

Introduction

I restricted myself to characteristic zero: for a short time, the quantum jump to $p \neq 0$ was beyond the range ... but it did not take me too long to make this jump.

—Oscar Zariski

The arithmetic of abelian varieties with complex multiplication over a number field is fascinating. However this will not be our focus. We study the theory of complex multiplication in mixed characteristic.

Abelian varieties over finite fields. In 1940 Deuring showed that an elliptic curve over a finite field can have an endomorphism algebra of rank 4 [33, §2.10]. For an elliptic curve in characteristic zero with an endomorphism algebra of rank 2 (rather than rank 1, as in the “generic” case), the j -invariant is called a *singular j -invariant*. For this reason elliptic curves with even more endomorphisms, in positive characteristic, are called *supersingular*.¹

Mumford observed as a consequence of results of Deuring that for any elliptic curves E_1 and E_2 over a finite field κ of characteristic $p > 0$ and any prime $\ell \neq p$, the natural map

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}(E_1, E_2) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_\ell[\mathrm{Gal}(\bar{\kappa}/\kappa)]}(T_\ell(E_1), T_\ell(E_2))$$

(where on the left side we consider only homomorphisms “defined over κ ”) is an isomorphism [118, §1]. The interested reader might find it an instructive exercise to reconstruct this (unpublished) proof by Mumford. Tate proved in [118] that the analogous result holds for all abelian varieties over a finite field and he also incorporated the case $\ell = p$ by using p -divisible groups. He generalized this result into his influential conjecture [117]:

An ℓ -adic cohomology class² that is fixed under the Galois group should be a \mathbb{Q}_ℓ -linear combination of fundamental classes of algebraic cycles when the ground field is finitely generated over its prime field.

Honda and Tate gave a classification of isogeny classes of simple abelian varieties A over a finite field κ (see [50] and [121]), and Tate refined this by describing

¹Of course, a supersingular elliptic curve isn’t singular. A purist perhaps would like to say “an elliptic curve with supersingular j -value”. However we will adopt the generally used terminology “supersingular elliptic curve” instead.

²The prime number ℓ is assumed to be invertible in the base field.

the structure of the endomorphism algebra $\text{End}^0(A)$ (working in the isogeny category over κ) in terms of the Weil q -integer of A , with $q = \#\kappa$; see [121, Thm. 1]. It follows from Tate's work (see 1.6.2.5) that an abelian variety A over a finite field κ admits sufficiently many complex multiplications in the sense that its endomorphism algebra $\text{End}^0(A)$ contains a CM subalgebra³ L of rank $2 \dim(A)$. We will call such an abelian variety (in any characteristic) a CM *abelian variety* and the embedding $L \rightarrow \text{End}^0(A)$ a CM *structure* on A .

Grothendieck showed that over any algebraically closed field K , an abelian variety that admits sufficiently many complex multiplications is isogenous to an abelian variety defined over a finite extension of the prime field [89]. This was previously known in characteristic zero (by Shimura and Taniyama), and in that case there is a number field $K' \subset K$ such that the abelian variety can be defined over K' (in the sense of 1.7.1). However in positive characteristic such abelian varieties can fail to be defined over a finite subfield of K ; examples exist in every dimension > 1 (see Example 1.7.1.2).

Abelian varieties in mixed characteristic. In characteristic zero, an abelian variety A gives a representation of the endomorphism algebra $D = \text{End}^0(A)$ on the Lie algebra $\text{Lie}(A)$ of A . If A has complex multiplication by a CM algebra L of degree $2 \dim(A)$ then the isomorphism class of the representation of L on $\text{Lie}(A)$ is called the CM *type* of the CM structure $L \hookrightarrow \text{End}^0(A)$ on A (see Lemma 1.5.2 and Definition 1.5.2.1).

As we noted above, every abelian variety over a finite field is a CM abelian variety. Thus, it is natural to ask whether every abelian variety over a finite field can be “CM lifted” to characteristic zero (in various senses that are made precise in 1.8.5). One of the obstacles⁴ in this question is that in characteristic zero there is the notion of CM type that is invariant under isogenies, whereas in positive characteristic whatever can be defined in an analogous way is *not* invariant under isogenies. For this reason we will use the terminology “CM type” only in characteristic zero.

