}{\varepsilon}}$ for which noise strength ε leads to random switches between the *most probable* potential wells near the (periodic) deterministic times when the role of the deepest well switches. But if μ is too big, occasional excursions into the higher well will destroy a truly periodic tuning with the potential (see Figure 1.12). Just the duration of the excursions, being exponentially smaller than the periods of dwelling in the deeper well, will not be noticed by the residence time criteria discussed. We therefore also need an upper bound for possible scales. In order to find this optimal tuning scale $\mu_R > \lambda$, we first have to measure *goodness* of periodic tuning of the trajectories of the solution. In the huge physics literature on stochastic resonance, two families of criteria can be distinguished. The first one is based on invariant measures and spectral properties of the infinitesimal generator associated with the diffusion X^ε. Now, X^ε is not time autonomous and consequently does not admit invariant measures. By taking into account deterministic motion of time in the interval of periodicity and considering the time autonomous process $Z^\varepsilon_t = (t \bmod T(\varepsilon), X^\varepsilon_t)$, $t \geq 0$, we obtain a Markov process with an invariant measure $\nu^\varepsilon_t(x)\,dt\,dx$. In particular, for $t \geq 0$ the law of $X^\varepsilon_t \sim \nu^\varepsilon_t(x)\,dx$ and the law of $X^\varepsilon_{t+T(\varepsilon)} \sim \nu^\varepsilon_{t+T(\varepsilon)}(x)\,dx$, under this measure are the same for all $t \geq 0$. Let us present the most important notions of quality of tuning (see Jung [**62**], or Gammaitoni et al. [**43**]):

- the *spectral power amplification* (SPA) which plays an eminent role in the physics literature and describes the energy carried by the spectral component of the averaged trajectories of X^ε corresponding to the period of the signal:

$$\eta^X(\varepsilon, T) = \Big|\int_0^1 \mathbb{E}_\nu X^\varepsilon_{sT} \cdot e^{2\pi i s}\,ds\Big|^2, \quad \varepsilon > 0,\ T > 0.$$

- the total energy of the averaged trajectories

$$\mathrm{En}^X(\varepsilon,T)=\int_0^1\left|\mathbb{E}_\nu X_{sT}\right|^2 ds,\quad \varepsilon>0,\ T>0.$$

The second family of criteria is more probabilistic. It refers to quality measures purely based on the location of transition times between domains of attraction of the local minima, and residence time distributions measuring the time spent in one well between two transitions, or interspike times. This family, to be discussed in more detail in Section 1.4 below is certainly less popular in the physics community.

As will turn out later, these physical notions of quality of periodic tuning of random trajectories exhibit one important drawback: they are not robust with respect to model resolution. It is here that an important concept of model reduction enters the stage. It is based on the conjecture that the effective dynamical properties of periodically forced diffusion processes as given by (1.3) can be traced back to finite state Markov chains periodically hopping between the stable equilibria of the potential function underlying the diffusion, for which the smallness parameter of the noise intensity is simply reflected in the transition matrix. These Markov chains should be designed to capture the essential information about the *inter-well* dynamics of the diffusion, while *intra-well* small fluctuations of the diffusion near the potential minima are neglected. Investigating goodness of tuning according to the different physical measures of quality makes sense both for the Markov chains as for the diffusions. This idea of model reduction was captured and followed in the physics literature in Eckmann and Thomas [**32**], McNamara and Wiesenfeld [**74**], and Nicolis [**83**]. In fact, theoretical work on the concept of stochastic resonance in the physics literature is based on the model reduction approach, see the surveys Anishchenko et al. [**1**], Gammaitoni et al. [**43, 44**], Moss et al. [**79**], and Wellens et al. [**108**].

As we shall see in Chapter 3, the optimal tuning relations between ε and T do not necessarily agree for Markov chains and diffusions. Even in the small noise limit discrepancies may persist that are caused by very subtle geometric properties of the potential function. It is our goal to present a notion of quality of periodic tuning which possesses this robustness property when passing from the Markov chains capturing the effective dynamics to the original diffusions. For this reason we shall study the different physical notions of quality of tuning first in the context of typical finite state Markov chains with periodically forced transition matrices.

1.3. Markov chains for the effective dynamics and the physical paradigm of spectral power amplification

To keep this heuristic exposition of the main ideas of our mathematical approach as simple as possible, besides allowing only two states for our Markov chain that play the role of the stable equilibria of the potential -1 and 1, let us also discretize time. We continue to assume as in the discussion of periodically switching potential states above that $U_1(-1)=U_2(1)=-\frac{V}{2}$, and $U_1(1)=U_2(-1)=-\frac{v}{2}$. In a setting better adapted to our continuous time diffusion processes, in Chapter 3 time continuous Markov chains switching between two states will capture the effective diffusion dynamics. Hence, we follow here Pavlyukevich [**86**] and Imkeller and Pavlyukevich [**58**] and shall assume in this section that the parameter T in our model describing the period length, is an even integer. So for $T\in 2\mathbb{N}$, $\varepsilon>0$, consider a Markov chain $Y^\varepsilon = (Y^\varepsilon(k))_{k\geq 0}$ on the state space $\mathcal{S} = \{-1, 1\}$. Let $P_T(k)$ be the matrix of one-step transition probabilities at time k. If we denote $p_T^-(k) = \mathbb{P}(Y^\varepsilon(k) = -1)$, $p_T^+(k) = \mathbb{P}(Y^\varepsilon(k) = 1)$, and write P^* for the transposed matrix, we have

$$\begin{pmatrix} p_T^-(k+1) \\ p_T^+(k+1) \end{pmatrix} = P_T^*(k) \begin{pmatrix} p_T^-(k) \\ p_T^+(k) \end{pmatrix}.$$

