CHAPTER 1

Introduction

1.1. What is a PET?

We begin by defining the main objects of study in this monograph. §2 gives more information about what we say here.

PETs: A *polytope exchange transformation* (or PET) is defined by a big polytope X which has been partitioned in two ways into small polytopes:

$$X = \bigcup_{i=1}^{m} A_i = \bigcup_{i=1}^{m} B_i. \tag{1.1}$$

What we mean is that, for each i, the two polytopes A_i and B_i are translation equivalent. That is, there is some vector V_i such that $B_i = A_i + V_i$. We always take the small polytopes to be convex, but sometimes X will not be convex.

We define a map $f : X \to X$ and its inverse $f^{-1} : X \to X$ by the formulas

$$f(x) = x + V_i \quad \forall x \in \mathrm{int}(A_i), \qquad f^{-1}(y) = y - V_i \quad \forall y \in \mathrm{int}(B_i). \tag{1.2}$$

f is not defined on points of ∂A_i and f^{-1} is not defined for points in ∂B_i. Even though f and f^{-1} are not everywhere defined, almost every point of x has a well-defined forwards and backwards orbit.

The Periodic Tiling: A point $p \in X$ is called *periodic* if $f^n(p) = p$ for some n. We will establish the following well-known results in §2: If p is a periodic point, then there is a maximal open convex polytope $U_p \subset X$ such that $p \in U_p$, and $f, ..., f^n$ are entirely defined on U_p, and every point of U_p is periodic with period n. We call U_p a *periodic tile*. We let Δ denote the union of periodic tiles. We call Δ the *periodic tiling*.

The Limit Set: When Δ is dense in X – and this happens in the cases of interest to us here – the *limit set* Λ consists of those points p such that every neighborhood of p intersects infinitely many tiles of Δ. Sometimes Λ is called the *residual set*. See §2.5 for a more general definition.

The Aperiodic Set: We let $\Lambda' \subset \Lambda$ denote the union of points with well-defined orbits. These orbits are necessarily aperiodic, so we call Λ' the *aperiodic set*.

1.2. Some Examples

The simplest examples of PETs are 1-dimensional systems, known as *interval exchange transformations* (IETs). Such a system is easy to produce: Partition an interval smaller intervals, then rearrange them. IETs have been extensively studied in the past 40 years. One very early paper is [**K**]; see papers [**Y**] and [**Z**] for surveys of the literature. The *Rauzy induction* [**R**] gives a satisfying renormalization theory for the family of IETs all having the same number of intervals in the partition. (There are other renormalization schemes as well.)

The simplest examples of higher dimensional polytope exchange transformations are products of IETs. In this case, all the polytopes are rectangular solids. More generally, one can consider PETs (not necessarily products) in which all the polytopes are rectangular solids. In 2 dimensions, these are called *rectangle exchanges*. The paper [**H**] establishes some foundational results about rectangle exchanges.

The paper [**AKT**] gives some early examples of piecewise isometric maps which are not rectangle exchanges. The main example analyzed in [**AKT**] produces locally the same tiling as outer billiards on the regular octagon, and also the same tiling as one of the examples studied here.

The papers [**T2**], [**AE**], [**Go**], [**Low1**], and [**Low2**] all treat a closely related set of systems with 5-fold symmetry which produce tilings by regular pentagons and/or regular decagons. The papers [**AG**], [**LKV**], [**Low1**] and [**Low2**] deal with the case of 7-fold symmetry, which is much more difficult. The difficulty comes from the fact that $\exp(2\pi i/7)$ is a cubic irrational, though one case with 7-fold symmetry is analyzed completely in [**Low2**].

Outer billiards on the regular n-gon furnishes an intriguing family of PETs. The cases $n = 3, 4, 6$ produce regular tilings of the plane, and the cases $n = 5, 7, 8, 10, 12$, where $\exp(2\pi/n)$ is a quadratic irrational, can be completely understood in terms of renormalization. See [**T2**] for the case $n = 5$, and [**BC**] for the other cases. There is partial information about the case $n = 7$, and the remaining cases are not understood at all.

Some definitive theoretical work concerning the entropy of PETs is done in [**GH1**], [**GH2**], and [**B**]. The main results here are that such systems have zero entropy, with a suitable definition of entropy.

The recent paper [**Hoo**] is very close in spirit to our work here. In [**Hoo**], the author works out a renormalization scheme for a 2-parameter family of (non-product) rectangle exchange transformations.

1.3. Goals of the Monograph

Multigraph PETs The first goal of this monograph is to give a general construction of PETs, based on decorated multigraphs. A *multigraph* is a graph in which one allows multiple edges connecting different vertices. The vertices are labelled by convex polytopes and the edges are labeled by Euclidean lattices, so that a vertex is incident to an edge iff the corresponding polytope is a fundamental domain for the corresponding lattice. Given such a multigraph G, we choose a base vertex x and we construct a functorial homomorphism

$$\pi_1(G, x) \to \mathrm{PET}(X). \tag{1.3}$$

Here $\pi_1(G, x)$ is the fundamental group of G, and $\mathrm{PET}(X)$ is the group of PETs whose domain is the polytope X corresponding to x. We call the resulting systems *multigraph PETs*. When G is a digon–i.e. two vertices connected by two edges, we call the system a *double lattice PET*. We will give a variety of nontrivial constructions of multigraph PETs, some related to outer billiards as in [**S2**] and some based on finite reflection groups.