For instance, the action of the endomorphism ring $R = \text{End}(A_0)$ of an abelian variety A_0 on the Lie algebra of A_0 in characteristic $p > 0$ defines a representation of R/pR on $\text{Lie}(A_0)$. Given an isogeny $f : A_0 \rightarrow B_0$ we get an identification $\text{End}^0(A_0) = \text{End}^0(B_0)$ of endomorphism algebras, but even if $\text{End}(A_0) = \text{End}(B_0)$ under this identification, the representations of this common endomorphism ring on $\text{Lie}(A_0)$ and $\text{Lie}(B_0)$ may well be non-isomorphic since $\text{Lie}(f)$ may not be an isomorphism. Moreover, if we have a lifting A of A_0 over a local domain of characteristic 0, in general the inclusion $\text{End}(A) \subset \text{End}(A_0)$ is not an equality. If the inclusion $\text{End}^0(A) \subset \text{End}^0(A_0)$ is an equality then the character of the representation of $\text{End}(A_0)$ on $\text{Lie}(A_0)$ is the reduction of the character of the representation of $\text{End}(A)$ on $\text{Lie}(A)$. This relation can be viewed as an obstruction to the existence of CM lifting with the full ring of integers of a CM algebra operating on the lift; see 4.1.2, especially 4.1.2.3–4.1.2.4, for an illustration.

In the case when $\text{End}(A_0)$ contains the ring of integers \mathcal{O}_L of a CM algebra $L \subset \text{End}^0(A_0)$ with $[L : \mathbb{Q}] = 2 \dim(A_0)$, the representation of $\mathcal{O}_L/p\mathcal{O}_L$ on $\text{Lie}(A_0)$ turns out to be quite useful, despite the fact that it is not an isogeny invariant. Its class in a suitable K-group will be called the *Lie type* of $(A_0, \mathcal{O}_L \hookrightarrow \text{End}(A_0))$.

³A CM algebra is a finite product of CM fields; see Definition 1.3.3.1.

⁴surely also part of the attraction

The above discrepancy between the theories in characteristic zero and characteristic $p > 0$ is the basic phenomenon underlying this entire book. Before discussing its content, we recall the following theorem of Honda and Tate ([50, §2, Thm. 1] and [121, Thm. 2]).

For an abelian variety A_0 over a finite field κ there is a finite extension κ' of κ and an isogeny $(A_0)_{\kappa'} \rightarrow B_0$ such that B_0 admits a CM lifting over a local domain of characteristic zero with residue field κ' .

This result has been used in the study of Shimura varieties, for settings where the ground field is an algebraic closure of \mathbb{F}_p and isogeny classes (of structured abelian varieties) are the objects of interest; see [135]. Our starting point comes from the following questions which focus on controlling ground field extensions and isogenies.

For an abelian variety A_0 over a finite field κ , to ensure the existence of a CM lifting over a local domain with characteristic zero and residue field κ' of finite degree over κ ,

- (a) *may we choose $\kappa' = \kappa$?*
- (b) *is an isogeny $(A_0)_{\kappa'} \rightarrow B_0$ necessary?*

These questions are formulated in various precise forms in 1.8.

An isogeny is necessary. Question (b) was answered in 1992 (see [93]) as follows.

There exist (many) abelian varieties over $\overline{\mathbb{F}}_p$ that do not admit any CM lifting to characteristic zero.

The main point of [93] is that a CM liftable abelian variety over $\overline{\mathbb{F}}_p$ can be defined over a small finite field. This idea is further pursued in Chapter 3, where the *size*, or more accurately the *minima*⁵ of the size, of all possible *fields of definition* of the p -divisible group of a given abelian variety over $\overline{\mathbb{F}}_p$ is turned into an obstruction for the existence of a CM lifting to characteristic 0. This is used to show (in 3.8.3) that in “most” isogeny classes of non-ordinary abelian varieties of dimension ≥ 2 over finite fields there is a member that has no CM lift to characteristic 0. (In dimension 1 a CM lift to characteristic 0 always exists, over the valuation ring of the minimal possible p -adic field, by Deuring Lifting Theorem; see 1.7.4.6.) We also provide effectively computable examples of abelian varieties over explicit finite fields such that there is no CM lift to characteristic 0.

A field extension might be necessary—depending on what you ask.

Bearing in mind the necessity to modify a given abelian variety over a finite field to guarantee the existence of a CM lifting, we rephrase question (a) in a more precise version (a)' below.