In order to model the periodic switching of the double-well potential in our Markov chains, we define the transition matrix P_T to be periodic in time with period T. More precisely,

$$P_T(k) = \begin{cases} Q_1, & 0 \leq k \bmod T \leq \frac{T}{2} - 1, \\ Q_2, & \frac{T}{2} \leq k \bmod T \leq T - 1, \end{cases}$$

with

$$\begin{gathered} Q_1 = \begin{pmatrix} 1-\varphi & \varphi \\ \psi & 1-\psi \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1-\psi & \psi \\ \varphi & 1-\varphi \end{pmatrix}, \\ \varphi = pe^{-V/\varepsilon}, \quad \psi = qe^{-v/\varepsilon}, \end{gathered} \tag{1.12}$$

where $0 \leq p, q \leq 1$, $0 < v < V < +\infty$, $0 < \varepsilon < \infty$.

The entries of the transition matrices clearly are designed to mimic transition rates between -1 and 1 or vice versa that correspond to the transition times of the diffusion processes between the meta-stable equilibria, given according to the preceding section by $\exp(\frac{V}{\varepsilon})$ resp. $\exp(\frac{v}{\varepsilon})$. The exponential factors in the one-step transition probabilities are just chosen to be the inverses of those mean transition times. This is exactly what elementary Markov chain theory requires *in equilibrium*. The phenomenological *prefactors* p and q, chosen between 0 and 1, add asymmetry to the picture.

It is well known that for a time-homogeneous Markov chain on $\{-1, 1\}$ with transition matrix P_T one can talk about *equilibrium*, given by the stationary distribution, to which the law of the chain converges exponentially fast. The stationary distribution can be found by solving the matrix equation $\pi = P_T^*\pi$ with norming condition $\pi^- + \pi^+ = 1$.

For time non homogeneous Markov chains with time periodic transition matrix, the situation is quite similar. Enlarging the state space $\mathcal{S}$ to $\mathcal{S}_T = \{-1, 1\} \times \{0, 1, \ldots, T-1\}$, we recover a time homogeneous chain by setting

$$Z^\varepsilon(k) = (Y^\varepsilon(k), k \bmod T), \quad k \geq 0,$$

to which the previous remarks apply. For convenience of notation, we assume $\mathcal{S}_T$ to be ordered in the following way:

$$\mathcal{S}_T = \Big((-1,0), (1,0), (-1,1), (1,1), \ldots, (-1,T-1), (1,T-1)\Big).$$

Writing A_T for the matrix of one-step transition probabilities of Z^ε, the stationary distribution $R = (r(i,j))^*$ is obtained as a normalized solution of the matrix equation $(A_T^* - E)R = 0$, E being the identity matrix. We shall be dealing with the following variant of stationary measure, which is not normalized in time. Let $\pi_T(k) = (\pi_T^-(k), \pi_T^+(k))^* = (r(-1,k), r(1,k))^*$, $0 \leq k \leq T-1$. We call the family $\pi_T = (\pi_T(k))_{0\leq k\leq T-1}$ the *stationary distribution* of the Markov chain Y^ε.

The matrix A_T of one-step transition probabilities of Z^ε is explicitly given by

$$A_T = \begin{pmatrix} 0 & Q_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & Q_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & Q_2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & Q_2 \\ Q_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

The matrix A_T has block structure. In this notation 0 means a 2×2-matrix with all entries equal to zero, Q_1, and Q_2 are the 2-dimensional matrices defined in (1.12).

Applying some algebra we see that the equation $(A_T^* - E)R = 0$ is equivalent to $A'_T R = 0$, where

$$A'_T = \begin{pmatrix} \widehat{Q} - E & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ Q_1^* & -E & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -E & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & P_2^* & -E & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & Q_2^* & -E \end{pmatrix}$$

and $\widehat{Q} = Q_2^* Q_2^* \cdots Q_1^* Q_1^* = (Q_2^*)^{\frac{T}{2}} (Q_1^*)^{\frac{T}{2}}$. But A'_T is a block-wise lower diagonal matrix, and so $A'_T R = 0$ can be solved in the usual way resulting in the following formulas.

For every $T \in 2\mathbb{N}$, the stationary distribution π_T of Y^ε with matrices of one-step probabilities defined in (1.12) is given by

$$\begin{cases} \pi_T^-(l) = \dfrac{\psi}{\varphi+\psi} + \dfrac{\varphi-\psi}{\varphi+\psi} \cdot \dfrac{(1-\varphi-\psi)^l}{1+(1-\varphi-\psi)^{\frac{T}{2}}}, \\ \pi_T^+(l) = \dfrac{\varphi}{\varphi+\psi} - \dfrac{\varphi-\psi}{\varphi+\psi} \cdot \dfrac{(1-\varphi-\psi)^l}{1+(1-\varphi-\psi)^{\frac{T}{2}}}, \\ \pi_T^-(l+\frac{T}{2}) = \pi_T^+(l), \\ \pi_T^+(l+\frac{T}{2}) = \pi_T^-(l), \qquad 0 \le l \le \frac{T}{2} - 1. \end{cases} \tag{1.13}$$