Structure of the Octagonal PETs: The octagonal PETs are the simplest example of our construction. They are planar double lattice PETs based on the order 8 dihedral reflection group. The second, and main, goal of this monograph is study the structure of the octagonal PETs. We state all our main results about the octagonal PETs in the sections following this one.

Connection to Outer Billiards: The third goal is to connect the octagonal PETs of outer billiards on semi-regular octagons. We will prove that outer billiards relative to any semi-regular octagon produces a periodic tiling locally isometric to the one produced by an octagonal PET at a suitable parameter. This fact allows us to give very detailed information about outer billiards on semi-regular octagons.

Connection to the Alternating Grid System: The fourth goal is to explain the connection between the octagonal PETs and a certain dynamical system in the plane (described below) that is defined by a pair of square grids. Each alternating grid system has a 4-dimensional compactification which is a double lattice PET, and the PET has an invariant 2-dimensional slice which is the octagonal PET at the same parameter. This is how we found the octagonal PETs.

1.4. The Octagonal PETs

Here we describe the octagonal PETs. Our construction depends on a parameter $s \in (0, \infty)$, but usually we take $s \in (0, 1)$. We suppress s from our notation for most of the discussion.

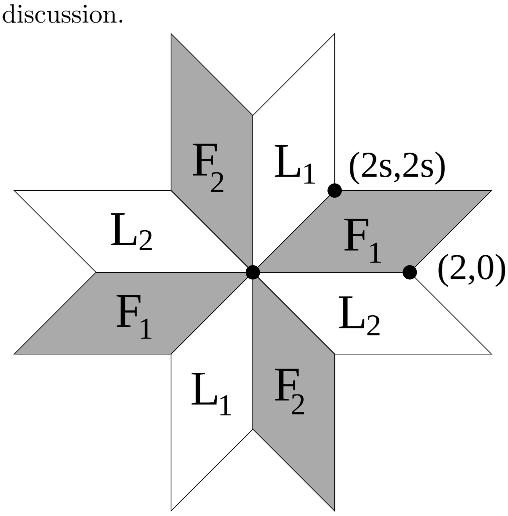

Figure 1.1: The scheme for the PET.

The 8 parallalograms in Figure 1.1 are the orbit of a single parallelogram under a dihedral group of order 8. Two of the sides of this parallelogram are determined by the vectors $(2, 0)$ and $(2s, 2s)$. For $j = 1, 2$, let F_j denote the parallelogram centered at the origin and translation equivalent to the ones in the picture labeled F_j. Let L_j denote the lattice generated by the sides of the parallelograms labeled L_j. (Either one generates the same lattice.)

In §2 we will check the easy fact that F_i is a fundamental domain for L_j, for all $i,j \in \{1,2\}$. We define a system (X', f'), with $X' = F_1 \cup F_2$, and $f' : X' \to X'$, as follows. Given $p \in F_j$ we let

$$f'(p) = p + V_p \in F_{3-j}, \qquad V_p \in L_{3-j}. \tag{1.4}$$

The choice of V_p is almost always unique, on account of F_{3-j} being a fundamental domain for L_{3-j}. When the choice is not unique, we leave f' undefined. When $p \in F_1 \cap F_2$ we have $V_p = 0 \in L_1 \cap L_2$. We will show in §2 that (X', f') is a PET.

We prefer the map $f = (f')^2$, which preserves both F_1 and F_2. We set $X = F_1$. Our system is $f : X \to X$, which we denote by (X, f).

1.5. The Main Theorem: Renormalization

We define the *renormalizaton map* $R : (0,1) \to [0,1)$ as follows.

- $R(x) = 1 - x$ if $x > 1/2$
- $R(x) = 1/(2x) - \text{floor}(1/(2x))$ if $x < 1/2$.

R relates to the $(2,4,\infty)$ reflection triangle triangle much in the way that the classical Gauss map

$$g(x) = 1/x - \text{floor}(1/x)$$

relates to the modular group. We will discuss the hyperbolic geometry connection in §16.6. The map R is an infinite-to-one map of the interval. Almost every orbit of R is dense, and the quadratic irrational numbers are all periodic points of R.

Define

$$Y = F_1 - F_2 = X - F_2 \subset X. \tag{1.5}$$

Figure 1.2 shows a picture of the set Y in one case.

For any subset $S \subset X$, let $f|S$ denote the first return map to S, assuming that this map is defined. When we use this notation, it means implicitly that the map is actually defined, at least away from a finite union of line segments. We call S *clean* if no point on ∂S has a well defined orbit. This means, in particular, that no tile of Δ crosses over ∂S.