(a)' *Given an abelian variety A_0 over a finite field κ of characteristic p , is it necessary to extend scalars to a strictly larger finite field $\kappa' \supset \kappa$ (depending on A_0) to ensure the existence of a κ' -rational isogeny $(A_0)_{\kappa'} \rightarrow B_0$ such that B_0 admits a CM lifting over a characteristic 0 local domain R with residue field κ' ?*

It turns out there are two quite different answers to question (a)', depending on whether one requires the local domain R of characteristic 0 to be normal. The subtle distinction between using normal or general local domains for the lifting

⁵The size of a finite field κ_1 is *smaller* than the size of a finite field κ_2 if κ_1 is isomorphic to a subfield of κ_2 , or equivalently if $\#\kappa_1 \mid \#\kappa_2$. Among the sizes of a family of finite fields there may not be a unique minimal element.

went unnoticed for a long time. Once this distinction came in focus, answers to the resulting questions became available.

If we ask for a CM lifting over a *normal* domain up to isogeny, in general a base field extension *before* modification by an isogeny is necessary. This is explained in 2.1.2, where we formulate the “residual reflex obstruction”, the idea for which goes as follows. Over an algebraically closed field K of characteristic zero, we know that a simple CM abelian variety B with K -valued CM type Φ (for the action of a CM field L) is defined over a number field in K containing the reflex field $E(\Phi)$ of Φ . Suppose that for every K -valued CM type Φ of L , the residue field of $E(\Phi)$ at any prime above p is not contained in the finite field κ with which we began in question (a). In such cases, for every CM structure $L \rightarrow \text{End}^0(A_0)$ on A_0 and any abelian variety B_0 over κ which is κ -isogenous to A_0 , there is no L -linear CM lifting of B_0 over a *normal* local domain R of characteristic zero with residue field κ .⁶ In 2.3.1–2.3.3 we give such an example, a supersingular abelian surface A_0 over \mathbb{F}_{p^2} with $\text{End}(A_0) = \mathbb{Z}[\zeta_5]$ for any $p \equiv \pm 2 \pmod{5}$. A much broader class of examples is given in 2.3.5, consisting of absolutely simple abelian varieties (with arbitrarily large dimension) over \mathbb{F}_p for infinitely many p .

Note that passing to the normalization of a complete local noetherian domain generally enlarges the residue field. Hence, if we drop the condition that the mixed characteristic local domain R be normal then the obstruction in the preceding consideration dissolves. And in fact we were put on the right track by mathematics itself. The phenomenon is best illustrated in the example in 4.1.2, which is the same as the example in 2.3.1 already mentioned: an abelian surface C_0 over \mathbb{F}_{p^2} with CM order $\mathbb{Z}[\zeta_5]$ that, even up to isogeny, is not CM liftable to a *normal* local domain of characteristic zero. On the other hand, this abelian surface C_0 is CM liftable to an abelian scheme C over a mixed characteristic non-normal local domain of characteristic zero, though the maximal subring of $\mathbb{Z}[\zeta_5]$ whose action lifts to C is non-Dedekind locally at p ; see 4.1.2.⁷ This example is easy to construct, and the proof of the existence of a CM lifting, possibly after applying an \mathbb{F}_{p^2} -rational isogeny, is not difficult either.

In Chapter 4 we show that the general question of existence of a CM lifting after an appropriate isogeny can be reduced to the same question for (a mild generalization of) the example in 4.1.2, enabling us to prove:

every abelian variety A_0 over a finite field κ admits an isogeny $A_0 \rightarrow B_0$ over κ such that B_0 admits a CM lifting to a mixed characteristic local domain with residue field κ .

There are refined lifting problems, such as specifying at the beginning which CM structure on A_0 is to be lifted, or even what its CM type should be on a geometric fiber in characteristic 0. These matters will also be addressed.

⁶The source of obstructions is that the *base field* κ might be too small to contain at least one characteristic p residue field of the reflex field $E(\Phi)$ for at least one CM type Φ on L . Thus, the field of definition of the generic fiber of the hypothetical lift may be too big. Likewise, an obstruction for question (b) is that the field of definition of the p -divisible group $A_0[p^\infty]$ may be too big (in a sense that is made precise in 3.8.3 and illustrated in 3.8.4–3.8.5).

⁷No modification by isogeny is necessary in this example, but the universal deformation for C_0 with its $\mathbb{Z}[\zeta_5]$ -action is a *non-algebraizable* formal abelian scheme over $W(\mathbb{F}_{p^2})$.

Our basic method is to “localize” various CM lifting problems to the corresponding problems for p -divisible groups. Although global properties of abelian varieties are often lost in this localization process, the non-rigid nature of p -divisible groups can be an advantage. In Chapter 3 the size of fields of definition of a p -divisible group in characteristic p appears as an obstruction to the existence of CM lifting. The reduction steps in Chapter 4 rely on a classification and descent of CM p -divisible groups in characteristic p with the help of their Lie types (see 4.2.2, 4.4.2). In addition, the “Serre tensor construction” is applied to p -divisible groups, both in characteristic p and in mixed characteristic $(0, p)$; see 1.7.4 and 4.3.1 for this general construction.

