

# Contents

Preface	ix
Chapter 1. Introduction	1
1.1. What is a PET?	1
1.2. Some Examples	1
1.3. Goals of the Monograph	2
1.4. The Octagonal PETs	3
1.5. The Main Theorem: Renormalization	4
1.6. Corollaries of The Main Theorem	6
1.7. Polygonal Outer Billiards	10
1.8. The Alternating Grid System	11
1.9. Computer Assists	13
1.10. Organization	13
<b>Part 1. Friends of the Octagonal PETs</b>	<b>15</b>
Chapter 2. Background	17
2.1. Lattices and Fundamental Domains	17
2.2. Hyperplanes	17
2.3. The PET Category	18
2.4. Periodic Tiles for PETs	19
2.5. The Limit Set	20
2.6. Some Hyperbolic Geometry	21
2.7. Continued Fractions	23
2.8. Some Analysis	24
Chapter 3. Multigraph PETs	27
3.1. The Abstract Construction	27
3.2. The Reflection Lemma	28
3.3. Constructing Multigraph PETs	29
3.4. Planar Examples	30
3.5. Three Dimensional Examples	30
3.6. Higher Dimensional Generalizations	31
Chapter 4. The Alternating Grid System	33
4.1. Basic Definitions	33
4.2. Compactifying the Generators	34
4.3. The PET Structure	36
4.4. Characterizing the PET	37
4.5. A More Symmetric Picture	38

4.6. Unbounded Orbits	40
4.7. The Complex Octagonal PETs	40
Chapter 5. Outer Billiards on Semiregular Octagons	43
5.1. The Basic Sets	43
5.2. The Far Partition	44
5.3. The First Return Map	45
5.4. The Necklace Orbits	47
5.5. Parallelograms, Halfbones, and Dogbones	47
5.6. The Dogbone Map	49
5.7. The First Conjugacy	50
5.8. The Second Conjugacy	54
Chapter 6. Quarter Turn Compositions	55
6.1. Basic Definitions	55
6.2. The Polytope Graph	55
6.3. QTCs and Polygon Graphs	57
6.4. QTCs and Outer Billiards	58
6.5. QTCs and Double Lattice PETs	60
<b>Part 2. Renormalization and Symmetry</b>	<b>63</b>
Chapter 7. Elementary Properties	65
7.1. Intersection of the Parallelograms	66
7.2. Intersection of the Lattices	66
7.3. Rotational Symmetry	66
7.4. Central Tiles	66
7.5. Inversion Symmetry	67
7.6. Insertion Symmetry	67
7.7. The Tiling in Trivial Cases	68
Chapter 8. Orbit Stability and Combinatorics	71
8.1. A Bound on Coefficients	71
8.2. Sharpness	71
8.3. The Arithmetic Graph	72
8.4. Orbit Stability	73
8.5. Uniqueness and Convergence	74
8.6. Ruling out Thin Regions	74
8.7. Joint Convergence	75
Chapter 9. Bilateral Symmetry	77
9.1. Pictures	77
9.2. Definitions and Formulas	78
Chapter 10. Proof of the Main Theorem	81
10.1. Discussion and Overview	81
10.2. Proof of Induction Lemma I	82
10.3. Proof of Induction Lemma II	84
Chapter 11. The Renormalization Map	87
11.1. Elementary Properties	87