The proof of (1.13) is easy and instructive, and will be contained in the following arguments. Note that $\pi_T(0)$ satisfies the matrix equation

$$\Big((Q_2^*)^{\frac{T}{2}} (Q_1^*)^{\frac{T}{2}} - E \Big) \pi_T(0) = 0$$

with additional condition $\pi_T^-(0) + \pi_T^+(0) = 1$. To calculate $(Q_2^*)^{\frac{T}{2}} (Q_1^*)^{\frac{T}{2}}$, we use a formula for the k-th power of 2×2-matrices $Q = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$, $a, b \in \mathbb{R}$, proved in a straightforward way by induction on k which reads

$$\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}^k = \frac{1}{a+b} \begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^k}{a+b} \begin{pmatrix} a & -a \\ -b & b \end{pmatrix}.$$

Using some more elementary algebra we find

$$
\begin{aligned}
(Q_2^*)^{\frac{T}{2}}(Q_1^*)^{\frac{T}{2}} &= \Big((Q_1)^{\frac{T}{2}}(Q_2)^{\frac{T}{2}}\Big)^* = \begin{pmatrix}1-\psi & \psi\\ \varphi & 1-\varphi\end{pmatrix}^{\frac{T}{2}}\begin{pmatrix}1-\varphi & \varphi\\ \psi & 1-\psi\end{pmatrix}^{\frac{T}{2}}\\
&= \frac{1}{\varphi+\psi}\begin{pmatrix}\varphi & \varphi\\ \psi & \psi\end{pmatrix} + (1-\varphi-\psi)^{\frac{T}{2}}\frac{\varphi-\psi}{\varphi+\psi}\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\\
&+ \frac{(1-\varphi-\psi)^T}{\varphi+\psi}\begin{pmatrix}\varphi & -\psi\\ -\varphi & \psi\end{pmatrix},
\end{aligned}
$$

from which another straightforward calculation yields

$$
\begin{cases}
\pi_T^-(0) = \dfrac{\varphi}{\varphi+\psi} + \dfrac{\psi}{\varphi+\psi}\cdot\dfrac{(1-\varphi-\psi)^{\frac{T}{2}}}{1+(1-\varphi-\psi)^{\frac{T}{2}}},\\[2ex]
\pi_T^+(0) = \dfrac{\psi}{\varphi+\psi} + \dfrac{\varphi}{\varphi+\psi}\cdot\dfrac{(1-\varphi-\psi)^{\frac{T}{2}}}{1+(1-\varphi-\psi)^{\frac{T}{2}}}.
\end{cases}
$$

To compute the remaining entries, we use $\pi_T(l) = (Q_1^*)^l\pi_T(0)$ for $0 \le l \le \frac{T}{2}-1$, and $\pi_T(l) = (Q_2^*)^l(Q_1^*)^{\frac{T}{2}}\pi_T(0)$ for $\frac{T}{2} \le l \le T-1$ to obtain (1.13). Note also the symmetry $\pi_T^-(l+\frac{T}{2}) = \pi_T^+(l)$ and $\pi_T^+(l+\frac{T}{2}) = \pi_T^-(l)$, $0 \le l \le \frac{T}{2}-1$.

To motivate the physical quality of tuning concept of *spectral power amplification*, we first remark that our Markov chain Y^ε can be interpreted as amplifier of the periodic input signal of period T. In the stationary regime, i.e. if the law of Y^ε is given by the measure π_T, the power carried by the output Markov chain at frequency a/T is a random variable

$$
\xi_T(a) = \frac{1}{T}\sum_{l=0}^{T-1} Y^\varepsilon(l)e^{\frac{2\pi i a}{T}l}.
$$

We define the *spectral power amplification (SPA)* as the relative expected power carried by the component of the output with (input) frequency $\frac{1}{T}$. It is given by

$$
\eta^Y(\varepsilon,T) = \Big|\mathbb{E}_{\pi_T}\xi_T(1)\Big|^2, \quad \varepsilon>0, T\in 2\mathbb{N}.
$$

Here $\mathbb{E}_{\pi_T}$ denotes expectation w.r.t. the stationary law π_T.

The explicit description of the invariant measure now readily yields an explicit formula for the spectral power amplification. In fact, using (1.13) one immediately gets

$$
\begin{aligned}
\mathbb{E}_{\pi_T}\xi_T(1) &= \frac{1}{T}\sum_{k=0}^{T-1}\mathbb{E}_{\pi_T}Y^\varepsilon(k)e^{\frac{2\pi i}{T}k} = \frac{1-e^{\pi i}}{T}\sum_{k=0}^{\frac{T}{2}-1}(\pi_T^+(k)-\pi_T^-(k))e^{\frac{2\pi i}{T}k}\\
&= \frac{4}{T}\frac{\varphi-\psi}{\varphi+\psi}\Big(\frac{1}{1-e^{\frac{2\pi i}{T}}} - \frac{1}{1-(1-\varphi-\psi)e^{\frac{2\pi i}{T}}}\Big).
\end{aligned}
$$

Elementary algebra then leads to the following description of the spectral power amplification coefficient of the Markov chain Y^ε for $\varepsilon>0$, $T\in 2\mathbb{N}$:

$$
\eta^Y(\varepsilon,T) = \frac{4}{T^2\sin^2(\frac{\pi}{T})}\cdot\frac{(\varphi-\psi)^2}{(\varphi+\psi)^2+4(1-\varphi-\psi)\sin^2(\frac{\pi}{T})}. \tag{1.14}
$$

