Introduction

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1. Elliptic cohomology

A ring-valued cohomology theory E is *complex orientable* if there is an 'orientation class' $x \in E^2(\mathbb{CP}^\infty)$ whose restriction along the inclusion $S^2 \cong \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^\infty$ is the element 1 in $E^0 S^0 \cong E^2\mathbb{CP}^1$. The existence of such an orientation class implies, by the collapse of the Atiyah–Hirzebruch spectral sequence, that

$$E^*(\mathbb{CP}^\infty) \cong E^*[[x]].$$

The class x is a universal characteristic class for line bundles in E-cohomology; it is the E-theoretic analogue of the first Chern class. The space $\mathbb{CP}^\infty$ represents the functor

$$X \mapsto \{\text{isomorphism classes of line bundles on } X\},$$

and the tensor product of line bundles induces a multiplication map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$. Applying E^* produces a ring map

$$E^*[[x]] \cong E^*(\mathbb{CP}^\infty) \to E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong E^*[[x_1, x_2]];$$

the image of x under this map is a formula for the E-theoretic first Chern class of a tensor product of line bundles in terms of the first Chern classes of the two factors. That ring map $E^*[[x]] \to E^*[[x_1, x_2]]$ is a (1-dimensional, commutative) formal group law—that is, a commutative group structure on the formal completion $\hat{\mathbb{A}}^1$ at the origin of the affine line $\mathbb{A}^1$ over the ring E^*.

A formal group often arises as the completion of a group scheme at its identity element; the dimension of the formal group is the dimension of the original group scheme. There are three kinds of 1-dimensional group schemes:

(1) the additive group $\mathbb{G}_a = \mathbb{A}^1$ with multiplication determined by the map $\mathbb{Z}[x] \to \mathbb{Z}[x_1, x_2]$ sending x to $x_1 + x_2$,
(2) the multiplicative group $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\}$ with multiplication determined by the map $\mathbb{Z}[x^{\pm 1}] \to \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ sending x to $x_1 x_2$, and
(3) elliptic curves (of which there are many isomorphism classes).

Ordinary cohomology is complex orientable, and its associated formal group is the formal completion of the additive formal group. Topological K-theory is also complex orientable, and its formal group is the formal completion of the multiplicative formal group. This situation naturally leads one to search for 'elliptic' cohomology theories whose formal groups are the formal completions of elliptic curves. These elliptic cohomology theories should, ideally, be functorial for morphisms of elliptic curves.

Complex bordism MU is complex orientable and the resulting formal group law is the universal formal group law; this means that ring maps from MU_* to R are in natural bijective correspondence with formal group laws over R. Given a commutative ring R and a map $MU_* \to R$ that classifies a formal group law over R, the functor

$$X \;\mapsto\; MU_*(X) \otimes_{MU_*} R$$

is a homology theory if and only if the corresponding map from $\mathrm{Spec}(R)$ to the moduli stack $\mathcal{M}_{FG}$ of formal groups is flat. There is a map

$$\mathcal{M}_{ell} \to \mathcal{M}_{FG}$$

from the moduli stack of elliptic curves to that of formal groups, sending an elliptic curve to its completion at the identity; this map is flat. Any flat map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$ therefore provides a flat map $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ and thus a homology theory, or equivalently, a cohomology theory (a priori only defined on finite CW-complexes). In other words, to any affine scheme with a flat map to the moduli stack of elliptic curves, there is a functorially associated cohomology theory.

The main theorem of Goerss–Hopkins–Miller is that this functor (that is, presheaf)

$$\big\{\text{flat maps from affine schemes to } \mathcal{M}_{ell}\big\} \to \big\{\text{multiplicative cohomology theories}\big\},$$

when restricted to maps that are étale, lifts to a sheaf

$$\mathcal{O}^{\mathrm{top}} :\; \big\{\text{étale maps to } \mathcal{M}_{ell}\big\} \to \big\{E_\infty\text{-ring spectra}\big\}.$$

(Here the subscript 'top' refers to it being a kind of 'topological', rather than discrete, structure sheaf.) The value of this sheaf on $\mathcal{M}_{ell}$ itself, that is the E_∞-ring spectrum of global sections, is the periodic version of the spectrum of topological modular forms:

$$\mathit{TMF} := \mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell}) = \Gamma(\mathcal{M}_{ell}, \mathcal{O}^{\mathrm{top}}).$$

The spectrum TMF owes its name to the fact that its ring of homotopy groups is rationally isomorphic to the ring

$$\mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 - 1728\Delta) \;\cong\; \bigoplus_{n \geq 0} \Gamma\big(\mathcal{M}_{ell}, \omega^{\otimes n}\big)$$

of weakly holomorphic integral modular forms. Here, the elements c_4, c_6, and Δ have degrees 8, 12, and 24 respectively, and ω is the sheaf of invariant differentials (the restriction to $\mathcal{M}_{ell}$ of the (vertical) cotangent bundle of the universal elliptic curve $\mathcal{E} \to \mathcal{M}_{ell}$). That ring of modular forms is periodic with period 24, and the periodicity is given by multiplication by the discriminant Δ. The discriminant is not an element in the homotopy groups of TMF, but its twenty-fourth power $\Delta^{24} \in \pi_{24^2}(\mathit{TMF})$ is, and, as a result, $\pi_*(\mathit{TMF})$ has a periodicity of order $24^2 = 576$.

One would like an analogous E_∞-ring spectrum whose homotopy groups are rationally isomorphic to the subring

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$$

of integral modular forms. For that, one observes that the sheaf $\mathcal{O}^{\mathrm{top}}$ is defined not only on the moduli stack of elliptic curves, but also on the Deligne–Mumford compactification $\overline{\mathcal{M}}_{ell}$ of the moduli stack—this compactification is the moduli stack of elliptic curves possibly with nodal singularities. The spectrum of global sections over $\overline{\mathcal{M}}_{ell}$ is denoted

$$Tmf := \mathcal{O}^{\mathrm{top}}(\overline{\mathcal{M}}_{ell}) = \Gamma(\overline{\mathcal{M}}_{ell}, \mathcal{O}^{\mathrm{top}}).$$

The element $\Delta^{24} \in \pi_{24^2}(Tmf)$ is no longer invertible in the homotopy ring, and so the spectrum Tmf is not periodic. This spectrum is not connective either, and the mixed capitalization reflects its intermediate state between the periodic version TMF and the connective version tmf, described below, of topological modular forms.