Survey of the contents. In **Chapter 1** we start with a survey of general facts about CM abelian varieties and their endomorphism algebras. In particular, we discuss the deformation theory of abelian varieties and p -divisible groups, and we review results in Honda-Tate theory that describe isogeny classes and endomorphism algebras of abelian varieties over a finite field in terms of Weil integers. We conclude by formulating various CM lifting questions in 1.8. These are studied in the following chapters. We will see that the questions can be answered with some precision.

In **Chapter 2** we formulate and study the “residual reflex condition”. Using this condition we construct several examples of abelian varieties over finite fields κ such that, even after applying a κ -isogeny, there is no CM lifting to a *normal local domain* with characteristic zero and residue field of finite degree over κ ; see 2.3. It is remarkable that many such examples exist, but we do not know whether we have characterized all possible examples; see 2.3.7.

We then study algebraic Hecke characters and review part of the theory of complex multiplication due to Shimura and Taniyama. Using the relationship between algebraic Hecke characters for a CM field L and CM abelian varieties with CM by L (the precise statement of which we review and prove), we use global methods to show that the residual reflex condition is the *only* obstruction to the existence of CM lifting up to isogeny over a *normal* local domain of characteristic zero. We also give another proof by local methods (such as p -adic Hodge theory).

In **Chapter 3** we take up methods described in [93]. In that paper classical CM theory in characteristic zero was used. Here we use p -divisible groups instead of abelian varieties and show that the size of fields of definition of a p -divisible group in characteristic p is a non-trivial obstruction to the existence of a CM lifting. In 3.3 we study the notion of isogeny for p -divisible groups over a base scheme (including its relation with duality). We show, in one case of the CM lifting problem left open in [93, Question C], that an isogeny is necessary. Our methods also provide effectively computed examples. Some facts about CM p -divisible groups explained in 3.7 are used in 3.8 to get an upper bound of a field of definition for the closed fiber of a CM p -divisible group.

In Appendix 3.9, we use the construction (in 3.7) of a p -divisible group with any given p -adic CM type over the reflex field to produce a semisimple abelian crystalline p -adic representation of the local Galois group such that its restriction to the inertia group is “algebraic” with algebraic part that we may prescribe arbitrarily in accordance with some necessary conditions (see 3.9.4 and 3.9.8).

In **Chapter 4** we show CM liftability after an isogeny over the finite ground field (lifting over a characteristic zero local domain that need not be normal). That is,

every CM structure $(A_0, L \rightarrow \text{End}^0(A_0))$ over a finite field κ has an isogeny over κ to a CM structure $(B_0, L \rightarrow \text{End}^0(B_0))$ that admits a CM lifting;

(see 4.1.1). This statement is immediately reduced to the case when L is a CM field (not just a CM algebra) and the whole ring \mathcal{O}_L of integers of L operates on A_0 , which we assume.

Our motivation comes from the proof in 4.1.2 (using an algebraization argument at the end of 4.1.3) that the counterexample in 2.3.1 to CM lifting over a *normal* local domain satisfies this property. In general, after an easy reduction to the isotypic case, we apply the Serre-Tate deformation theorem to localize the problem at p -adic places v of the maximal totally real subfield L^+ of a CM field $L \subseteq \text{End}^0(A_0)$ of degree $2 \dim(A_0)$. This reduces the existence of a CM lifting for the abelian variety A_0 to a corresponding problem for the CM p -divisible group $A_0[v^\infty]$ attached to v .⁸

We formulate several properties of v with respect to the CM field L ; any one of them ensures the existence of a CM lifting of $A_0[v^\infty]_{\bar{\kappa}}$ after applying a κ -isogeny to $A_0[v^\infty]$ (see 4.1.6, 4.1.7, and 4.5.7). These properties involve the ramification and residue fields of L and L^+ relative to v . If v violates all of these properties then we call it *bad* (with respect to L/L^+ and κ). Let $L_v := L \otimes_{L^+} L_v^+$. After applying a preliminary κ -isogeny to arrange that $\mathcal{O}_L \subset \text{End}(A_0)$, for v that are not bad we apply an \mathcal{O}_L -linear κ -isogeny to arrange that the Lie type of the $\mathcal{O}_{L,v}$ -factor of $\text{Lie}(A_0)$ (i.e., its class in a certain K-group of $(\mathcal{O}_{L,v}/(p)) \otimes \kappa$ -modules) is “self-dual”. Under the self-duality condition (defined in 4.4.3) we produce an $\mathcal{O}_{L,v}$ -linear CM lifting of $A_0[v^\infty]_{\bar{\kappa}}$ by specializing a suitable $\mathcal{O}_{L,v}$ -linear CM v -divisible group in mixed characteristic; see 4.4.6. We use an argument with deformation rings to eliminate the intervention of $\bar{\kappa}$: if every p -adic place v of L^+ is not bad then there exists a κ -isogeny $A_0 \rightarrow B_0$ such that $\mathcal{O}_L \subset \text{End}(B_0)$ and the pair $(B_0, \mathcal{O}_L \hookrightarrow \text{End}(B_0))$ admits a lift to characteristic 0 without increasing κ .