THEOREM 1.1 (Main). *Suppose $s \in (0,1)$ and $t = R(s) \in (0,1)$. There is a clean set $Z_s \subset X_s$ such that $f_t|Y_t$ is conjugate to $f_s^{-1}|Z_s$ by a map ϕ_s.*

1. *ϕ_s commutes with reflection in the origin and maps the acute vertices of X_t to the acute vertices of X_s.*
2. *When $s < 1/2$, the restriction of ϕ_s to each component of Y_t is an orientation reversing similarity, with scale factor $s\sqrt{2}$.*
3. *When $s < 1/2$, either half of ϕ_s extends to the trivial tile of Δ_t and maps it to a tile in Δ_s which has period 2.*
4. *When $s < 1/2$, the only nontrivial orbits which miss Z_s are contained in the ϕ_s-images of the trivial tile of Δ_t. These orbits have period 2.*
5. *When $s > 1/2$ the restriction of ϕ_s to each component of Y_s is a translation.*
6. *When $s > 1/2$, all nontrivial orbits intersect Z_s.*

Our main result is an example of a situation where a picture says a thousand words. Figures 1.2 and 1.3 show the Main Theorem in action for $s < 1/2$. Figures 1.4 and 1.5 show the Main Theorem in action for $s > 1/2$.

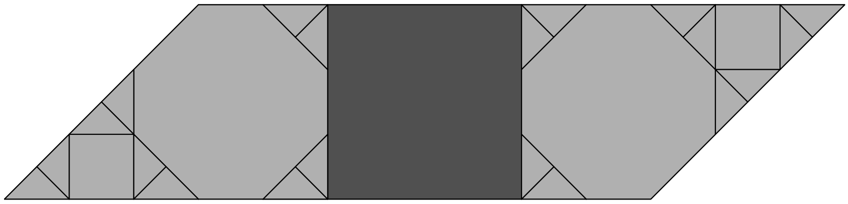

Figure 1.2: Y_t lightly shaded for $t = 3/10 = R(5/13)$.

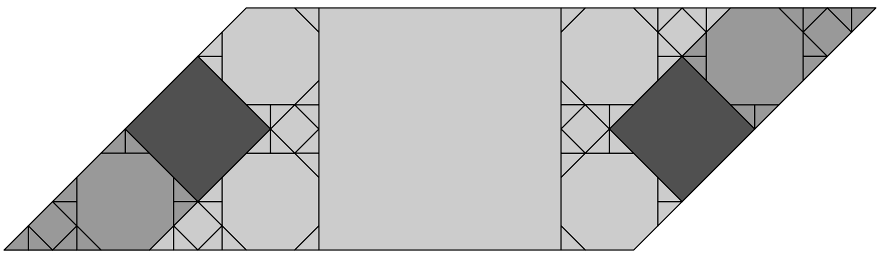

Figure 1.3: Z_s lightly shaded $s = 5/13$.

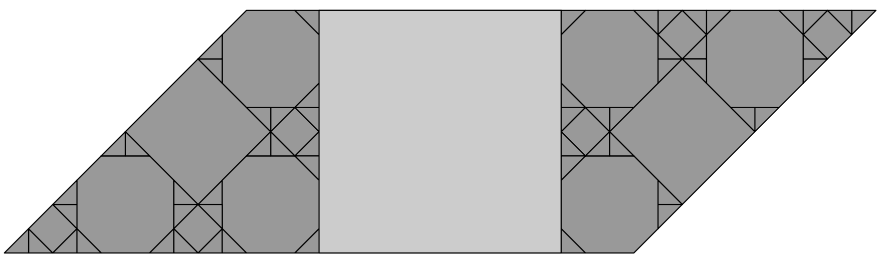

Figure 1.4: Y_t lightly shaded for $t = R(8/13) = 5/13$.

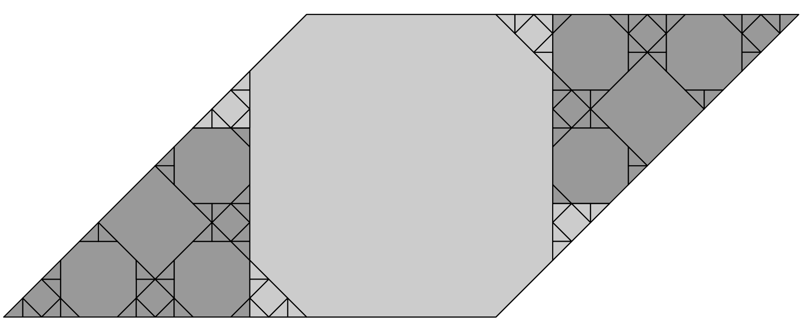

Figure 1.5: Z_s lightly shaded for $s = 8/13$.

1.6. Corollaries of The Main Theorem

1.6.1. Structure of the Tiling. A *semi-regular octagon* is an octagon with 8-fold dihedral symmetry. When $s \in (1/2, 1)$, the intersection

$$O_s = (F_1)_s \cap (F_2)_s \tag{1.6}$$

is the semi-regular octagon with vertices

$$(\pm s, \pm(1-s)), \qquad (\pm(1-s), \pm s). \tag{1.7}$$

When $s \in (0, 1/2)$, the interection O_s is the square with vertices $(\pm s, \pm s)$.

Given $s \in (0,1)$, let $s_n = R^n(s)$. We call the index n *good* if $s_{n-1} < 1/2$ or if $n = 0$.