In positive degrees, the homotopy groups of Tmf are rationally isomorphic to the ring $\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$. The homotopy groups $\pi_{-1}, \ldots, \pi_{-20}$ are all zero, and the remaining negative homotopy groups are given by:

$$\pi_{-n}(Tmf) \;\cong\; \big[\pi_{n-21}(Tmf)\big]_{\textit{torsion-free}} \oplus \big[\pi_{n-22}(Tmf)\big]_{\textit{torsion}}.$$

This structure in the homotopy groups is a kind of Serre duality reflecting the properness (compactness) of the moduli stack $\overline{\mathcal{M}}_{ell}$.

If we take the (-1)-connected cover of the spectrum Tmf, that is, if we kill all its negative homotopy groups, then we get

$$tmf := Tmf\langle 0\rangle,$$

the connective version of the spectrum of topological modular forms. This spectrum is now, as desired, a topological refinement of the classical ring of integral modular forms. Note that one can recover TMF from either of the other versions by inverting the element Δ^{24} in the 576^{th} homotopy group:

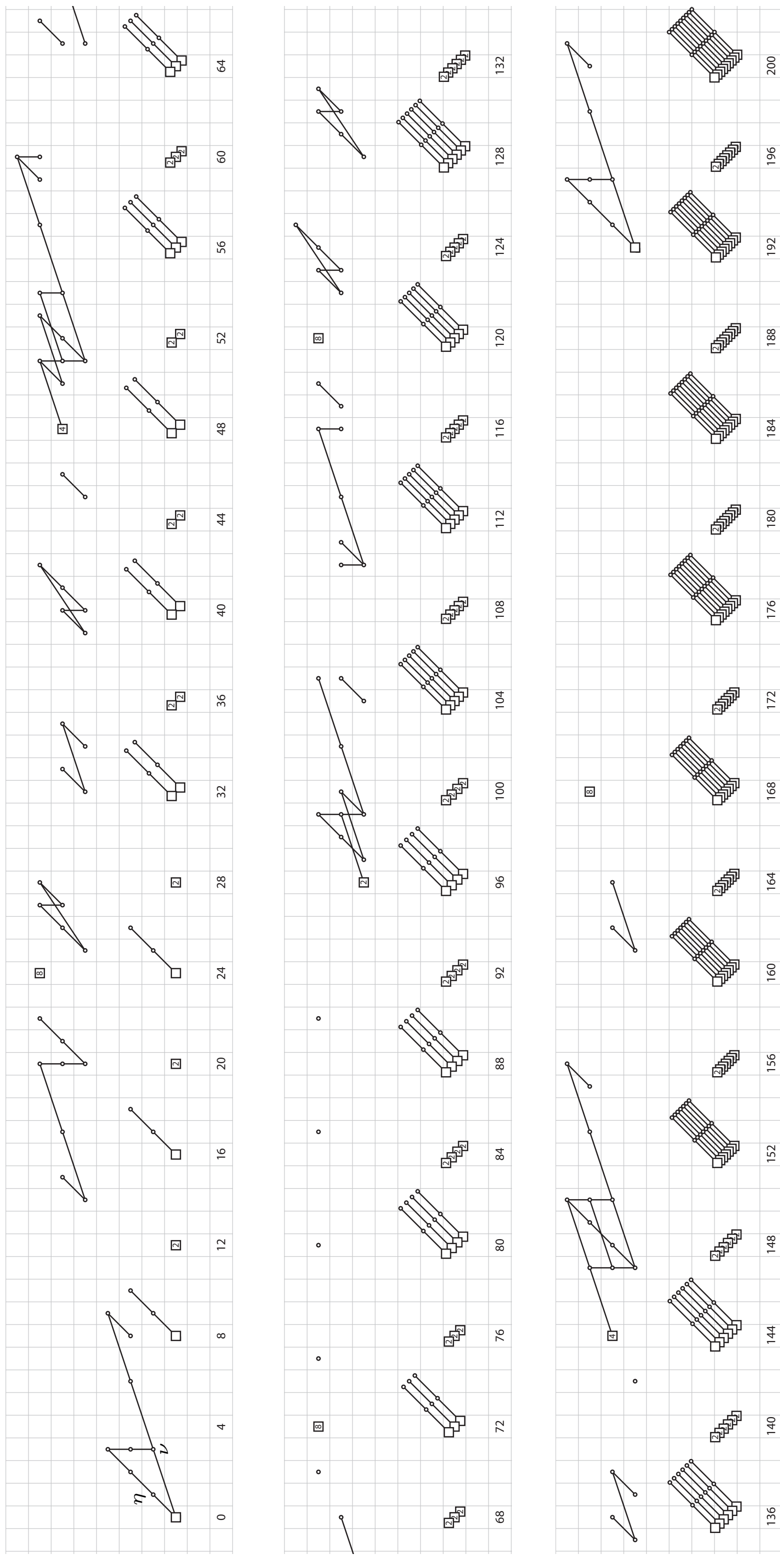
$$TMF = tmf[\Delta^{-24}] = Tmf[\Delta^{-24}].$$

There is another moduli stack worth mentioning here, the stack $\overline{\mathcal{M}}^{+}_{ell}$ of elliptic curves with possibly nodal or cuspidal singularities. There does not seem to be an extension of $\mathcal{O}^{\mathrm{top}}$ to that stack. However, if there were one, then a formal computation, namely an elliptic spectral sequence for that hypothetical sheaf, shows that the global sections of the sheaf over $\overline{\mathcal{M}}^{+}_{ell}$ would be the spectrum tmf. That hypothetical spectral sequence is the picture that appears before the preface. It is also, more concretely, the Adams–Novikov spectral sequence for the spectrum tmf.

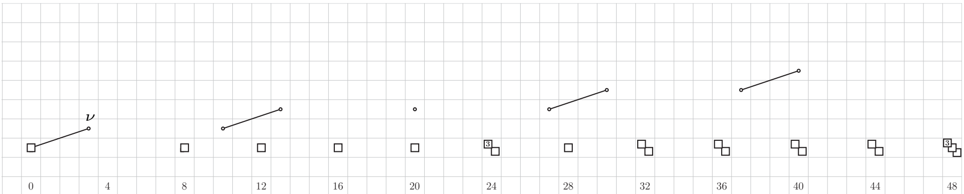
So far, we have only mentioned the connection between tmf and modular forms. The connection of tmf to the stable homotopy groups of spheres is equally strong and the unit map from the sphere spectrum to tmf detects an astounding amount of the 2- and 3-primary parts of the homotopy $\pi_*(\mathbb{S})$ of the sphere.