If some p -adic place v of the totally real field L^+ is bad then the above argument does not work because in that case no member of the $\mathcal{O}_{L,v}$ -linear κ -isogeny class of the p -divisible group $A_0[v^\infty]$ has a self-dual Lie type. Instead we change $A_0[v^\infty]$ by a suitable $\mathcal{O}_{L,v}$ -linear κ -isogeny so that its Lie type becomes as symmetric as possible, a condition whose precise formulation is called “striped”. Such a p -divisible group is shown to be isomorphic to the Serre tensor construction applied to a special class of 2-dimensional p -divisible groups of height 4 that are similar to the ones arising from the abelian surface counterexamples in 2.3.1; we call these *toy models* (see 4.1.3, especially 4.1.3.2).

These “toy models” are sufficiently special that we can analyze their CM lifting properties directly; see 4.2.10 and 4.5.15(iii). After this key step we deduce the existence of a CM lifting of $A_0[v^\infty]_{\bar{\kappa}}$ from corresponding statements for (the p -divisible group version of) toy models. In the final step, once again we use deformation theory to produce an abelian variety B_0 isogenous to (the original) A_0 over κ and a CM lifting of B_0 over a possibly *non-normal* 1-dimensional complete local noetherian domain of characteristic 0 with residue field κ . Although \mathcal{O}_L acts

⁸See 1.4.5.3 for the statement of the Serre–Tate deformation theorem, and 2.2.3 and 4.6.3.1 for a precise statement of the algebraization criterion that is used in this localization step.

on the closed fiber, we can only ensure that a subring of \mathcal{O}_L of finite index⁹ acts on the lifted abelian scheme (see 4.6.4).

Appendix A. In Appendix A.1 we provide a self-contained development of the proof of the p -part of Tate’s isogeny theorem for abelian varieties over finite fields of characteristic p , as well as a proof of Tate’s formula for the local invariants at p -adic places for endomorphism algebras of simple abelian varieties over such fields. (An exposition of these results is also given in [79]; our treatment uses less input from non-commutative algebra.) Appendices A.2 and A.3 provide purely algebraic proofs of the Main Theorem of Complex Multiplication for abelian varieties, as well as a converse result, both of which are used in essential ways in Chapter 2. In Appendix A.4 we use Shimura’s method to show that an algebraic Hecke character with a given algebraic part can be constructed over the field of moduli of the algebraic part, with control over places of bad reduction.

In the special case of the reflex norm of a CM type (L, Φ) , combining this construction of algebraic Hecke characters with the converse to the Main Theorem of CM in A.3 proves that over the associated field of moduli $M \subset \overline{\mathbb{Q}}$ (a subfield of the Hilbert class field of the reflex field $E(L, \Phi)$) there exists a CM abelian variety A with CM type (L, Φ) such that A has good reduction at all p -adic places of M ; see A.4.6.5. Since M is the smallest possible field of definition given (L, Φ) , this existence result is optimal in terms of its field of definition. Typically $M \neq E(L, \Phi)$, and this is regarded as a “class group obstruction” to finding A with its CM structure by L over $E(L, \Phi)$, a well-known phenomenon in the classical CM theory of elliptic curves.

(In the “local” setting of CM p -divisible groups over p -adic integer rings there are no class group problems and one gets a better result: in 3.7 we use the preceding global construction over the field of moduli to prove that for any p -adic CM type (F, Φ) and the associated p -adic reflex field $E \subset \overline{\mathbb{Q}}_p$ over \mathbb{Q}_p there exists a CM p -divisible group over \mathcal{O}_E with p -adic CM type (F, Φ) .)