Note now that the one-step probabilities Q_1 and Q_2 depend on the parameters noise level ε. Our next goal is to *tune* this parameter to a value which maximizes the amplification coefficient $\eta^Y(\varepsilon, T)$ as a function of ε. So the *stochastic resonance* point is marked by the maximum of the spectral power amplification coefficient as a function of ε. To calculate it, substitute $e^{-1/\varepsilon} = x$, and differentiate the explicit formula (1.14). The resulting relationship between period length $T(\varepsilon)$ and *noise intensity* ε marking the stochastic resonance point can be recast in the formula

$$T(\varepsilon) \cong \frac{1}{2\pi}\sqrt{pq\frac{V-v}{v}}\exp\left(\frac{V+v}{2\varepsilon}\right).$$

The maximal value of spectral power amplification is given by

$$\lim_{\varepsilon\to 0}\eta^Y(\varepsilon, T(\varepsilon)) = \frac{4}{\pi^2}$$

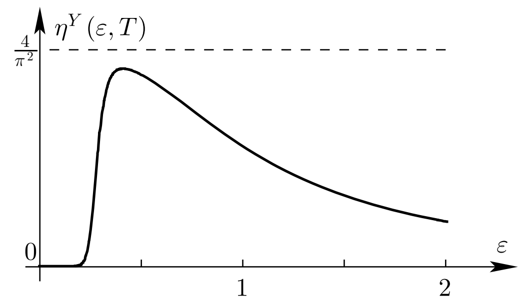

FIGURE 1.13. The coefficient of the spectral power amplification $\varepsilon \mapsto \eta^Y(\varepsilon, T)$ for $p = q = 0.5$, $V = 2$, $v = 1$, $T = 10\,000$.

We also see that the spectral power amplification as a measure of quality of stochastic resonance allows to distinguish a unique time scale, and find its exponential rate $\frac{V+v}{2}$ together with the pre-exponential factor. The optimal exponential rate is therefore given by the arithmetic mean of the two potential barriers marked by the deep and shallow well of our double well potential. This basic relationship will appear repeatedly at different stages of our mathematical elaboration of concepts of optimal tuning.

We may summarize our findings so far for discrete Markov chains that capture the effective dynamics of the potential diffusions which are our main subject of interest. Following the physics literature (e.g. Gammaitoni et al. [**43**] and McNamara and Wiesenfeld [**74**]) we understand *stochastic resonance* as *optimal spectral power amplification*. The closely related notion of *signal-to-noise ratio* and other reasonable concepts based on quality measures such as the relative entropy of invariant laws are discussed for Markov chains in Chapter 3 (see Section 3.2). The spectral power amplification coefficient measures the power carried by the expected Fourier coefficient in equilibrium of the Markov chain switching between the stable equilibria of the potential landscape of the diffusion which corresponds to the frequency of the underlying periodic deterministic signal.

1.4. Diffusions with continuously varying potentials

The concept of spectral power amplification is readily extended to Markov chains in continuous time, still designed to capture the effective diffusion dynamics in higher dimensions, as well as to potential diffusions themselves. This will be done in detail in Chapter 3. However, it will turn out that diffusions and their reduced dynamics Markov chains are not as similar as expected. Indeed, in a reasonably large time window around the resonance point for Y^ε, the tuning picture of the spectral power amplification for the diffusion is different. Under weak regularity conditions on the potential, it exhibits strict monotonicity in the window. Hence optimal tuning points for diffusion and Markov chain differ essentially. In other words, the diffusion's SPA tuning behavior is not robust for passage to the reduced model (see Chapter 3, subsection 3.4.4). This strange deficiency is difficult to explain. The main reason of this subtle effect appears to be that the diffusive nature of the Brownian particle is neglected in the reduced model. In order to point out this feature, we may compute the SPA coefficient of $g(X^\varepsilon)$ where g is a particular function designed to cut out the small fluctuations of the diffusion in the neighborhood of the bottoms of the wells, by identifying all states there. So $g(x) = -1$ (resp. 1) in some neighborhood of -1 (resp. 1) and otherwise g is the identity. This results in

$$\tilde{\eta}^X(\varepsilon, T) = \Big| \int_0^1 \mathbb{E}_\nu g(X^\varepsilon_{sT})\, e^{2\pi i s}\, ds \Big|^2, \quad \varepsilon > 0,\ T \geq 0.$$

In the small noise limit this quality function admits a local maximum close to the resonance point of the reduced model: the growth rate of $T_{\mathrm{opt}}(\varepsilon)$ is also given by the arithmetic mean of the wells' depths. So the lack of robustness seems to be due to the small fluctuations of the particle in the wells' bottoms.

In any case, this clearly calls for other quality measures to be used to transfer properties of the reduced model to the original one. Our discussion indicates that due to their emphasis on the pure transition dynamics, a second more probabilistic family of quality measures should be used. This will be made mathematically rigorous in Chapter 4. The family is composed of quality measures based on transition times between the domains of attraction of the local minima, residence times distributions measuring the time spent in one well between two transitions, or interspike times. To explain its main features there is no need to restrict to landscapes frozen in time independent potential states on half period intervals. So from now on the potential $U(t, x)$ is a continuous function in (t, x). For simplicity — remaining in the one-dimensional case — we further suppose that its local minima are given by ± 1, and its only saddle point by 0, independently of time. So the only meta-stable states on the time axis are ± 1. Let us denote by $\frac{v_-(t)}{2}$ (resp. $\frac{v_+(t)}{2}$ the depth of the left (resp. right) well. These function are continuous and 1-periodic. We shall assume that they are strictly monotonous between their global extrema. Let us now consider the motion of the Brownian particle in this landscape. As in the preceding case, according to Freidlin's law of quasi-deterministic motion its trajectory gets close to the global minimum, if the period is large enough. The exponential rate of the period should be large enough to permit transitions: if $T(\varepsilon) = e^{\mu/\varepsilon}$ with $\mu \geq \max_{i=\pm} \sup_{t \geq 0} v_i(t)$ meaning that μ is larger than the maximal work needed to cross the barrier, then the particle often switches between the two wells and should stay close to the deepest position in the landscape. By defining $\phi(t) = 2\mathbb{I}_{\{v_+(t) > v_-(t)\}} - 1$, in the small noise limit the Lebesgue measure of the set