The homotopy groups of tmf are as follows at the prime 2:

The homotopy groups of tmf at the prime 2.



and as follows at the prime 3:



Here, a square indicates a copy of $\mathbb{Z}$ and a dot indicates a copy of $\mathbb{Z}/p$. A little number n drawn in a square indicates that the copy of $\mathbb{Z}$ in $\pi_*(tmf)$ maps onto an index n subgroup of the corresponding $\mathbb{Z}$ in the ring of modular forms. A vertical line between two dots indicates an additive extension, and a slanted line indicates the multiplicative action of the generator $\eta \in \pi_1(tmf)$ or $\nu \in \pi_3(tmf)$. The y-coordinate, although vaguely reminiscent of the filtration degree in the Adams spectral sequence, has no meaning in the above charts.

Note that, at the prime 2, the pattern on the top of the chart (that is, above the expanding ko pattern on the base) repeats with a periodicity of $192 = 8 \cdot 24$. A similar periodicity (not visible in the above chart) happens at the prime 3, with period $72 = 3 \cdot 24$. Over $\mathbb{Z}$, taking the least common multiple of these two periodicities results in a periodicity of $24 \cdot 24 = 576$.

2. A brief history of *tmf*

In the sixties, Conner and Floyd proved that complex K-theory is determined by complex cobordism: if X is a space, then its K-homology can be described as $K_*(X) \cong MU_*(X) \otimes_{MU_*} K_*$, where K_* is a module over the complex cobordism ring of the point via the Todd genus map $MU_* \to K_*$. Following this observation, it was natural to look for other homology theories that could be obtained from complex cobordism by a similar tensor product construction. By Quillen's theorem (1969), MU_* is the base ring over which the universal formal group law is defined; ring maps $MU_* \to R$ thus classify formal groups laws over R.

Given such a map, there is no guarantee in general that the functor $X \mapsto MU_*(X) \otimes_{MU_*} R$ will be a homology theory. If R is a flat MU_*-module, then long exact sequences remain exact after tensoring with R and so the functor in question does indeed define a new homology theory. However, the condition of being flat over MU_* is quite restrictive. Landweber's theorem (1976) showed that, because arbitrary MU_*-modules do not occur as the MU-homology of spaces, the flatness condition can be greatly relaxed. A more general condition, Landweber exactness, suffices to ensure that the functor $MU_*(-) \otimes_{MU_*} R$ satisfies the axioms of a homology theory. Shortly after the announcement of Landweber's result, Morava applied that theorem to the formal groups of certain elliptic curves and constructed the first elliptic cohomology theories (though the term 'elliptic cohomology' was coined only much later).

In the mid-eighties, Ochanine introduced certain genera (that is homomorphisms out of a bordism ring) related to elliptic integrals, and Witten constructed a genus that took values in the ring of modular forms, provided the low-dimensional characteristic classes of the manifold vanish. Landweber–Ravenel–Stong made explicit the connection between elliptic genera, modular forms, and elliptic cohomology by identifying the target of the universal Ochanine elliptic genus with the coefficient ring of the homology theory $X \mapsto MU_*(X) \otimes_{MU_*} \mathbb{Z}[\frac{1}{2}][\delta, \epsilon, \Delta^{-1}]$ associated to the Jacobi quartic elliptic curve $y^2 = 1 - 2\delta x^2 + \epsilon x^4$ (here, Δ is the discriminant of the polynomial in x). Segal had also presented a picture of the relationship between elliptic cohomology and Witten's physics-inspired index theory on loop spaces. In hindsight, a natural question would have been whether there existed a form of elliptic cohomology that received Witten's genus, thus explaining its integrality and modularity properties. But at the time, the community's attention was on Witten's rigidity conjecture for elliptic genera (established by Bott and Taubes), and on finding a geometric interpretation for elliptic cohomology—a problem that remains open to this day, despite a tantalizing proposal by Segal and much subsequent work.

Around 1989, inspired in part by work of McClure and Baker on A_∞ structures and actions on spectra and by Ravenel's work on the odd primary Arf invariant, Hopkins and Miller showed that a certain profinite group known as the Morava stabilizer group acts by A_∞ automorphisms on the Lubin–Tate spectrum E_n (the representing spectrum for the Landweber exact homology theory associated to the universal deformation of a height n formal group law). Of special interest was the action of the binary tetrahedral group on the spectrum E_2 at the prime 2. The homotopy fixed point spectrum of this action was called EO_2, by analogy with the real K-theory spectrum KO being the homotopy fixed points of complex conjugation on the complex K-theory spectrum.

Mahowald recognized the homotopy of EO_2 as a periodic version of a hypothetical spectrum with mod 2 cohomology $A/\!/A(2)$, the quotient of the Steenrod algebra by the submodule generated by Sq^1, Sq^2, and Sq^4. It seemed likely that there would be a corresponding connective spectrum eo_2 and indeed a bit later Hopkins and Mahowald produced such a spectrum; (in hindsight, that spectrum eo_2 is seen as the 2-localization of *tmf*). However, Davis–Mahowald (1982) had proved, by an intricate spectral sequence argument, that it is impossible to realize $A/\!/A(2)$ as the cohomology of a spectrum. This conundrum was resolved only much later, when Mahowald found a missing differential around the 55^{th} stem of the Adams spectral sequence for the sphere, invalidating the earlier Davis–Mahowald argument.

In the meantime, computations of the cohomology of $MO\langle 8\rangle$ at the prime 2 revealed an $A/\!/A(2)$ summand, suggesting the existence of a map of spectra from $MO\langle 8\rangle$ to eo_2. While attempting to construct a map $MO\langle 8\rangle \to EO_2$, Hopkins (1994) thought to view the binary tetrahedral group as the automorphism group of the supersingular elliptic curve at the prime 2; the idea of a sheaf of ring spectra over the moduli stack of elliptic curves quickly followed—the global sections of that sheaf, *TMF*, would then be an integral version of EO_2.

The language of stacks, initially brought to bear on complex cobordism and formal groups by Strickland, proved crucial for even formulating the question *TMF* would answer. In particular, the stacky perspective allowed a reformulation of Landweber's exactness criterion in a more conceptual and geometric way: $MU_* \to R$ is Landweber exact if and only if the corresponding map to the moduli stack of formal groups, $\operatorname{Spec}(R) \to \mathcal{M}_{FG}$, is flat. From this viewpoint, Landweber's theorem defined a presheaf of homology theories on the flat site of the moduli stack $\mathcal{M}_{FG}$ of formal groups. Restricting to those formal groups coming from elliptic curves then provided a presheaf of homology theories on the moduli stack of elliptic curves.