Appendix B. In Appendices B.1 and B.2, we give two versions of a more direct (but more complicated) proof of the existence of CM liftings for a higher-dimensional generalization of the toy model.¹⁰ The first version uses Raynaud’s theory of group schemes of type (p, \dots, p) . The second version uses recent developments in p -adic Hodge theory. We hope that material described there will be useful in the future. In Appendix B.3 we compare several Dieudonné theories over a perfect base field of characteristic $p > 0$. In Appendix B.4 we give a formula for the Dieudonné module of the closed fiber of a finite flat commutative group scheme, constructed using integral p -adic Hodge theory; this formula is used in B.2.

⁹This subring of finite index can be taken to be $\mathbb{Z} + p\mathcal{O}_L$.

¹⁰In the original proof of our main CM lifting result in 4.1.1, the case of a bad place $v|p$ of L^+ was reduced through the Serre tensor construction to this existence result. Both B.1 and B.2 are logically independent of results in Chapter 4. Readers who cannot wait to see a proof of the existence of a CM lifting (without modification by any isogeny) for such a higher-dimensional toy model may proceed directly to B.1 or B.2, after consulting 4.2 for the definition of the Lie type of an \mathcal{O} -linear p -divisible group and related notation.

References

- (1) ABELIAN VARIETIES. In Mumford's book [82] the theory of abelian varieties is developed over an algebraically closed base field, and we need the theory over a general field; references addressing this extra generality are Milne's article [76] (which rests on [82]) and the forthcoming book [45]. Since [45] is not yet in final form we do not refer to it in the main text, but the reader should keep in mind that many results for which we refer to [82] and [76] are also treated in [45]. We refer the reader to [83, Ch. 6, §1–§2] for a self-contained development of the elementary properties of abelian schemes, which we freely use. (For example, the group law is necessarily commutative and is determined by the identity section, as in the theory over a field.)
- (2) SEMISIMPLE ALGEBRAS. We assume familiarity with the classical theory of finite-dimensional semisimple algebras over fields (including the theory of their splitting fields and maximal commutative subfields). A suitable reference for this material is [53, §4.1–4.6]; another reference is [11]. In 1.2.2–1.2.4 we review some of the facts that we need from that theory.
- (3) DESCENT THEORY AND FORMAL SCHEMES. In many places, we need to use the techniques of descent theory and Grothendieck topologies (especially the fppf topology, though in some situations we use the fpqc topology to perform descent from a completion). This is required for arguments with group schemes, even over a field, such as in considerations with short exact sequences. For accounts of descent theory, we refer the reader to [10, §6.1–6.2], and to [39, Part 1] for a more exhaustive discussion. These techniques are discussed in a manner well-suited to group schemes in [98] and [30, Exp. IV–VI_A].

Our arguments with deformation theory rest on the theory of formal schemes, especially Grothendieck's formal GAGA and algebraization theorems. A succinct overview of these matters is given in [39, Part 4], and the original references [34, I, §10; III₁, §4–§5] are also highly recommended.

- (4) DIEUDONNÉ THEORY AND p -DIVISIBLE GROUPS. To handle p -torsion phenomena in characteristic $p > 0$ we use Dieudonné theory and p -divisible groups. Brief surveys of some basic definitions and properties in this direction are given in 1.4, 3.1.2–3.1.6, and B.3.5.1–B.3.5.5. We refer the reader to [119], [71] and [110, §6] for more systematic discussions of basic facts concerning p -divisible groups, and to [29] and [41, Ch. II–III] for self-contained developments of (contravariant) Dieudonné theory, with applications to p -divisible groups. Contravariant Dieudonné theory is used in Chapters 1–4.

Covariant Dieudonné theory is used in Appendix B.1 because the alternative proof there of the main result of Chapter 4 uses a covariant version of p -adic Hodge theory. A brief summary of covariant Dieudonné can be found in B.3.5.6–B.3.6.7. We recommend [136] for Cartier theory; an older standard reference is [69].

A very useful technique within the deformation theory of p -divisible groups is Grothendieck–Messing theory, which is developed from scratch in [75]. Although we do not provide an introduction to this topic, we hope that our applications of it may inspire an interested reader who is not familiar with this technique to learn more about it.

