Preface

The present volume contains some applications of noncommutative and nonassociative algebras to constructing unusual (nonclassical and singular) solutions to fully nonlinear elliptic partial differential equations of second order. Here the solutions are to be understood in a weak (viscosity) sense. Using such algebras to construct exotic or specific analytic and geometric structures is not new. One can mention here, for instance, the constructions of exotic spheres by Milnor [**163**], of singular solutions to the minimal surface system by Lawson and Osserman [**144**], the ADHM and constuction of instantons by Atiyah, Drinfel'd, Hitchin, and Manin [**17**], and the construction of singular coassociative manifolds by Harvey and Lawson [**103**], all four using quaternions, as well as the recent constructions of unusual solutions of the Ginzburg-Landau system by Farina and Ge-Xie [**86**], [**93**], using isoparametric polynomials and thus, implicitly, Jordan or Clifford algebras.

However, our applications of quaternions, octonions, and Jordan algebras to elliptic partial differential equations of second order are new; they allow us to solve a longstanding problem of the existence of truly weak viscosity solutions, which are not smooth (= classical) ones. Moreover, in some sense, they give (albeit along with some other arguments) an almost complete description of homogeneous solutions to fully nonlinear elliptic equations. In fact, a major part of the book is devoted to the simplest class of fully nonlinear uniformly elliptic equations, namely those of the form

$$F(D^2u) = 0, \tag{0.1}$$

F being a nonlinear sufficiently smooth functional on symmetric matrices and D^2u being the Hessian of a putative solution u. Those are "constant coefficient" fully nonlinear elliptic equations. Moreover, often we impose a rather drastic condition that F depends only on the eigenvalues of the Hessian (so-called "Hessian equations"). In that case F is a function of only n values of symmetric functions of D^2u rather than of $n(n+1)/2$ partial derivatives, n being the dimension of the ambient space. Our methods show that even in that very restricted setting in five and more dimensions (some of) those equations admit homogeneous δ-order solutions with any $\delta \in]1,2]$, that is, of all orders compatible with known regularity results by Caffarelli and Trudinger [**37**], [**255**] for viscosity solutions of fully nonlinear uniformly elliptic equations, proving the optimality of these regularity results. To the contrary, the situation in four and fewer dimensions is completely different. First of all, in two dimensions the classical result by L. Nirenberg [**186**] guarantees the regularity of all viscosity solutions, homogeneous or not. In Section 1.6 we prove that in four (and thus three) dimensions there are no homogeneous order 2 solutions to fully nonlinear uniformly elliptic equations, at least in the analytic setting, which suggests strongly that there are no nonclassical homogeneous solutions at all in four and three dimensions. If so, we get a complete list of dimensions where nonclassical homogeneous solutions to fully nonlinear uniformly elliptic equations do exist. One can compare this with the situation of, say, ten years ago, when the very existence of nonclassical viscosity solutions was not known.

We should repeat once more that this result of fundamental importance for the theory of partial differential equations is obtained by applications of relatively elementary algebraic (and differential geometric) means, thus stressing once more that studying relations between apparently disconnected mathematical areas can often be very fruitful.

Furthermore, there are some cases where (singular) solutions of some classes of nonlinear elliptic equations and some nonassociative algebras are interrelated even more strongly, leading in certain circumstances to the equivalence of those objects. A study of these relations and their applications to classifying both classes of objects is the second theme of the book, intimately related to the previous one.

Our exposition is as follows. Since we hope that our work can be of use to a rather diversified mathematical audience, we devote the first three chapters to the basics of nonlinear elliptic equations and of noncommutative and nonassociative algebraic structures used in our constructions.

In Chapter 1 we recall basic facts about nonlinear elliptic equations and their viscosity solutions. The material in the first five sections is quite traditional in many papers devoted to viscosity solutions. However, in Section 1.2 we also formulate two recent results on partial regularity of solutions, in Section 1.3 we expose a recent result concerning the difference of viscosity solutions, and in Section 1.4 we give a recent result on the regularity of solutions to axially symmetric Dirichlet problems for Hessian equations. Section 1.6 is devoted to recent results and conjectures for homogeneous solutions to fully nonlinear uniformly elliptic equations. Section 1.7 gives some Liouville type results and various results on removable singularities for solutions of fully nonlinear elliptic equations, including a recent result describing viscosity solutions of a uniformly elliptic Hessian equation in a punctured ball.