Hopkins and Miller conceived of the problem as lifting this presheaf of homology theories to a sheaf of spectra. In the 80s and early 90s, Dwyer, Kan, Smith, and Stover had developed an obstruction theory for rigidifying a diagram in a homotopy category (here a diagram of elliptic homology theories) to an honest diagram (here a sheaf of spectra). Hopkins and Miller adapted the Dwyer–Kan–Stover theory to treat the seemingly more difficult problem of rigidifying a diagram of multiplicative cohomology theories to a diagram of A_∞-ring spectra. The resulting multiplicative obstruction groups vanished, except at the prime 2—Hopkins addressed that last case by a direct construction in the category of $K(1)$-local E_∞-ring spectra. Altogether the resulting sheaf of spectra provided a universal elliptic cohomology theory, the spectrum *TMF* of global sections (and its connective version *tmf*). Subsequently, Goerss and Hopkins upgraded the A_∞ obstruction theory to an obstruction theory for E_∞-ring spectra, leading to the definitive theorem of Goerss–Hopkins–Miller: the presheaf of elliptic homology theories on the compactified moduli stack of elliptic curves lifts to a sheaf of E_∞-ring spectra.

Meanwhile, Ando–Hopkins–Strickland (2001) established a systematic connection between elliptic cohomology and elliptic genera by constructing, for every elliptic cohomology theory E, an E-orientation for almost complex manifolds with certain vanishing characteristic classes. This collection was expected to assemble into a single unified multiplicative *tmf*-orientation. Subsequently Laures (2004) built a $K(1)$-local E_∞-map $MO\langle 8\rangle \to tmf$ and then finally Ando–Hopkins–Rezk produced the expected integral map of E_∞-ring spectra $MO\langle 8\rangle \to tmf$ that recovers Witten's genus at the level of homotopy groups.

Later, an interpretation of *tmf* was given by Lurie (2009) using the theory of spectral algebraic geometry, based on work of Töen and Vezzosi. Lurie interpreted the stack $\mathcal{M}_{ell}$ with its sheaf $\mathcal{O}^{\mathrm{top}}$ as a stack not over commutative rings but over E_∞-ring spectra. Using Goerss–Hopkins–Miller obstruction theory and a spectral form of Artin's representability theorem, he identified that stack as classifying oriented elliptic curves over E_∞-ring spectra. Unlike the previous construction of *tmf* and of the sheaf $\mathcal{O}^{\mathrm{top}}$, this description specifies the sheaf and therefore the spectrum *tmf* up to a contractible space of choices.

3. Overview

Part I

<u>Chapter 1</u>: **Elliptic genera and elliptic cohomology.** One-dimensional formal group laws entered algebraic topology though complex orientations, in answering the question of which generalized cohomology theories E carry a theory of Chern classes for complex vector bundles. In any such theory, the E-cohomology of $\mathbb{CP}^\infty$ is isomorphic to $E^*[[c_1]]$, the E-cohomology ring of a point adjoin a formal power series generator in degree 2. The tensor product of line bundles defines a map $\mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$, which in turn defines a comultiplication on $E^*[[c_1]]$, i.e., a formal group law. Ordinary homology is an example of such a theory; the associated formal group is the additive formal group, since the first Chern class of the tensor product of line bundles is the sum of the respective Chern classes, $c_1(L \otimes L') = c_1(L) + c_1(L')$. Complex K-theory is another example of such a theory; the associated formal group is the multiplicative formal group.

Complex cobordism also admits a theory of Chern classes, hence a formal group. Quillen's theorem is that this is the universal formal group. In other words, the formal group of complex cobordism defines a natural isomorphism of MU^* with the Lazard ring, the classifying ring for formal groups. Thus, a one-dimensional formal group over a ring R is essentially equivalent to a complex genus, that is, a ring homomorphism $MU^* \to R$. One important example of such a genus is the Todd genus, a map $MU^* \to K^*$. The Todd genus occurs in the Hirzebruch–Riemann–Roch theorem, which calculates the index of the Dolbeault operator in terms of the Chern character. It also determines the K-theory of a finite space X from its complex cobordism groups, via the Conner–Floyd theorem: $K^*(X) \cong MU^*(X) \otimes_{MU^*} K^*$.

Elliptic curves form a natural source of formal groups, and hence complex genera. An example of this is Euler's formal group law over $\mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$ associated to Jacobi's quartic elliptic curve; the corresponding elliptic cohomology theory is given on finite spaces by $X \mapsto MU^*(X) \otimes_{MU^*} \mathbb{Z}[\frac{1}{2}, \delta, \epsilon]$. Witten defined a genus $MSpin \to \mathbb{Z}[[q]]$ (not a complex genus, because not a map out of MU^*) which lands in the ring of modular forms, provided the characteristic class $\frac{p_1}{2}$ vanishes. He also gave an index theory interpretation of this genus, at a physical level of rigor, in terms of Dirac operators on loop spaces. It was later shown, by Ando–Hopkins–Rezk, that the Witten genus can be lifted to a map of ring spectra $MString \to tmf$. The theory of topological modular forms can therefore be seen as a solution to the problem of finding a kind of elliptic cohomology that is connected to the Witten genus in the same way that the Todd genus is to K-theory.

Chapter 2: Elliptic curves and modular forms. An elliptic curve is a non-singular curve in the projective plane defined by a Weierstrass equation:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Elliptic curves can also be presented abstractly, as pointed genus one curves. They are equipped with a group structure, where one declares the sum of three points to be zero if they are collinear in $\mathbb{P}^2$. The bundle of Kähler differentials on an elliptic curve, denoted ω, has a one-dimensional space of global sections.

When working over a field, one-dimensional group varieties can be classified into additive groups, multiplicative groups, and elliptic curves. However, when working over an arbitrary ring, the object defined by a Weierstrass equation will typically be a combination of those three cases. The general fibers will be elliptic curves, some fibers will be nodal (multiplicative groups), and some cuspidal (additive groups).

By a 'Weierstrass curve' we mean a curve defined by a Weierstrass equation—there is no smoothness requirement. An *integral modular form* can then be defined, abstractly, to be a law that associates to every (family of) Weierstrass curves a section of $\omega^{\otimes n}$, in a way compatible with base change. Integral modular forms form a graded ring, graded by the power of ω. Here is a concrete presentation of that ring:

$$\mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta).$$

In the context of modular forms, the degree is usually called the *weight*: the generators c_4, c_6, and Δ have weight 4, 6, and 12, respectively. As we will see, those weights correspond to the degrees 8, 12, and 24 in the homotopy groups of *tmf*.