Notation and terminology

- Numerical labeling of text items and displayed expressions.
 - We use “ $x.y.z$ ”, “ $x.y.z.w$ ”, etc. for text items (sub-subsections, results, remarks, definitions, etc.), arranged lexicographically without repetition.
 - Any labeling of displayed expressions (equations, commutative diagrams, etc.) is indicated with parentheses, so “see $(x.y.z)$ ” means that one should look at the z th displayed expression in subsection $x.y$. This convention avoids confusion with the use of “ $x.y.z$ ” to label a text item.
 - Any label for a text item is *uniquely assigned*, so even though “see $x.y.z$ ” does not indicate if it is a sub-subsection or theorem (or lemma, etc.), there is no ambiguity for finding it in this book.
- Convention on notation.
 - p denotes a prime number.
 - CM fields are usually denoted by L .
 - K often stands for an arbitrary field, κ is usually used to denote either a residue field or a finite field of characteristic p .
 - V^\vee denotes the dual of a finite-dimensional vector space V over a field.
 - k denotes a perfect field, often of characteristic $p > 0$. In 4.2–4.6, k is an algebraically closed field of characteristic p .
 - K_0 is the fraction field of $W(k)$, where k is a perfect field of characteristic $p > 0$ and $W(k)$ is the ring of p -adic Witt vectors with entries in k .
 - Abelian varieties are usually written as A , B , or C , and p -divisible groups are often denoted as G or as X or Y .
 - The p -divisible group attached to an abelian variety or an abelian scheme A is denoted by $A[p^\infty]$; its subgroup scheme of p^n -torsion points is $A[p^n]$.
- Fields and their extensions.
 - For a field K , we write \overline{K} to denote an algebraic closure and K_s to denote a separable closure.
 - An extension of fields K'/K is *primary* if K is separably algebraically closed in K' (i.e., the algebraic closure of K in K' is purely inseparable over K).
 - For a number field L we write \mathcal{O}_L to denote its ring of integers. Similar notation is used for non-archimedean local fields.
 - If q is a power of a prime p , \mathbb{F}_q denotes a finite field with size q (sometimes understood to be the unique subfield of order q in a fixed algebraically closed field of characteristic p). If κ and κ' are abstract finite fields with respective sizes $q = p^n$ and $q' = p^{n'}$ for integers $n, n' \geq 1$ then $\kappa \cap \kappa'$ denotes the unique subfield of either κ or κ' with size $p^{\gcd(n, n')}$; the context will always make clear if this is being considered as a subfield of either κ or κ' . Likewise, $\kappa\kappa'$ denotes $\kappa \otimes_{\kappa \cap \kappa'} \kappa'$, a common extension of κ and κ' with size $p^{\text{lcm}(n, n')}$.
- Base change.
 - If $T \rightarrow S$ is a map of schemes and S' is an S -scheme, then $T_{S'}$ denotes the S' -scheme $T \times_S S'$ if S is understood from context.
 - When $S = \text{Spec}(R)$ and $S' = \text{Spec}(R')$ are affine, we may write $T_{R'}$ to denote $T \otimes_R R' := T \times_{\text{Spec}(R)} \text{Spec}(R')$ when R is understood from context.