THEOREM 1.2. *When s is irrational, a polygon appears in Δ_s if and only it is similar to O_{s_n} for a good index n. When s is rational, a polygon appears in Δ_s if and only if it is a square, a right-angled isosceles triangle, or similar to O_{s_n} for a good index n.*

Remark: Theorem 1.2 is a consequence of a more precise and technical result, Theorem 12.1, which describes the tiles of Δ_s up to translation.

According to Theorem 1.2, the tiling Δ_s is an infinite union of squares and semi-regular octagons when s is irrational. Here is some more information in the irrational case.

THEOREM 1.3. *The following is true when $s \in (0,1)$ is irrational.*

(1) *If Δ_s has no squares then $s = \sqrt{2}/2$. If Δ_s has finitely many squares, then $s \in \boldsymbol{Q}[\sqrt{2}]$.*
(2) *Δ_s has only squares if and only if the continued fraction expansion of s has the form $(0, a_0, a_1, a_2, ...)$ where a_k is even for all odd k. This happens if and only if $R^n(s) < 1/2$ for all n.*
(3) *Δ_s has infinitely many squares and a dense set of shapes of semi-regular octagons for almost all s.*

According to Theorem 1.2, when s is rational, Δ_s consists entirely of squares, semi-regular octagons, and right-angled isosceles triangles. We say that a periodic tile τ_s of Δ_s is *stable* if, for all parameters r sufficiently close to s, there is a tile τ_r of Δ_r consisting of points having the same period and dynamical behavior as the points in τ_s. See §8 for more information.

THEOREM 1.4. *When s is rational, a tile of Δ_s is unstable if and only if it is a triangle. All the unstable tiles are isometric to each other, and they are arranged into 4 orbits, all having the same period.*

1.6.2. Structure of the Limit Set. Here are some results about the limit set. The quantity $\dim(\Lambda_s)$ denotes the Hausdorff dimension of the limit set.

THEOREM 1.5. *Suppose s is irrational.*

(1) *Λ_s has zero area.*
(2) *The projection of Λ_s onto a line parallel to any 8th root of unity contains a line segment. Hence* $\dim(\Lambda_s) \geq 1$.
(3) *Λ_s is not contained in a finite union of lines.*
(4) *Λ'_s is dense in Λ_s.*

Theorem 1.4 implies, if $s \in \boldsymbol{Q}$, that there is a single number $N(s)$ such that Δ_s has $4N(s)$ triangular tiles, and all unstable periodic orbits of (X_s, f_s) have period $N(s)$. In §18, we will give a kind of formula for $N(s)$. We will then use this formula to get some upper bounds on the $\dim(\Lambda_s)$.

THEOREM 1.6. *For any irrational s, there is a sequence $\{p_n/q_n\}$ of rational parameters, converging to s, such that*

$$\dim(\Lambda_s) \leq \limsup \frac{\log N(p_n/q_n)}{\log q_n}. \tag{1.8}$$

In particular

- $\dim(\Lambda_s) = 1$ *if* $\lim R^n(s) = 0$.
- $\dim(\Lambda_s) \leq 1 + (\log 8/\log 9)$ *in all cases.*

Remarks:
(i) The sequence $\{p_n/q_n\}$ in the theorem is closely related to the sequence of continued fraction approximations to s. Technically, we have $p_n/q_n \to s$ in the sense of §11.5.
(ii) Computer experiments with our formula for $N(s)$ suggest that

$$\dim(\Lambda_s) \leq \frac{2\log(1+\sqrt{2})}{\log(2+\sqrt{3})}, \qquad \forall\ s \in (0,1). \tag{1.9}$$

The bound is attained when $s = \sqrt{3}/2 - 1/2$. Figure 1.6 below shows the picture for this parameter. The case when $R^n(s) > 1/2$ finitely often boils down to a question about the dynamics of the Gauss map. In a private communication, Pat Hooper sketched a proof for me that the result holds in this case.
(iii) Theorem 1.6 says, in particular, that $\dim(\Lambda_s) < 2$, which of course implies that Λ_s has zero area. However, we prove that Λ_s has zero area separately because the proof of Theorem 1.6 is rather involved.

Now we turn to questions about the topololy of Λ_s. Here is the complete classification of the topological types.

THEOREM 1.7. *Let $s \in (0,1)$ be irrational.*

(1) *Λ_s is a disjoint union of two arcs if and only if Δ_s contains only squares. This happens if and only if $R^n(s) < 1/2$ for all n.*
(2) *Λ_s is a finite forest if and only if Δ_s contains finitely many octagons. This happens if and only if $R^n(s) > 1/2$ for finitely many n.*
(3) *Λ_s is a Cantor set if and only if Δ_s contains infinitely many octagons. This happens if and only if $R^n(s) > 1/2$ for infinitely many n.*

Figures 1.6-1.8 illustrate the three cases of Theorem 1.7. The curve in Figure 1.6 is isometric to one of the curves which appears in Pat Hooper's Truchet tile systems. The notation in Figure 1.7 refers to the even expansion of s. See §11 for a definition. The tiling in Figure 1.8 is the same as the main example in [**AKT**].