$$\{t \in [0,1] \colon |X_{tT} - \phi(t)| > \delta\}$$

converges to 0 in probability for any $\delta > 0$. But in this case many transitions occur in practice, and the trajectory looks chaotical instead of periodic. So we have to choose smaller periods even if we cannot assure that the particle stays close to the global minimum since it needs some time to cross the barrier. Let us study the transition times. For this we assume that the starting point is -1 corresponding to the bottom of the deepest well. If the depth of the well is always larger than $\mu = \varepsilon \ln T(\varepsilon)$, then the particle does not have enough time during one period to climb the barrier and should therefore stay in the starting well. On the contrary if the depth of the starting well becomes smaller than μ, the transition can and will happen. More precisely, for $\mu \in (\inf_{t\geq 0} v_-(t), \sup_{t\geq 0} v_-(t))$ we define

$$a_\mu^-(s) = \inf\{t \geq s \colon v_-(t) \leq \mu\}.$$

The first transition time from -1 to 1 denoted τ_+ has the following asymptotic behavior in the small noise limit: $\tau_+/T(\varepsilon) \to a_\mu^-(0)$. The second transition which lets the particle return to the starting well will appear near the deterministic time $a_\mu^+(a_\mu^-(s))T(\varepsilon)$. The definitions of the coefficients a_μ^- and a_μ^+ are similar, the depth of the left well just being replaced by that of the right well. In order to observe periodic behavior of the trajectory, the particle has to stay a little time in the right well before going back. This will happen under the assumption $v_+(a_\mu(0)) > \mu$, that is, the right well is the deepest one at the transition time. In fact we can then define the *resonance interval* I_R, the set of all values μ such that the trajectories look periodic in the small noise limit:

$$I_R = \Big(\max_{i=\pm} \inf_{t\geq 0} v_i(t), \inf_{t\geq 0} \max_{i=\pm} v_i(t)\Big).$$

On this interval trajectories approach some deterministic periodic limit. We now outline the construction of a quality measure that is based on these observations, to be optimized in order to obtain *stochastic resonance* as the best possible response to periodic forcing. The measure we consider is based on the probability that a random transition of the diffusion happens during a small time window around the limiting deterministic transition time. Recall the transition times $\tau_{\pm 1}^\varepsilon$ of X^ε to ± 1. For $h > 0, \varepsilon > 0, T \geq 0$ let

$$\mathcal{M}^h(\varepsilon, T) = \min_{i=\pm} \mathbb{P}_i\Big(\frac{\tau_{\mp 1}^\varepsilon}{T(\varepsilon)} \in [a_\mu^i - h, a_\mu^i + h]\Big).$$

In the small noise limit, this quality measure tends to 1 and optimal tuning can be obtained due to its asymptotic behavior described by the formula

$$\lim_{\varepsilon\to 0} \varepsilon \ln(1 - \mathcal{M}^h(\varepsilon, T)) = \max_{i=\pm}\{\mu - v_i(a_\mu^i - h)\}$$

for $\mu \in I_R$, uniformly on each compact subset. This property results from classical large deviation techniques applied to an approximation of the diffusion which is supposed to be locally time homogeneous, and will be derived in Chapter 4. Now we minimize the term on the left hand side in the preceding equality. In fact, if the window length $2h$ is small then $\mu - v_i(a_\mu^i - h) \approx 2hv_i'(a_\mu^i)$ since $v_i(a_\mu^i) = \mu$ by definition. The value $v_i'(a_\mu^i)$ is of course negative. Thus the position in which its absolute value is maximal should be identified. At this position the depth of the starting well drops most rapidly below the level μ.

It is clear that for h small the eventually existing global minimizer $\mu_R(h)$ is a good candidate for the resonance point. To get rid of the dependence on h, we shall consider the limit of $\mu_R(h)$ as $h \to 0$ denote by μ_R. This limit, if it exists, is called the *resonance point* of the diffusion with time periodic landscape U. Let us note that for $v_-(t) = \frac{V+v}{4} + \frac{V-v}{4}\cos(2\pi t)$ and $v_+(t) = v_-(t+\pi)$, which corresponds to the case of periodically switching wells' depths between $\frac{v}{2}$ to $\frac{V}{2}$ as in the *frozen* landscape case described above. Then the optimal tuning is $T(\varepsilon) = \exp(\frac{\mu_R}{\varepsilon})$ with $\mu_R = \frac{v+V}{2}$. This optimal rate is equivalent to the optimal rate given by the SPA coefficient.