Chapter 3: The moduli stack of elliptic curves. We next describe the geometry of the moduli stack of elliptic curves over fields of prime characteristic, and over the integers. At large primes, the stack $\mathcal{M}_{ell}$ looks rather like it does over $\mathbb{C}$:

general elliptic curves have an automorphism group of order two, and there are two special curves with automorphism groups of orders four and six. That picture needs to be modified when dealing with small primes. At the prime $p = 3$ (respectively $p = 2$), there is only one special 'orbifold point', and the automorphism group of the corresponding elliptic curve has order 12 (respectively 24). The automorphism group of that curve is given by $\mathbb{Z}/4 \ltimes \mathbb{Z}/3$ at the prime 3, and by $\mathbb{Z}/3 \ltimes Q_8$ (also known as the binary tetrahedral group) at the prime 2.

In characteristic p, there is an important dichotomy between *ordinary* and *supersingular* elliptic curves. An elliptic curve C is ordinary if its group of p-torsion points has p connected components, and supersingular if the group of p-torsion points is connected. This dichotomy is also reflected in the structure of the multiplication-by-p map, which is purely inseparable in the supersingular case, and the composite of an inseparable map with a degree p covering in the case of an ordinary elliptic curve. The supersingular elliptic curves form a zero-dimensional substack of $(\mathcal{M}_{ell})_{\mathbb{F}_p}$—the stack of elliptic curves in characteristic p—whose cardinality grows roughly linearly in p. If one counts supersingular curves with a multiplicity equal to the inverse of the order of their automorphism group, then there are exactly $(p-1)/24$ of them.

The stratification of $(\mathcal{M}_{ell})_{\mathbb{F}_p}$ into ordinary and supersingular elliptic curves is intimately connected to the stratification of the moduli stack of formal groups by the *height* of the formal group. A formal group has height n if the first non-zero coefficient of the multiplication-by-p map is that of x^{p^n}. The ordinary elliptic curves are the ones whose associated formal group has height 1, and the supersingular elliptic curves are the ones whose associated formal group has height 2. Higher heights cannot occur among elliptic curves.

Chapter 4: The Landweber exact functor theorem. The next main result is that there this a presheaf Ell of homology theories on the (affines of the) flat site of the moduli stack of elliptic curves—the category whose objects are flat maps $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$. That presheaf is defined as follows. Given an elliptic curve C over a ring R, classified by a flat map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$, the associated formal group $\widehat{C}$ corresponds to a map $MP_0 \to R$, where $MP_* = \bigoplus_{n\in\mathbb{Z}} MU_{*+2n}$ is the periodic version of complex cobordism. Ell^C is then defined by

$$Ell^C(X) := MP_*(X) \underset{MP_0}{\otimes} R.$$

We claim that for every elliptic curve C whose classifying map $\mathrm{Spec}(R) \to \mathcal{M}_{ell}$ is flat, the functor Ell^C is a homology theory, i.e., satisfies the exactness axiom. An example of an elliptic curve whose classifying map is flat, and which therefore admits an associated elliptic homology theory, is the universal smooth Weierstrass curve.

The proof of this claim is a combination of several ingredients. The main one is the Landweber exact functor theorem, which provides an algebraic criterion (Landweber exactness, which is weaker than flatness) on a ring map $MP_0 \to R$, that ensures the functor $X \mapsto MP_*(X) \otimes_{MP_0} R$ satisfies exactness. The other ingredients, due to Hopkins and Miller, relate the geometry of $\mathcal{M}_{ell}$ and $\mathcal{M}_{FG}$ to the Landweber exactness criterion. These results are the following: (1) A formal group law $MP_0 \to R$ over R is Landweber exact if and only if the corresponding map $\mathrm{Spec}(R) \to \mathcal{M}_{FG}$ is flat; together with Landweber exactness, this gives a presheaf of homology theories on the flat site of the moduli stack of formal groups $\mathcal{M}_{FG}$. (2) The map of stacks, $\mathcal{M}_{ell} \to \mathcal{M}_{FG}$ defined by sending an elliptic curve to its associated formal group, is flat.

Chapter 5: Sheaves in homotopy theory. By the above construction, using the Landweber exact functor theorem, we have a presheaf $\mathcal{O}^{\mathrm{hom}}$ of homology theories (previously called *Ell*) on the moduli stack of elliptic curves. One might try to define a single 'universal elliptic homology theory' as the limit $\lim_{U\in\mathfrak{U}} \mathcal{O}^{\mathrm{hom}}(U)$, where $\mathfrak{U}$ is an affine cover of the moduli stack. However, the category of homology theories does not admit limits. If, though, we can rigidify the presheaf $\mathcal{O}^{\mathrm{hom}}$ of homology theories to a presheaf $\mathcal{O}^{\mathrm{top}}$ of spectra, then we can use instead a homotopy limit construction in the category of spectra. The main theorem is that there does indeed exist such a presheaf, in fact a sheaf, of spectra.

THEOREM (Goerss–Hopkins–Miller). *There exists a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞-ring spectra on $(\mathcal{M}_{ell})_{ét}$, the étale site of the moduli stack of elliptic curves (whose objects are étale maps to $\mathcal{M}_{ell}$), such that the associated presheaf of homology theories, when restricted to those maps whose domain is affine, is the presheaf $\mathcal{O}^{\mathrm{hom}}$ built using the Landweber exact functor theorem.*

In this theorem, it is essential that the sheaf $\mathcal{O}^{\mathrm{top}}$ is a functor to a point-set-level, not homotopy, category of E_∞-ring spectra. Moreover, the functor is defined on *all* étale maps $\mathcal{N} \to \mathcal{M}_{ell}$, not just those where $\mathcal{N}$ is affine; (in fact, $\mathcal{N}$ can be itself a stack, as long as the map to $\mathcal{M}_{ell}$ is étale). Given a cover $\mathfrak{N} = \{\mathcal{N}_i \to \mathcal{N}\}$ of an object $\mathcal{N}$, we can assemble the n-fold 'intersections' $\mathcal{N}_{ij} := \mathcal{N}_i \times_{\mathcal{N}} \mathcal{N}_j$, $\mathcal{N}_{ijk} := \mathcal{N}_i \times_{\mathcal{N}} \mathcal{N}_j \times_{\mathcal{N}} \mathcal{N}_k$, and so forth, into a simplicial object

$$\mathfrak{N}_\bullet \;=\; \Big[\coprod \mathcal{N}_i \leftleftarrows \coprod \mathcal{N}_{ij} \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\; \coprod \mathcal{N}_{ijk} \;\overset{\leftarrow}{\underset{\leftarrow}{\leftleftarrows}}\; \cdots\Big].$$

The sheaf condition is that the natural map from the totalization (homotopy limit) of the cosimplicial spectrum

$$\mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet) \;=\; \Big[\mathcal{O}^{\mathrm{top}}(\coprod \mathcal{N}_i) \rightrightarrows \mathcal{O}^{\mathrm{top}}(\coprod \mathcal{N}_{ij}) \;\overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}}\; \mathcal{O}^{\mathrm{top}}(\coprod \mathcal{N}_{ijk}) \cdots\Big]$$

to $\mathcal{O}^{\mathrm{top}}(\mathcal{N})$ is an equivalence.