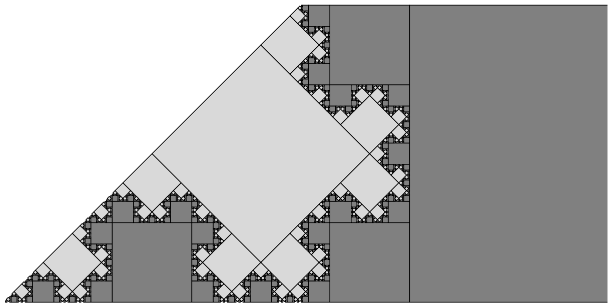

Figure 1.6: The left half of Δ_s for $s = \sqrt{3}/2 - 1/2$.

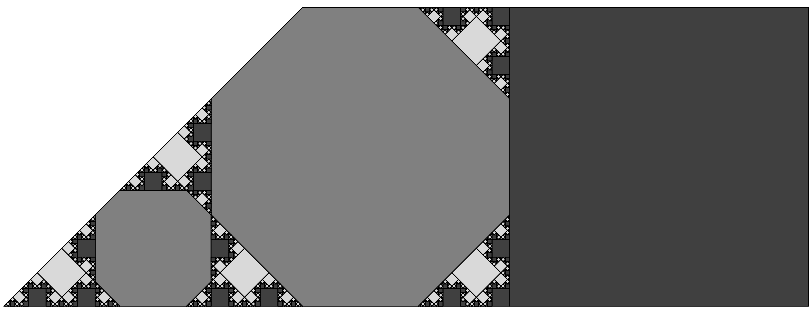

Figure 1.7: The left half of Δ_s for $s = (0, 3, 1, 3, 1, 2, 2, 2...)$.

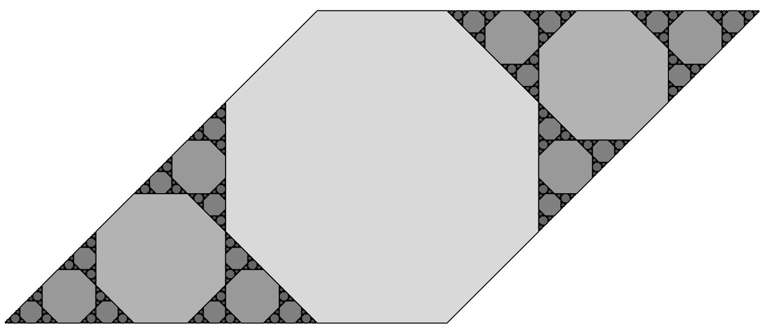

Figure 1.8: The left half of Δ_s for $s = \sqrt{2}/2$.

We will investigate Case 1 of Theorem 1.7 in more detail.

THEOREM 1.8. *Suppose that the continued fraction of s is $(0, a_1, a_2, a_3, ...)$ with a_k even for all odd k. Then the restriction $f_s|\Lambda_s$ is a $\boldsymbol{Z}/2$ extension of the irrational rotation with rotation number* $(0, 2a_2, a_3, 2a_4, a_5, 2a_6, a_7, ...)$.

Remark: Theorem 1.8, and the more precise version, Theorem 25.1, are similar to forthcoming results of Pat Hooper. Indeed, conversations with Hooper inspired me to formulate and prove Theorems 1.8 and 25.1.

Now we discuss the dependence of the limit set on the parameter. When s is rational, we let U_s denote the closure of the union of the unstable tiles. Let $\mathcal{K}$ denote the set of compact subsets of $\boldsymbol{R}^2$. We equip $\mathcal{K}$ with the *Hausdorff metric*.

The distance between two subsets is the infimal ϵ such that each is contained in the ϵ-tubular neighborhood of the other.

We have a map $\Xi : (0,1) \to \mathcal{K}$ defined as follows.

- $\Xi_s = U_s$ when s is rational
- $\Xi_s = \Lambda_s$ when s is irrational.

THEOREM 1.9. *The map Ξ is continuous at irrational points of $(0,1)$.*

Remark: Theorem 1.9 says, in particular, that the union of unstable orbits in the rational case gives a good approximation to the limit set in the irrational case. Theorem 17.9, which is the main tool we use to prove Theorem 1.6, gives another view of this same idea.

1.6.3. Hyperbolic Symmetry. The Main Theorem above implies that there is an underlying hyperbolic symmetry to the family of octagonal PETs. (See §2.6 for some background information on hyperbolic geometry.)

Let $\boldsymbol{H}^2 \subset \boldsymbol{C}$ denote the upper half plane model of the hyperbolic plane. Let Γ denote the $(2,4,\infty)$ reflection triangle group, generated by reflections in the sides of the ideal hyperbolic triangle with vertices

$$\frac{i}{\sqrt{2}}, \qquad \frac{1}{2}+\frac{i}{2}, \qquad \infty. \tag{1.10}$$

We extend our parameter range so that our system is defined for all $s \in \boldsymbol{R}$. The systems at s and $-s$ are identical. Γ acts on the parameter set by linear fractional transformations.