The big advantage of the quality measure based on the transition times is its robustness. Let us therefore consider the reduced model consisting in a two-state Markov chain with the infinitesimal generator

$$Q(t) = \begin{pmatrix} -\varphi(t) & \varphi(t) \\ \psi(t) & -\psi(t) \end{pmatrix},$$

where $\varphi(t) = \exp(-\frac{v_-(t/T)}{\varepsilon})$ and $\psi(t) = \exp(-\frac{v_+(t/T)}{\varepsilon})$. The distribution of transition times of this Markov chain is well known (see Chapter 4) and, divided by the period length, converges to a^i_μ. The reduced dynamics of the diffusion is captured by the Markov chain, and the optimization of the quality measure $\mathcal{M}^h(\varepsilon, T)$ for the Markov chain and the diffusion leads to the same resonance points.

Our investigation focuses essentially on two criteria: one concerning the family of spectral measures, especially the spectral power amplification coefficient, and the other one dealing with transitions between the local minima of the potential. Many other criteria for optimal tuning between weak periodic signals in dynamical systems and stochastic response can be employed (see Chapter 3). The relation between long deterministic periods and noise intensity usually is expressed in exponential form $T(\varepsilon) = \exp(\frac{\mu}{\varepsilon})$, since the particle needs exponentially large times to cross the barrier separating the wells. This approach relies on the basic assumption that the barrier height is bounded below uniformly in time. This assumption which seems natural in the simple energy balance model of climate dynamics may be questionable in other situations. If the barrier height becomes small periodically on a scale related to the noise intensity, the Brownian particle does not need to wait an exponentially long time to climb it. In this scaling trajectories may appear periodic in the small noise limit. The modulation is assumed to be slow, but the time dependence does not have to be assumed exponentially slow in the noise intensity. In a series of papers [**8, 9, 10, 11, 12**] and in a monograph [**13**], Berglund and Gentz study the case in which the barrier between the wells becomes low twice per period: at time zero the right-hand well becomes almost flat and at the same time the bottom of the well and the saddle approach each other; half a period later, the scenario with the roles of the wells switched occurs. Even in this situation, there is a threshold value for the noise intensity under which transitions are unlikely and, above this threshold, trajectories typically exhibit two transitions per period. In this particular situation, optimal tuning can be described in terms of the concentration of sample paths in small space-time sets.

1.5. Stochastic resonance in models from electronics to biology

As described in the preceding sections, the paradigm of stochastic resonance can quite generally and roughly be seen as the optimal amplification of a weak periodic signal in a dynamical system triggered by random forcing. In this section, we shall briefly deviate from the presentation of our mathematical approach of optimal tuning by large deviations methods, illustrate the ubiquity of the phenomenon of stochastic resonance. We will briefly discuss some prominent examples of dynamical systems arising in different areas of natural sciences in which it occurs, following several big reviews on stochastic resonance from the point of view of natural sciences such as [**1, 43, 44, 79, 108**]. We refer the reader to these references for ample further information on a huge number of examples where stochastic resonance appears. Finally we will briefly comment on computational aspects of stochastic resonance that are important in particular in high dimensional applications.

1.5.1. Resonant activation and Brownian ratchets. The two popular examples we mention here are elementary realizations of transition phenomena corresponding roughly to our paradigm of an overdamped Brownian particle in a potential landscape subject to weak periodic variation of some parameters. Here we face the examples of one-well potentials resp. asymmetric periodic multi-well potentials.

The effect of the so-called *resonant activation* arises in the simple situation in which an overdamped Brownian particle exits from a single potential well with randomly fluctuating potential barrier. In the case we consider the potential barrier can be considered to undergo weak periodic deterministic fluctuations in contrast. Even in the simplest situation, in which the height of the potential barrier is given by a Markov chain switching between two states, one can observe a non-linear dependence of the mean first exit time from the potential well and the intensity of the switching (see e.g. Doering and Elston [**28**]).

Noise induced transport in Brownian ratchets addresses the directed motion of the Brownian particle in a spatially asymmetric periodic potential having the shape of a long chain of downward directed sawtooths of equal length. It arises as another exit time phenomenon, since random exits over the lower potential barrier on the right hand side of the particle's actual position are highly favored. For instance in the context of an electric conductor, this effect creates a current in the downward direction indicated, see Doering et al. [**28, 27**] and Reimann [**91**]. An important application of this effect is the biomolecular cargo transport, see e.g. Elston and Peskin [**35**] and Vanden–Eijnden et al. [**80, 26**].

1.5.2. Threshold models and the Schmitt trigger. Models of stochastic resonance based on a bistable weakly periodic dynamical system of the type (0.1) are often referred to as dynamical models in contrast to the so-called non-dynamical or *threshold models*. These are models usually consisting of a biased deterministic input which may be periodic or not, and a multi-state output. In the simplest situation, the output takes a certain value as the input crosses a critical threshold. The simplest model of this type is the Schmitt trigger, an *electronic device* studied first by Fauve and Heslot [**38**] and Melnikov [**76**] (see also [**1, 43, 69, 70, 74**]). It is given by a well-known electronic circuit, characterized by a two-state output and a hysteretic loop. The circuit is supplied with the input voltage $w = w_t$, which is an arbitrary function of time. In the ideal Schmitt trigger the output voltage $Y = Y_t$ has only two possible values, say $-V$ and V. Let w increase from $-\infty$. Then $Y = V$ until w reaches the critical voltage level V_+. As this happens, the output jumps instantaneously to the level $-V$. Decreasing w does not affect the output Y until w reaches the critical voltage V_-. Then Y jumps back. Therefore, the Schmitt trigger is a bistable system with hysteresis, see Figure 1.14. The width of the hysteresis loop is $V_+ - V_-$. Applying a periodic voltage of small amplitude a