Now consider a cover $\mathfrak{N} = \{\mathcal{N}_i \to \mathcal{M}_{ell}\}$ of $\mathcal{M}_{ell}$ by affine schemes. The aforementioned cosimplicial spectrum for this cover has an associated tower of fibrations

$$\ldots \to \mathrm{Tot}^2\, \mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet) \to \mathrm{Tot}^1\, \mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet) \to \mathrm{Tot}^0\, \mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet)$$

whose inverse limit is $\mathrm{Tot}\, \mathcal{O}^{\mathrm{top}}(\mathfrak{N}_\bullet) = \mathcal{O}^{\mathrm{top}}(\mathcal{M}_{ell}) = TMF$. The spectral sequence associated to this tower has as E^2 page the Cech cohomology $\check{H}^q_{\mathfrak{N}}(\mathcal{M}_{ell}, \pi_p\mathcal{O}^{\mathrm{top}})$ of $\mathfrak{N}$ with coefficients in $\pi_p\mathcal{O}^{\mathrm{top}}$. Since $\mathfrak{N}$ is a cover by affines, the Cech cohomology of that cover is the same as the sheaf cohomology of $\mathcal{M}_{ell}$ with coefficients in the sheafification $\pi_p^\dagger\mathcal{O}^{\mathrm{top}}$ of $\pi_p\mathcal{O}^{\mathrm{top}}$; (that sheafification happens to agree with $\pi_p\mathcal{O}^{\mathrm{top}}$ on maps to $\mathcal{M}_{ell}$ whose domain is affine). Altogether, we get a spectral sequence, the so-called descent spectral sequence, that converges to the homotopy groups of the spectrum of global sections:

$$E^2_{pq} = H^q(\mathcal{M}_{ell}, \pi_p^\dagger\mathcal{O}^{\mathrm{top}}) \;\Rightarrow\; \pi_{p-q}\, TMF.$$

Chapter 6: Bousfield localization and the Hasse square. We would like a sheaf of spectra $\mathcal{O}^{\mathrm{top}}$ on the moduli stack of elliptic curves $\mathcal{M}_{ell}$. As we will see, this moduli stack is built out of its p-completions $\mathcal{O}_p^{\mathrm{top}}$ and its rationalization. The p-completion $\mathcal{O}_p^{\mathrm{top}}$ is in turn built from certain localizations of $\mathcal{O}^{\mathrm{top}}$ with respect to the first and second Morava K-theories.

Localizing a spectrum X at a spectrum E is a means of systematically ignoring the part of X that is invisible to E. A spectrum A is called E-acyclic if $A \wedge X$ is contractible. A spectrum B is called E-local if there are no nontrivial maps from an E-acyclic spectrum into B. Finally, a spectrum Y is an E-localization of X if it is E-local and there is a map $X \to Y$ that is an equivalence after smashing with E. This localization is denoted $L_E X$ or X_E.

The localization $L_p X := L_{M(\mathbb{Z}/p)} X$ of a spectrum X at the mod p Moore spectrum is the p-completion of X (when X is connective); we denote this localization map $\eta_p : X \to L_p X$. The localization $L_{\mathbb{Q}} X := L_{H\mathbb{Q}} X$ at the rational Eilenberg–MacLane spectrum is the rationalization of X; we denote this localization map $\eta_{\mathbb{Q}} : X \to L_{\mathbb{Q}} X$.

Any spectrum X can be reconstructed from its p-completion and rationalization by means of the 'Sullivan arithmetic square', which is the following homotopy pullback square:

$$\begin{array}{ccc} X & \xrightarrow{\prod \eta_p} & \prod_p L_p X \\ {\scriptstyle \eta_{\mathbb{Q}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathbb{Q}}} \\ L_{\mathbb{Q}} X & \xrightarrow{L_{\mathbb{Q}}(\prod \eta_p)} & L_{\mathbb{Q}}\left(\prod_p L_p X\right). \end{array}$$

The above pullback square is a special case of the localization square

$$\begin{array}{ccc} L_{E\vee F} X & \xrightarrow{\eta_E} & L_E X \\ {\scriptstyle \eta_F}\downarrow & & \downarrow{\scriptstyle \eta_F} \\ L_F X & \xrightarrow{L_F(\eta_E)} & L_F L_E X, \end{array}$$

which is a homotopy pullback square if one assumes that $E_*(L_F X) = 0$.

An application of this localization square gives the so-called 'chromatic fracture square':

$$\begin{array}{ccc} L_{K(1)\vee K(2)} X & \xrightarrow{\eta_{K(2)}} & L_{K(2)} X \\ {\scriptstyle \eta_{K(1)}}\downarrow & & \downarrow{\scriptstyle \eta_{K(1)}} \\ L_{K(1)} X & \xrightarrow{L_{K(1)}(\eta_{K(2)})} & L_{K(1)} L_{K(2)} X. \end{array}$$

Here $K(1)$ and $K(2)$ are the first and second Morava K-theory spectra.

When the spectrum in question is an elliptic spectrum, the above square simplifies into the 'Hasse square': for any elliptic spectrum E, there is a pullback square

$$\begin{array}{ccc} L_pE & \longrightarrow & L_{K(2)}E \\ \big\downarrow & & \big\downarrow \\ L_{K(1)}E & \longrightarrow & L_{K(1)}L_{K(2)}E. \end{array}$$

By means of the arithmetic square, the construction of the sheaf $\mathcal{O}^{\text{top}}$ is reduced to the construction of its p-completions, of its rationalization, and of the comparison map between the rationalization and the rationalization of the product of the p-completions. In turn, via the Hasse square, the construction of the p-completion $\mathcal{O}_p^{\text{top}}$ of the sheaf $\mathcal{O}^{\text{top}}$ is reduced to the construction of the corresponding $K(1)$- and $K(2)$-localizations and of a comparison map between the $K(1)$-localization and the $K(1)$-localization of the $K(2)$-localization.