We say that (X_s, f_s) is *locally modelled on* (X_t, f_t) at $p_s \in X_s$ if there is some $p_t \in X_t$ and a similarity $g : \Delta_s \cap U_s \to \Delta_t \cap U_t$, with U_s and U_t being neighborhoods of p_s and p_t respectively. We say that (X_s, f_s) and (X_t, f_t) are *locally equivalent* if there are finite collections of lines L_s and L_t such that (X_s, f_s) is locally modelled on (X_t, f_t) for all $p_t \in \Lambda_t - L_t$ and (X_t, f_t) is locally modelled on (X_t, f_s) for all $p_s \in \Lambda_s - L_s$. Intuitively, the tilings of locally equivalent systems have the same fine-scale structure. In particular, the limit sets of locally equivalent systems have the same Hausdorff dimension.

THEOREM 1.10. *Suppose s and t are in the same orbit of Γ. Then (X_s, f_s) and (X_t, f_t) are locally equivalent. In particular, the function $s \to \dim(\Lambda_s)$ is a Γ-invariant function.*

Remarks:
(i) Γ is contained with index 4 in the group generated by reflections in the ideal triangle with vertices $0, 1, \infty$. Using this fact, together with a classic result about continued fractions, we we will show that the forward orbit $\{R^n(s)\}$ is dense in $(0,1)$ for almost all $s \in (0,1)$.
(ii) The need to exempt a finite union of lines in the definition of local equivalence seems partly to be an artifact of our proof, but in general one needs to disregard some points to make everything work.
(iii) Given the ergodic nature of the action of Γ, we can say that there is some number δ_0 such that $\dim(\Lambda_s) = \delta_0$ for almost all s. However, we don't know the value of δ_0.

1.7. Polygonal Outer Billiards

B. H. Neumann [**N**] introduced outer billiards in the late 1950s and J. Moser [**M1**] popularized the system in the 1970s as a toy model for celestial mechanics. Outer billiards is a discrete self-map of $\boldsymbol{R}^2 - P$, where P is a bounded convex planar set as in Figure 1.9 below. Given $p_1 \in \boldsymbol{R}^2 - P$, one defines p_2 so that the segment $\overline{p_1 p_2}$ is tangent to P at its midpoint and P lies to the right of the ray $\overrightarrow{p_1 p_2}$. The map $p_1 \to p_2$ is called *the outer billiards map*.

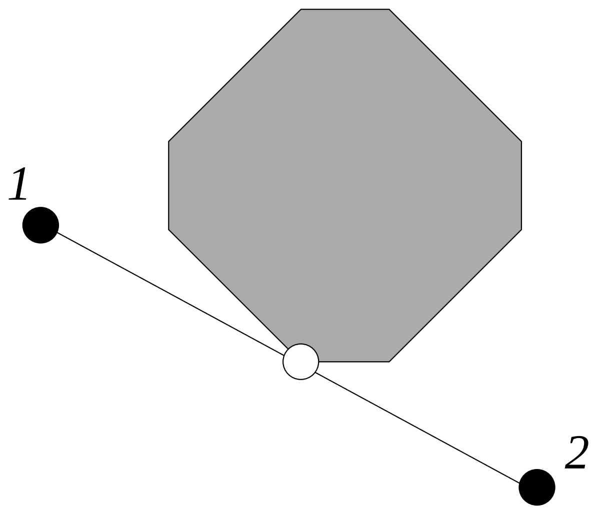

Figure 1.9: Outer billiards relative to a semi-regular octagon.

The second iterate of the outer billiards map is a PET whose domain is the entire plane. (We make the map the identity inside the central shape.) In particular, we define $\Delta(P)$ and $\Lambda(P)$ as above.

Much work on outer billiards has been done in recent years. See, for instance, [**BC**], [**Bo**], [**DT1**], [**DT2**], [**G**], [**GS**], [**Ko**], [**S2**], [**S3**], [**T1**], [**T2**], and [**VS**]. There is a survey in [**T1**], and a more recent one in [**S3**].

The sets $\Delta(P)$ and $\Lambda(P)$ are well understood when P is a regular n-gon for $n = 3, 4, 5, 8, 12$. These cases correspond to some of the piecewise isometric systems discussed above. (The case $n = 12$ has not really been worked out, but it is quite similar to the cases $n = 5, 8$.) Many pictures have been drawn for other values of n, but there are no theoretical results.

I have given some information in [**S4**] about $\Delta(P)$ when P is the Penrose kite, but the information is far from complete. In all other cases, nothing is known about the tiling and the limit set.

The only thing known about outer billiards on semi-regular octagons is that all orbits are bounded. This follows from the general result in [**GS**], [**Ko**], and [**VS**]. Here we will give fairly complete information about outer billiards on semi-regular octagons.

We parameterize semi-regular octagons as in Equation 1.6. The complement $\boldsymbol{R}^2 - O_s$ is tiled, in two ways, by isometric copies of $X_s = (F_1)_s$. The two ways are mirror reflections of each other. We first divide the $\boldsymbol{R}^2 - O_s$ into 8 cones, and then we fill each cone with parallel copies of X_s, in the pattern (partially) shown.