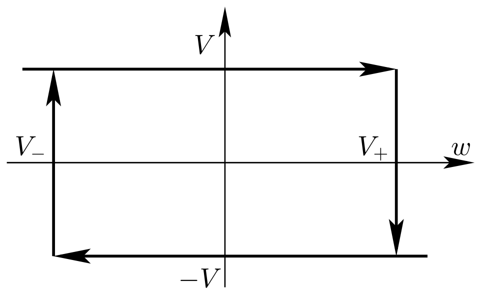

FIGURE 1.14. The input-output characteristic (hysteresis loop) of the Schmitt trigger.

and period $T > 0$, for example, to V_+, we periodically modulate the critical level. After adding a random noise at the input, the system is able to jump between the two states $\pm V$. As in the example of glacial cycles we can consider a discontinuous modulation, for instance given by $V_+(t) = a\,\mathrm{sign}(\sin(\frac{2\pi t}{T}))$. The whole picture is now similar to the one in (1.3). Here the periodic modulation of the reference voltage corresponds to the tilting of the potential wells.

Fauve and Heslot [**38**] studied the power spectrum of the system and, as in the glacial cycle example, established that the energy carried by the spectral component of Y at a given driving frequency has a local maximum for a certain intensity of the input noise.

The Schmitt trigger provides another interpretation to the phenomenon of stochastic resonance. A system displaying stochastic resonance can be considered as a random amplifier. The weak periodic signal which cannot be detected in the absence of noise, can be successfully recovered if the system (the Schmitt trigger or (1.3)) is appropriately tuned. In other words, the weak underlying periodicity is exhibited at appropriately chosen non-zero levels of noise, and gets lost if noise is either too small or too large.

To date, the most important application area of threshold models is neural dynamics (see Bulsara et al. [**17**], Douglass et al. [**29**], Patel and Kosko [**85**]) and transmission of information (see Neiman et al. [**81**], Simonotto et al. [**99**], Stocks [**101**], Moss et al. [**79**]). The recent book [**73**] by McDonnell et al. gives a very complete account on the theory of non-dynamic or threshold stochastic resonance.

1.5.3. The paddlefish. In this well known and frequently discussed example stochastic resonance appears in the noise-enhanced feeding behavior of the paddlefish *Polyodon spathula* (see Greenwood et al. [**47**], Russel et al. [**95**], Freund et al. [**42**]). This species of fish lives in the Midwest of the United States and in the Yangtze River in China, and feeds on the zooplankton Daphnia. To detect its prey animals under limited visibility conditions at river bottoms, the paddlefish uses the long rostrum in front of its mouth as an electrosensory antenna. The frequency range of sensitivity of the rostrum's electroreceptors well overlaps with the range of frequencies produced by the prey. Roughly, the capture probability is observed as a function of the position of the prey relative to the rostrum. In experiments, external noise was generated by electrodes connected to an electric noise generator. It was observed that the spatial distribution and number of strike locations is a function of the external noise intensity, with a maximum of captures of more distant plankton at some optimal external noise intensity. If experimentally noisy electric signals improve the sensitivity of the electroreceptors, nature itself should also provide sources of noise. In [**95**] it was conjectured that, besides the signal, such a noise might be produced by the populations of prey animals themselves. In [**42**] this conjecture was confirmed by measurements of the noise strength produced by single Daphnia in the vicinity of a swarm. In the simple quantitative approaches, quality of tuning is measured by Fisher information, a concept that may be comparable to the entropy notions in Chapter 3, Section 3.2.

1.5.4. The FitzHugh–Nagumo system. A more detailed modeling of neural activities of living systems underlies this well known and studied example. It deals with action potentials and electric currents transmitted through systems of ion channels provided by the axons in neural networks, triggered by their mutual interaction and the interaction of the system with the biological environment. Neurons communicate with each other or with muscle cells by means of electric signals. Each single neuron can be modeled as an excitable dynamical system: in the rest state characterized by a negative potential gap with respect to the extracellular environment, no current flows through the membrane of the neuronal cell. If this threshold potential barrier disappears due to noisy perturbations created by the environment (neighboring cells, external field), ion channels through the membrane are opened and currents appear in form of a spike or firing, followed by a deterministic recovery to the rest state. During a finite (refractory) time interval, the membrane potential is hyperpolarized by the current flow, and any firing impossible. The theory that captures the above-mentioned features of neuronal dynamics, including the finite refractory time, is described by the FitzHugh–Nagumo (FHN) equations (see Kanamaru et al. [**63**]). In the diffusively coupled form, a system of N coupled neurons is described by the system of equations (see [**63**])

$$\begin{aligned}
\tau\dot{u}_i(t) &= \Big(-v_i + \Big(u_i - \frac{u_i^3}{3}\Big) + S(t)\Big) + \sqrt{\varepsilon}\,\dot{W}_i(t) + \frac{1}{N}\sum_{j=1}^{N}(u_i - u_j),\\
\dot{v}_i(t) &= u_i - \beta v_i + \gamma.
\end{aligned}$$