- Abelian varieties and homomorphisms between them.
 - The dual of an abelian variety A is denoted A^t .
 - For an abelian variety A over a field K and a prime ℓ not divisible by $\text{char}(K)$, upon choosing a separable closure K_s of K (often understood from context) the ℓ -adic Tate module $T_\ell(A)$ denotes $\varprojlim A[\ell^n](K_s)$ and $V_\ell(A)$ denotes $\mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} T_\ell(A)$.
 - For any abelian varieties A and B over a field K , $\text{Hom}(A, B)$ denotes the group of homomorphisms $A \rightarrow B$ over K , and $\text{Hom}^0(A, B)$ denotes $\mathbb{Q} \otimes_{\mathbb{Z}} \text{Hom}(A, B)$.
(Since $\text{Hom}(A, B) \rightarrow \text{Hom}(A_{\overline{K}}, B_{\overline{K}})$ is injective, $\text{Hom}(A, B)$ is a finite free \mathbb{Z} -module since the same holds over \overline{K} by [82, §19, Thm. 3].)
 - When $B = A$ we write $\text{End}(A)$ and $\text{End}^0(A)$ respectively, and call $\text{End}^0(A)$ the *endomorphism algebra* of A (over K). The endomorphism algebra $\text{End}^0(A)$ is an invariant which only depends on A up to isogeny over K , in contrast with the endomorphism ring $\text{End}(A)$.
 - We write $A \sim B$ to denote that abelian varieties A and B over K are K -isogenous.
 - To avoid any possible confusion with notation found in the literature, we emphasize that what we call $\text{Hom}(A, B)$ and $\text{Hom}^0(A, B)$ are sometimes denoted by others as $\text{Hom}_K(A, B)$ and $\text{Hom}_K^0(A, B)$.¹¹
- Adeles and local fields.
 - We write \mathbb{A}_L to denote the adèle ring of a number field L , $\mathbb{A}_{L,f}$ to denote the factor ring of finite adeles, and \mathbb{A} and \mathbb{A}_f in the case $L = \mathbb{Q}$.
 - If v is a place of a number field L then L_v denotes the completion of L with respect to v ; $\mathcal{O}_{L,v}$ denotes the valuation ring \mathcal{O}_{L_v} of L_v in case v is non-archimedean, with residue field κ_v whose size is denoted q_v .
 - For a place w of \mathbb{Q} we define $L_w := \mathbb{Q}_w \otimes_{\mathbb{Q}} L = \prod_{v|w} L_v$, and in case w is the ℓ -adic place for a prime ℓ we define $\mathcal{O}_{L,\ell} := \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathcal{O}_L = \prod_{v|\ell} \mathcal{O}_{L,v}$.
- Class field theory and reciprocity laws.
 - The Artin maps of local and global class field theory are taken with the arithmetic normalization, which is to say that local uniformizers are carried to arithmetic Frobenius elements.¹²
 - $\text{rec}_L : \mathbb{A}_L^\times / L^\times \rightarrow \text{Gal}(L^{\text{ab}}/L)$ denotes the arithmetically normalized global reciprocity map for a number field L .
 - The composition of $\mathbb{A}_L^\times \rightarrow \mathbb{A}_L^\times / L^\times$ with rec_L is denoted r_L .
 - For a non-archimedean local field F we write $r_F : F^\times \rightarrow \text{Gal}(F^{\text{ab}}/F)$ to denote the arithmetically normalized local reciprocity map.
- Frobenius and Verschiebung.
 - For a commutative group scheme N over an \mathbb{F}_p -scheme S , $N^{(p)}$ denotes the base change of N by the absolute Frobenius endomorphism of S . The relative Frobenius homomorphism is denoted $\text{Fr}_{N/S} : N \rightarrow N^{(p)}$, and the

¹¹with the notation $\text{Hom}(A, B)$ and $\text{Hom}^0(A, B)$ then reserved to mean the analogues for $A_{\overline{K}}$ and $B_{\overline{K}}$ over \overline{K} , or equivalently for A_{K_s} and B_{K_s} over K_s (see Lemma 1.2.1.2).

¹²Recall that for a non-archimedean local field F with residue field of size q , an element of $\text{Gal}(F_s/F)$ is called an arithmetic (resp. geometric) Frobenius element if its effect on the residue field of F_s is the automorphism $x \mapsto x^q$ (resp. $x \mapsto x^{1/q}$); this automorphism of the residue field is likewise called the arithmetic (resp. geometric) Frobenius automorphism. We choose the arithmetic normalization of class field theory so that uniformizers correspond to Frobenius endomorphisms of abelian varieties in the Main Theorem of Complex Multiplication.

Verschiebung homomorphism for S -flat N of finite presentation denoted $\text{Ver}_{N/S}: N^{(p)} \rightarrow N$ see [30, VII_A, 4.2–4.3]). If S is understood from context then we may denote these as Fr_N and Ver_N respectively.

For $n \geq 1$, the p^n -fold relative Frobenius and Verschiebung homomorphisms $N \rightarrow N^{(p^n)}$ and $N^{(p^n)} \rightarrow N$ are respectively denoted $\text{Fr}_{N/S, p^n}$ and $\text{Ver}_{N/S, p^n}$.

- For a perfect field k with $\text{char}(k) = p > 0$ and the unique lift $\sigma: W(k) \rightarrow W(k)$ of the Frobenius automorphism $y \mapsto y^p$ of k , a *Dieudonné module* over k is a $W(k)$ -module M equipped with additive endomorphisms $\mathcal{F}: M \rightarrow M$ and $\mathcal{V}: M \rightarrow M$ such that $\mathcal{F} \circ \mathcal{V} = [p]_M = \mathcal{V} \circ \mathcal{F}$, $\mathcal{F}(cm) = \sigma(c)\mathcal{F}(m)$, and $c\mathcal{V}(m) = \mathcal{V}(\sigma(c)m)$ for all $c \in W(k)$ and $m \in M$; these are the left modules over the *Dieudonné ring* D_k (see 1.4.3.1).
- The semilinear operators \mathcal{F} and \mathcal{V} on a Dieudonné module M correspond to respective $W(k)$ -linear maps $M^{(p)} \rightarrow M$ and $M \rightarrow M^{(p)}$, where $M^{(p)} := W(k) \otimes_{\sigma, W(k)} M$.