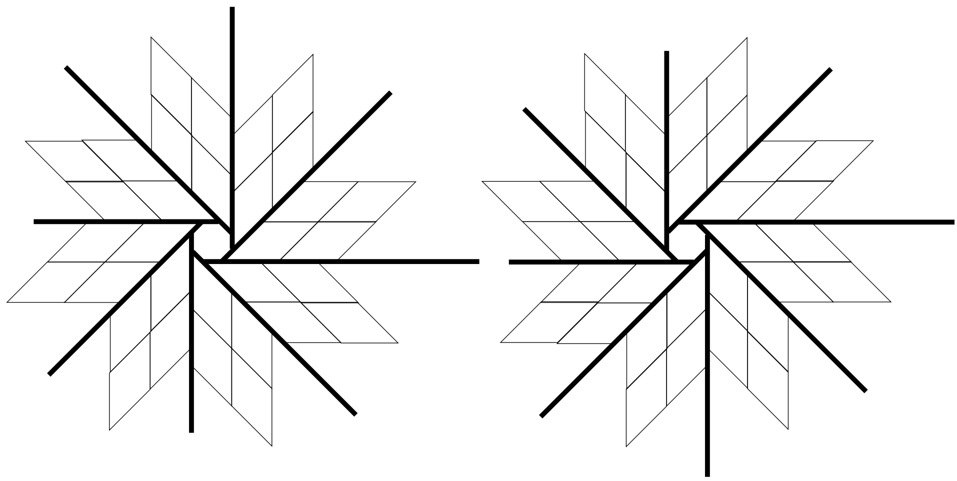

Figure 1.10: The tiling of $\boldsymbol{R}^2 - O_s$.

THEOREM 1.11. *Suppose $s \in (1/2,1)$. Let Y be any parallelogram in either of the two tilings of $\boldsymbol{R}^2 - O_s$. Let Δ_s and Λ_s be the periodic tiling and the limit set for the octagonal PET (X_s, f_s). Then $\Delta(O_s) \cap Y$ and $\Lambda(O_s) \cap Y$ are isometric copies of Δ_s and Λ_s respectively.*

Thanks to Theorem 1.11, all the results mentioned above for the octagonal PETs have analogous statements for outer billiards on semi-regular octagons. For instance, almost every point is periodic and the Hausdorff dimension of the limit set is strictly less than 2. The map R does not preserve the interval $(1/2,1)$ and indeed there are many points in $(1/2,1)$, such as $s = 3/2 - \sqrt{3}/2$, for which $R^n(s)$ never returns to $(1/2,1)$. Thus, the family of outer billiards systems in semi-regular octagons really only has a renormalization scheme associated to it when it is included in the larger family of octagonal PETs.

In §3.4 we will see that the octagonal PETs correspond to the case $n = 4$ of what we call *dihedral PETs*, a more general construction which works for $n = 3,4,5,\ldots$. The case $n = 3$, corresponding to outer billiards on semi-regular hexagons, should have a theory very similar to what we do for $n = 4$ in this monograph.

1.8. The Alternating Grid System

Let $G_{s,z}$ denote the grid of squares in the plane, with side length s, having $z \in \boldsymbol{C}$ as a vertex. We take the sides of the squares to be parallel to the coordinate axes. Let $T_{s,z}$ denote the piecewise isometric transformation which rotates each square of $G_{s,z}$ counterclockwise by $\pi/2$ about its center. The map $T_{s,z}$ is not defined on the edges of $G_{s,z}$.

The alternating grid system is based on the composition of two maps like this, based on differently sized grids. The standard grid is $G_{1,0}$, and we consider the infinite group $\langle T_{1,0}, T_{s,z}\rangle$. This group acts on the plane by piecewise isometries. To get a dynamical system in the traditional sense, one can choose a word

$$T_{1,0}^{e_1} T_{s,z}^{e_2} \ldots T_{1,0}^{e_{n-1}} T_{s,z}^{e_n}, \qquad e_1, \ldots, e_n \in \{0,1,2,3\} \tag{1.11}$$

and study the action of this map on $\boldsymbol{R}^2$. It seems reasonable to require that the exponents sum to 0 mod 4. With this restriction, the map is a PET whose domain is the whole plane.

It is interesting to vary the choice of word and see how the system changes. In [**S6**] we consider this question in detail, and show that most words lead to systems having very few periodic points. We make a case in [**S6**] that the word

$$F_{s,z} = T_{1,0}T_{s,z}T_{1,0}T_{s,z} \tag{1.12}$$

is the most interesting word to study. We call the system $(\boldsymbol{R}^2, F_{s,z})$ the *Alternating Grid System*, or *AGS* for short. When we discuss the AGS in §4, we will present some interesting experimental observations about these systems which go beyond what we can actually prove.

In [**S6**] (which I wrote after completing the first version of this monograph) we show that the noncompact system given by the map in Equation 1.11 always has a higher dimensional compactification, in the sense discussed below. In the case of the AGS, the compactification is 4 dimensional. This is the case we discuss here. The compactification of the system $(\boldsymbol{R}^2, F_{s,z})$ does not depend on z, though the map into the compactification does depend (mildly) on z.