Chapter 2 is devoted to the construction and elementary properties of the real division algebras $\mathbb{H}, \mathbb{O}$, Clifford algebras, spinor groups, and some exceptional Lee groups, especially G_2. We also discuss cross products in the algebra $\mathbb{O}$ and the resulting calibrations (in their algebraic form).

In Chapter 3 we give an overview of Jordan algebras in their relation to special cubics and some partial differential equations (of first order). Most of its material is classical, but some new facts concerning relations between cubic Jordan algebras and the so-called eiconal differential equation, $|\nabla f(x)|^2 = c|x|^4$, are proven.

In Chapters 4 and 5 we give our main constructions of nonclassical and singular solutions to fully nonlinear, uniformly elliptic equations, often of Hessian or of Isaacs type. In fact, all our nonclassical solutions are of the form $P(x)/|x|^\alpha$ with a homogeneous polynomial $P(x), x \in \mathbb{R}^n$, of degree 3, 4, or 6 and a suitable α. Chapter 4 contains the constructions based on trialities, which use real division algebras: quaternions and octonions; there $n = 12$ or 24, $\deg P = 3$, $\alpha \in [1, 2[$. Chapter 5 gives constructions based on isoparametric polynomials $P(x)$, $x \in \mathbb{R}^n$, $n \geq 5$, of degrees 3, 4, or 6 coming from Jordan and Clifford algebras. The constructions of Chapter 4 are more elementary in that they use less of algebraic theory but need more calculations than those of Chapter 5. The arguments in these chapters are based on several closely related criteria for solutions of fully nonlinear uniformly elliptic equations in terms of appropriate combinations of the spectrum for their Hessians. Those conditions are extremely restrictive and one needs rather elaborated arguments and/or calculations to verify them, which is obtained partially by using MAPLE calculations. One notes, however, that the calculations in Chapters 4 and 5 (and in Chapter 7) which use MAPLE extensively are completely rigorous since there MAPLE is used to verify algebraic identities, albeit rather cumbersome ones.

Chapter 6 is devoted to a classification of cubic minimal cones, that is, the simplest nontrivial solutions to the minimal surface system which is (almost) complete under a natural additional condition (the case of radial eigencubics). The main method there is to construct a certain nonassociative algebra from a given minimal cubic cone in such a way that the differential-analytical structure of the cones becomes transparent from the algebraic side, and vice versa. The main tool for this is the so-called Freudenthal multiplication. It associates to any fixed cubic form u on a vector space V carrying a symmetric nondegenerate bilinear form Q the multiplication $(x, y) \to xy$ by setting $\partial_x \partial_y u|_z = Q(xy; z)$. The algebra $V(u)$ defined in this way is called the Freudenthal-Springer algebra of the cubic form u. In the basic case of a radial eigencubic the corresponding Freudenthal-Springer algebra leads to a so-called radial eigencubic algebra, or just a *REC algebra*. Thus, the classification of radial eigencubics becomes equivalent to that of REC algebras. There exist two principal classes of REC algebras, namely those coming from Clifford and Jordan structures, respectively. Applying standard methods of nonassociative algebra such as Pierce decomposition and a thorough study of certain defining relations in REC algebras, one eventually gets their complete classification. Note, however, that the algebraic techniques of this chapter are elaborated more than in the other chapters and assume more advanced knowledge of the nonassociative algebraic systems.

In Chapter 7 we treat elliptic equations arising in calibrated geometry [**103**], namely, the special Lagrangian, associative, coassociative, and Cayley equations; they are not uniformly, but only strictly, elliptic, and we recall briefly their constructions in Section 7.1. One notes, however, that the construction of singular coassociative 4-folds given by Harvey and Lawson in [**103**] and recalled in Section 7.2 resembles strongly the constructions in Chapter 4. It would be very interesting to understand a possible common ground of constructions in Chapters 4 and 7 (and, presumably, in Chapters 5 and 6) and eventually find some other situations where it works. Sections 7.3 and 7.4 are devoted to constructions of some singular solutions to the special Lagrangian equations (SLE) in the nonconvex case, in three dimensions. Note that in the convex (or concave) case those solutions are smooth in any dimension by [**48**] and that in two dimensions these equations cannot be nonconvex. These constructions also lead to examples of a failure of the maximum principle for the Hessian of solutions to a uniformly elliptic equation in three and more dimensions as well as to examples of solutions to the minimal surface system with a notably low regularity.

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