11.2.	The Even Expansion	87
11.3.	Oddly Even Numbers	88
11.4.	The Even Expansion and Continued Fractions	89
11.5.	Diophantine Approximation	89
11.6.	Dense Orbits	90
11.7.	Proof of the Triangle Lemma	91
Chapter 12.	Properties of the Tiling	93
12.1.	Tedious Special Cases	93
12.2.	Classification of Tile Shapes	93
12.3.	Classification of Stable Orbits	95
12.4.	Existence of Square Tiles	96
12.5.	The Oddly Even Case	97
12.6.	Density of Shapes	97
<b>Part 3.</b>	<b>Metric Properties</b>	<b>99</b>
Chapter 13.	The Filling Lemma	101
13.1.	The Layering Constant	101
13.2.	The Filling Lemma, Part 1	101
13.3.	The Filling Lemma, Part 2	104
Chapter 14.	The Covering Lemma	105
14.1.	The Main Result	105
14.2.	Some Additional Pictures	110
Chapter 15.	Further Geometric Results	111
15.1.	The Area Lemma	111
15.2.	Tiles in Symmetric Pieces	112
15.3.	Pyramids	113
Chapter 16.	Properties of the Limit Set	115
16.1.	Elementary Topological Properties	115
16.2.	Zero Area	116
16.3.	Projections of the Limit Set	116
16.4.	Finite Unions of Lines	118
16.5.	Existence of Aperiodic Points	119
16.6.	Hyperbolic Symmetry	119
Chapter 17.	Hausdorff Convergence	121
17.1.	Results about Patches	121
17.2.	Convergence	121
17.3.	Covering	123
Chapter 18.	Recurrence Relations	127
Chapter 19.	Hausdorff Dimension Bounds	131
19.1.	The Upper Bound Formula	131
19.2.	A Formula in the Oddly Even Case	131
19.3.	One Dimensional Examples	132
19.4.	A Warm-Up Case	133

19.5.	Most of The General Bound	133
19.6.	Dealing with the Exceptions	136
<b>Part 4.</b>	<b>Topological Properties</b>	<b>139</b>
Chapter 20.	Controlling the Limit Set	141
20.1.	The Shield Lemma	141
20.2.	Another Version of the Shield Lemma	143
20.3.	The Pinching Lemma	145
20.4.	Rational Oddly Even Parameters	147
Chapter 21.	The Arc Case	149
21.1.	The Easy Direction	149
21.2.	A Criterion for Arcs	150
21.3.	Elementary Properties of the Limit Set	153
21.4.	Verifying the Arc Criterion	153
Chapter 22.	Further Symmetries of the Tiling	157
22.1.	Zones	157
22.2.	Symmetry of Zones	158
22.3.	Intersections with Zones	158
22.4.	Folding	160
Chapter 23.	The Forest Case	163
23.1.	Reduction to the Loops Theorem	163
23.2.	Proof of the Loops Theorem	163
23.3.	An Example	164
Chapter 24.	The Cantor Set Case	167
24.1.	Unlikely Sets	167
24.2.	Tails and Anchored Paths	167
24.3.	Acute Crosscuts	168
24.4.	The Main Argument	172
24.5.	Pictorial Explanation	173
Chapter 25.	Dynamics in the Arc Case	175
25.1.	The Main Result	175
25.2.	Intersection with the Partitions	176
25.3.	The Rational Case	178
25.4.	Measures of Symmetric Pieces	179
25.5.	Controlling the Measures	180
25.6.	The End of the Proof	181
<b>Part 5.</b>	<b>Computational Details</b>	<b>183</b>
Chapter 26.	Computational Methods	185
26.1.	The Fiber Bundle Picture	185
26.2.	Avoiding Computational Error	186
26.3.	Dealing with Polyhedra	187
26.4.	Verifying the Partition	188
26.5.	Verifying Outer Billiards Orbits	189

26.6. A Planar Approach	190
26.7. Generating the Partitions	191
Chapter 27. The Calculations	193
27.1. Calculation 1	193
27.2. Calculation 2	194
27.3. Calculation 3	194
27.4. Calculation 4	196
27.5. Calculation 5	196
27.6. Calculation 6	197
27.7. Calculation 7	198
27.8. Calculation 8	198
27.9. Calculation 9	198
27.10. Calculation 10	198
27.11. Calculation 11	199
27.12. Calculation 12	200
Chapter 28. The Raw Data	203
28.1. A Guide to the Files	203
28.2. The Main Domain	203
28.3. The Symmetric Pieces	203
28.4. Period Two Tiles	204
28.5. The Domains from the Main Theorem	204
28.6. The Polyhedra in the Partition	204
28.7. The Action of the Map	206
28.8. The Partition for Calculation 11	207
28.9. The First Partition for Calculation 12	208
28.10. The Second Partition for Calculation 12	209
Bibliography	211