Theorem 1.12. *The system $(\boldsymbol{R}^2, F_{s,z})$ has a 4-dimensional compactification $(\widetilde{X}_s, \widetilde{F}_s)$, where $\widetilde{X}_s \subset \boldsymbol{R}^4$ is a parallelotope and $\widetilde{F}_s$ is a polytope exchange transformation.*

By *compactification*, we mean that there is an injective piecewise affine map $\widetilde{\Psi}_{s,z} : \boldsymbol{R}^2 \to \widetilde{X}_s$ such that

$$\widetilde{F}_s \circ \Psi_{s,z} = \widetilde{\Psi}_{s,z} \circ F_{s,z} \tag{1.13}$$

When s is irrational, $\Psi_{s,z}$ has a dense image. This map depends on z, but any two choices of z lead to maps which differ by a translation of $\widetilde{X}_s$. We think of $\boldsymbol{R}^2$ as an "irrational plane" sitting inside $\widetilde{X}_s$, as fundamental domain for a 4-dimensional torus. (The choice of plane depends on z.) The map $\widetilde{F}_{s,z}$ acts in such a way as to preserve this irrational plane, and thus induces the action of $F_{s,z}$ on $\boldsymbol{R}^2$.

The pair $(\widetilde{X}_s, \widetilde{F}_s)$ is a 4-dimensional double lattice PET. The next result says that the octagonal PET (X_s, f_s) is the invariant slice of the 4-dimensional compactification of the AGS.

Theorem 1.13. *The system $(\widetilde{X}_s, \widetilde{F}_s)$ commutes with an involution I_s of $\widetilde{F}_s$. The two eigen-planes of I_s are invariant under the system, and the restriction of $\widetilde{F}_s$ to each one is a copy of the octagonal PET at parameter s.*

Theorem 1.13 combines with Theorem 1.5 to prove the following result.

Theorem 1.14. *For every irrational $s \in (0,1)$, there is some choice of z (depending on s) such that $(\boldsymbol{R}^2, F_{s,z})$ has unbounded orbits.*

Let $K(s)$ denote the set of z such that $(R^2, F_{s,z})$ has unbounded orbits. Our result above says that $K(s)$ is nonempty. We think that $K(s) = \boldsymbol{C}$.

Combining Theorem 1.13 and Theorem 1.11, we get the following result.

Corollary 1.15. For any $s \in (1/2, 1)$ the tiling produced by outer billiards on the semi-regular octagon O_s is locally isometric to the one which appears in the invariant slice of the system $(\widetilde{X}_s, \widetilde{F}_s)$.

In short, one "sees" outer billiards on semi-regular octagons inside the AGS.

So far, I do not have a good understanding of the whole AGS, but I think that the AGS is quite rich and mysterious. It certainly deserves further study. This monograph is really an outgrowth of my attempt to understand the AGS. The few results here about the AGS should really just be the beginning of the story.

1.9. Computer Assists

The proofs we give in Parts I-IV of the monograph rely on 12 computer calculations, which we explain in Part V. These calculations involve showing that various pairs of convex integer polyhedra are either disjoint or nested. Everything is done with integer arithmetic, so that there is no roundoff error. We control the sizes of the integers, so that there is also no overflow error.

My interactive java programs OctaPET and BonePET [1] do all the calculations. OctaPET does all the calculations connected to our main result, and BonePET does all the calculations connected to Theorem 1.11. The programs can be downloaded from the URLs
http://www.math.brown.edu/∼res/Java/OCTAPET.tar
http://www.math.brown.edu/∼res/Java/BONEPET.tar
These are tarred directories, which untar to a directories called OctaPET and BonePET. These directories contain the programs, as well as instructions for compiling and running them.

All the pictures shown in this monograph are taken from these programs. We strongly advise the reader to use OctaPET and BonePET while reading the monograph. In the monograph we can only illustrate the important phenomena with a few pictures whereas the reader can see the picture for essentially any parameter using the programs. For the sake of giving a readable exposition, we omit a number of routine geometric calculations. Such calculations are all exercises in plane geometry. Rather than write out these calculations, we illustrate them with pictures from our programs. Again, such calculations will be all the more obvious to the reader who is using the programs while looking at the monograph.

Finally, we mention that our proof of Theorem 1.6 requires a small amount of Mathematica [**W**] code, which we include in the a directory called **Mathematica** in the source code for our program OctaPET.

1.10. Organization

Following a chapter containing some background material, this monograph is organized into 5 parts.

- Part I deals with the relation of the octagonal PET to other PETs. In particular, we prove Theorems 1.11, 1.12, and 1.13. We also introduce the multigraph PETs, which are a functorial way of producing PETs.
- In Part II we establish some elementary properties of the octagonsl PETs and then prove the Main Theorem. Following a discussion of the properties of the renormalization map R from the Main Theorem, we prove Theorems 1.2, 1.3, and 1.4.
- In Part III we elaborate on the Main Theorem, by explaining more precisely how the tiling at the parameter s is related to the parameter $R(s)$. We use these results to prove Theorems 1.5, 1.6, 1.9, and 1.10,

[1]My daughter helped me name the second program. The name derives from the fact that the domains for the PETs we use for Theorem 1.11 look like dog bones.

- In Part IV we investigate the topology of the limit set, and at the end, some of the dynamics on the limit set. In particular, we prove the Theorems 1.7 and 1.8,
- In Part V, we present all the computer assisted calculations.

At the beginning of each part of the monograph, we will give a more detailed overview of that part.