Chapter 7: The local structure of the moduli stack of formal groups. By Landweber's theorem, flat maps $\text{Spec}(R) \to \mathcal{M}_{FG}$ to the moduli stack of one-dimensional formal groups give rise to even-periodic homology theories:

$$X \mapsto MP_*(X) \otimes_{MP_0} R.$$

Here, MP is periodic complex bordism, $MP_0 = MU_* \cong \mathbb{Z}[u_1, u_2, \ldots]$ is the Lazard ring, and the choice of a formal group endows R with the structure of an algebra over that ring. We wish to understand the geometry of $\mathcal{M}_{FG}$ with an eye towards constructing such flat maps.

The geometric points of $\mathcal{M}_{FG}$ can be described as follows. If k is a separably closed field of characteristic $p > 0$, then formal groups over k are classified by their height, where again a formal group has height n if the first non-trivial term of its p-series (the multiplication-by-p map) is the one involving x^{p^n}. Given a formal group of height n, classified by $\text{Spec}(k) \to \mathcal{M}_{FG}$, one may consider 'infinitesimal thickenings' $\text{Spec}(k) \hookrightarrow B$, where B is the spectrum of a local (pro-)Artinian algebra with residue field k, along with an extension

$$\begin{array}{ccc} \text{Spec}(k) & \longrightarrow & \mathcal{M}_{FG}. \\ \big\downarrow & \nearrow & \\ B & & \end{array}$$

This is called a deformation of the formal group. The Lubin–Tate theorem says that a height n formal group admits a universal deformation (a deformation with a unique map from any other deformation), carried by the ring $\mathbb{W}(k)[[v_1, \ldots v_{n-1}]]$. Here, $\mathbb{W}(k)$ denotes the ring of Witt vectors of k. Moreover, the map from $B := \text{Spf}(\mathbb{W}(k)[[v_1, \ldots v_{n-1}]])$ to $\mathcal{M}_{FG}$ is flat.

The formal groups of interest in elliptic cohomology come from elliptic curves. The Serre–Tate theorem further connects the geometry of $\mathcal{M}_{ell}$ with that of $\mathcal{M}_{FG}$, in the neighborhood of supersingular elliptic curves. According to this theorem, the deformations of a supersingular elliptic curve are equivalent to the deformations of its associated formal group. The formal neighborhood of a point $\text{Spec}(k) \to \mathcal{M}_{ell}$ classifying a supersingular elliptic curve is therefore isomorphic to $\text{Spf}(\mathbb{W}(k)[[v_1]])$, the formal spectrum of the universal deformation ring.

Chapter 8: Goerss–Hopkins obstruction theory. Goerss–Hopkins obstruction theory is a technical apparatus for approaching questions such as the following: for a ring spectrum E and a commutative E_*-algebra A in E_*E-comodules, is there an E_∞-ring spectrum X such that E_*X is equivalent to A? What is the homotopy type of the space of all such E_∞-ring spectra X?

That space is called the realization space of A and is denoted $B\mathcal{R}(A)$. There is an obstruction theory for specifying points of $B\mathcal{R}(A)$, and the obstructions live in certain André–Quillen cohomology groups of A. More precisely, there is a Postnikov-type tower

$$\ldots \to B\mathcal{R}_n(A) \to B\mathcal{R}_{n-1}(A) \to \ldots \to B\mathcal{R}_0(A)$$

with inverse limit $B\mathcal{R}(A)$ whose layers are controlled by the André–Quillen cohomology groups of A, as follows. If we let $\mathcal{H}^{n+2}(A;\Omega^n A)$ be the André–Quillen cohomology space (the Eilenberg–MacLane space for the André–Quillen cohomology group) of the algebra A with coefficients in the nth desuspension of A, then $\mathcal{H}^{n+2}(A;\Omega^n A)$ is acted on by the automorphism group of the pair $(A,\Omega^n A)$ and we can form, by the Borel construction, a space $\widehat{\mathcal{H}}^{n+2}(A;\Omega^n A)$ over the classifying space of $Aut(A,\Omega^n A)$. This is a bundle of pointed spaces and the base points provide a section $BAut(A,\Omega^n A) \to \widehat{\mathcal{H}}^{n+2}(A;\Omega^n A)$. The spaces $B\mathcal{R}_n(A)$ then fit into homotopy pullback squares

$$\begin{array}{ccc} B\mathcal{R}_n(A) & \longrightarrow & BAut(A,\Omega^n A) \\ \downarrow & & \downarrow \\ B\mathcal{R}_{n-1}(A) & \longrightarrow & \widehat{\mathcal{H}}^{n+2}(A;\Omega^n A). \end{array}$$

Chapter 9: From spectra to stacks. We have focussed on constructing spectra using stacks, but one can also go the other way, associating stacks to spectra. Given a commutative ring spectrum X, let $\mathcal{M}_X$ be the stack associated to the Hopf algebroid

$$(MU_*X, MU_*MU \otimes_{MU_*} MU_*X).$$

If X is complex orientable, then $\mathcal{M}_X$ is the scheme $\mathrm{Spec}(\pi_*X)$—the stackiness of $\mathcal{M}_X$ therefore measures the failure of complex orientability of X. The canonical Hopf algebroid map $(MU_*, MU_*MU) \to (MU_*X, MU_*MU \otimes_{MU_*} MU_*X)$ induces a map of stacks from $\mathcal{M}_X$ to $\mathcal{M}^{(1)}_{FG}$, the moduli stack of formal groups with first order coordinate. Moreover, under good circumstances, the stack associated to a smash product of two ring spectra is the fiber product over $\mathcal{M}^{(1)}_{FG}$:

$$\mathcal{M}_{X\wedge Y} \cong \mathcal{M}_X \times_{\mathcal{M}^{(1)}_{FG}} \mathcal{M}_Y.$$

It will be instructive to apply the above isomorphism to the case when Y is *tmf*, and X is one of the spectra in a filtration

$$\mathbb{S}^0 = X(1) \to X(2) \to \cdots X(n) \to \cdots \to MU$$

of the complex cobordism spectrum. By definition, $X(n)$ is the Thom spectrum associated to the subspace $\Omega SU(n)$ of $\Omega SU \simeq BU$; the spectrum $X(n)$ is an E_2-ring spectrum because $\Omega SU(n)$ is a double loop space. Recall that for a complex orientable theory R, multiplicative maps $MU \to R$ correspond to coordinates on the formal group of R. There is a similar story with $X(n)$ in place of MU, where the formal groups are now only defined modulo terms of degree $n+1$, and multiplicative maps $X(n) \to R$ correspond to coordinates up to degree n. Using this description, one can show that $\mathcal{M}_{X(n)}$ is the stack $\mathcal{M}_{FG}^{(n)}$, the classifying stack of formal groups with a coordinate up to degree n. The map from $\mathcal{M}_{FG}^{(n)}$ to $\mathcal{M}_{FG}^{(1)}$ is the obvious forgetful map.