Here u_i describes the membrane potential of neuron i, v_i a variable describing whether and to which degree neuron i is in the refractory interval of time after firing, S describes an external periodic pulse acting on the potential levels, while $W_1, \ldots, W_n$ is a vector of independent Brownian motions. Finally, τ, β, ε and γ are system parameters. In the infinite particle limit, the system becomes a stochastic partial differential equation. Roughly, total throughput current will be a function of the model parameters, and stochastic resonance appears as its optimal value for a suitable parameter choice (synchronization). The paper by Wiesenfeld et al. [**109**] reports about a much simplified form of this system, in which action potentials of single mechanoreceptor cells of the crayfish Procambarus clarkii are concerned. The mechanosensory system of the crayfish consists of hairs located on its tailfan, connected to mechanoreceptor cells. Streaming water moves the hair and so provides the external excitation that causes the mechanoreceptor cell to fire. Experimentally (see [**109**]), a piece of tailfan containing the hair and sensory neuron was extracted and put into a saline solution environment. Then, periodic pressure modulations and random noise were imposed on the environment. The firings produced by the mechanoreceptor cell were recorded for different noise levels, and show clear stochastic resonance peaks as functions of noise intensity. Similar phenomena are encountered on a much more general basis in the exchange of substances or information through ionic channels on cell membranes in living organisms.

1.5.5. Physiological systems. The fact that sensory neurons are excitable systems leading to the FitzHugh–Nagumo equations in the preceding subsection, is also basic for many suggestions of how to make use of the phenomenon of stochastic resonance in medicine. Disfunctions arising in sensory organs responsible for hearing, tactile or visual sensations or for balance control could result from relatively higher sensitivity thresholds compared with those of healthy organs. To raise the sensitivity level, a natural idea seems to be to apply the right amount of external noise to these dysfunctional organs, in order to let stochastic resonance effects amplify weak signal responses. In experiments reported in Collins et al. [**21**], local indentations were applied to the tips of digits of test persons who had to correctly identify whether a stimulus was presented. Stimuli generated by subthreshold signals garnished with noise led to improvements in correct identification, with some optimal noise level indicating a stochastic resonance point. Results like this may be used for designing practical devices, as for instance gloves, for individuals with elevated cutaneous sensory thresholds. Similarly, randomly vibrating shoe inserts may help restoring balance control (see Priplata et al. [**89**]). Stochastic resonance effects may be used for treating disfunctions of the human blood pressure system (baroreflex system) featuring a negative feedback between blood pressure and heart rate resp. width of blood vessels. Blood pressure is monitored by two types of receptors, for arteries and veins. In Hidaka [**54**], a weak periodic input was introduced at the venous blood pressure receptor, whereas noise was added to the arterial receptor. It was shown that the power of the output signal of the heart rate (measured by an electrocardiogram) as a function of noise intensity exhibits a bell-shape form, typical for a curve with a stochastic resonance point. Another group of possible medical applications of the amplification effects of stochastic resonance is related to the human brains information processing activity (see Mori and Kai [**78**]). In an experiment in Usher and Feingold [**103**] the effect of stochastic resonance in the speed of memory retrieval was exhibited. Test persons were proven to perform single digit calculations (e.g. $7 \times 8 =?$) significantly faster when exposed to an optimal level of acoustic noise (via headphones).

1.5.6. Optical systems. In optical systems, stochastic resonance was first observed in McNamara et al. [**75**] and Vemuri and Roy [**104**] in a bidirectional ring laser, i.e. a ring resonator with a dye as lasing medium. This laser system supports two meta-stable states realized as modes of the same frequency that travel in opposite directions. They are strongly coupled to each other by the lasing medium, thus permitting a bistable operation. When the pumping exceeds the lasing threshold, either clockwise or counterclockwise modes propagate in the laser, with switchings between those two modes initiated by spontaneous emission in the active medium, and fluctuations of the pump laser. The net gains of the two propagating modes in opposite directions can be controlled by an acousto-optical modulator inside the cavity, which thus can be used both to impose a periodic switching rate between the modes and to inject noise. Therefore the resulting semiclassical laser equations are equivalent to those describing overdamped motion of a particle in a periodically modulated double well potential, as described in the prototypical example of Section 1.2.

The choice of examples we discussed in more detail is rather selective. The effects of stochastic resonance have been found in a big number of dynamical systems in various further areas of the sciences, and studied by a variety of physical measures of quality of tuning. We just mention a big field of applications in microscopic systems underlying the laws of quantum mechanics in which intrinsic quantum tunneling effects interfere with the interpretation of potential barrier tunneling that can be seen as causing noise induced transitions in diffusion dynamics. See [**108**] for a comprehensive survey. Stochastic resonance has further been observed in passive optical bistable systems [**30**], in experiments with magnetoelastic ribbons [**100**], in chemical systems [**67**], as well as in further biological ones [**94, 60, 41**]. Stochastic resonance may even be observed in more general systems in which the role of periodic deterministic signals is taken by some other physical mechanisms (see [**108**]).

1.5.7. Computational aspects of large deviation theory related to stochastic resonance. Our theoretical approach aimed at explaining stochastic resonance conceptually by means of space-time large deviations of weakly periodic dynamical systems does not touch at all the field of numerical algorithms and scientific computing for stochastic resonance related quantities which become very important for applications especially in high dimensions. In the framework of the classical Freidlin–Wentzell theory, first exit time estimates as well as large deviations rates are analytically expressed by the *quasi-potential* (see Chapter 2) which can be calculated more or less explicitly for gradient systems. To determine and minimize the quasi-potential in high-dimensional scenarios is an analytically hardly accessible task. In Vanden–Eijnden et al. [**31, 53**] practically relevant algorithms with numerous applications for this task have been developed. They have been applied to various problems in different areas of application of stochastic resonance.