The stack $\mathcal{M}_{tmf}$ associated to *tmf* is the moduli stack of generalized elliptic curves (both multiplicative and additive degenerations allowed) with first order coordinate. The stack $\mathcal{M}_{X(4)\wedge tmf}$ can therefore be identified with the moduli stack of elliptic curves together with a coordinate up to degree 4. The pair of an elliptic curve and such a coordinate identifies a Weierstrass equation for the curve, and so this stack is in fact a scheme:

$$\mathcal{M}_{X(4)\wedge tmf} \cong \operatorname{Spec} \mathbb{Z}[a_1, a_2, a_3, a_4, a_6].$$

Here, a_1, a_2, a_3, a_4, a_6 are the coefficients of the universal Weierstrass equation. By considering the products $X(4) \wedge \ldots \wedge X(4) \wedge tmf$, one can furthermore identify the whole $X(4)$-based Adams resolution of *tmf* with the cobar resolution of the Weierstrass Hopf algebroid.

Chapter 10: The string orientation. The string orientation, or σ-orientation of *tmf* is a map of E_∞-ring spectra

$$MO\langle 8\rangle \to tmf.$$

Here, $MO\langle 8\rangle = MString$ is the Thom spectrum of the 7-connected cover of BO, and its homotopy groups are the cobordism groups of string manifolds (manifolds with a chosen lift to $BO\langle 8\rangle$ of their tangent bundle's classifying map). At the level of homotopy groups, the map $MO\langle 8\rangle \to tmf$ is the Witten genus, a homomorphism $[M] \mapsto \phi_W(M)$ from the string cobordism ring to the ring of integral modular forms. Note that $\phi_W(M)$ being an element of $\pi_*(tmf)$ instead of a mere modular form provides interesting congruences, not visible from the original definition of the Witten genus.

Even before having a proof, there are hints that the σ-orientation should exist. The Steenrod algebra module $H^*(tmf, \mathbb{F}_2) = A/\!/A(2)$ occurs as a summand in $H^*(MString, \mathbb{F}_2)$. This is reminiscent of the situation with the Atiyah–Bott–Shapiro orientation $MSpin \to ko$, where $H^*(ko, \mathbb{F}_2) = A/\!/A(1)$ occurs as a summand of $H^*(MSpin, \mathbb{F}_2)$.

Another hint is that, for any complex oriented cohomology theory E with associated formal group G, multiplicative (not E_∞) maps $MO\langle 8\rangle \to E$ correspond to sections of a line bundle over G^3 subject to a certain cocycle condition. If G is the completion of an elliptic curve C, then that line bundle is naturally the restriction of a bundle over C^3. That bundle is trivial, and because C^3 is proper, its space of sections is one dimensional (and there is even a preferred section). Thus, there is a preferred map $MO\langle 8\rangle \to E$ for every elliptic spectrum E.

A not-necessarily E_∞ orientation $MO\langle 8\rangle \to tmf$ is the same thing as a nullhomotopy of the composite

$$BO\langle 8\rangle \to BO \xrightarrow{J} BGL_1(\mathbb{S}) \to BGL_1(tmf),$$

where $BGL_1(R)$ is the classifying space for rank one R-modules. An E_∞ orientation $MO\langle 8\rangle \to tmf$ is a nullhomotopy of the corresponding map of spectra

$$bo\langle 8\rangle \to bo \xrightarrow{J} \Sigma gl_1(\mathbb{S}) \to \Sigma gl_1(tmf).$$

In order to construct that nullhomotopy, one needs to understand the homotopy type of $gl_1(tmf)$—this is done one prime at a time. The crucial observation is that there is a map of spectra $gl_1(tmf)^{\wedge}_p \to tmf^{\wedge}_p$, the 'topological logarithm', and a homotopy pullback square

$$\begin{array}{ccc} gl_1(tmf) & \xrightarrow{\log} & tmf \\ \downarrow & & \downarrow \\ L_{K(1)}(tmf) & \xrightarrow{1-U_p} & L_{K(1)}(tmf) \end{array}$$

where U_p is a topological refinement of Atkin's operator on p-adic modular forms. The fiber of the topological logarithm is particularly intriguing: Hopkins speculates that it is related to exotic smooth structures on free loop spaces of spheres.

Chapters 11 and 12: The sheaf of E_∞ ring spectra and **The construction of *tmf*.** We outline a roadmap for the construction of *tmf*, the connective spectrum of topological modular forms. The major steps in the construction are given in reverse order.

- The spectrum *tmf* is the connective cover of the nonconnective spectrum *Tmf*,

$$tmf := \tau_{\geq 0} Tmf,$$

and *Tmf* is the global sections of a sheaf of spectra,

$$Tmf := \mathcal{O}^{top}(\overline{\mathcal{M}}_{ell}),$$

where $\overline{\mathcal{M}}_{ell}$ is the moduli stack of elliptic curves with possibly nodal singularities. This stack is the Deligne–Mumford compactification of the moduli stack of smooth elliptic curves. Here, $\mathcal{O}^{top}$ is a sheaf on $\overline{\mathcal{M}}_{ell}$ in the étale topology. Also, *TMF* is the global sections of $\mathcal{O}^{top}$ over the substack $\mathcal{M}_{ell}$ of smooth elliptic curves,

$$TMF := \mathcal{O}^{top}(\mathcal{M}_{ell}).$$

The uppercase '*T*' in *Tmf* signifies that the spectrum is no longer connective (but it is also not periodic). The '*top*' stands for topological, and $\mathcal{O}^{top}$ can be viewed as a kind of structure sheaf for a spectral version of $\overline{\mathcal{M}}_{ell}$.

We are left now to construct the sheaf of spectra $\mathcal{O}^{top}$. The first step is to isolate the problem at every prime p and at $\mathbb{Q}$. That is, one constructs $\mathcal{O}^{top}_p$, a sheaf of spectra on the p-completion $(\overline{\mathcal{M}}_{ell})_p$ and then pushes this sheaf forward along the inclusion map $\iota_p : (\overline{\mathcal{M}}_{ell})_p \to \overline{\mathcal{M}}_{ell}$. One then assembles these pushforwards to obtain $\mathcal{O}^{top}$, as follows.

- The sheaf $\mathcal{O}^{top}$ is the limit in a diagram

$$\begin{array}{ccc} \mathcal{O}^{top} & \longrightarrow & \prod_p \iota_{p,*}\mathcal{O}^{top}_p \\ \downarrow & & \downarrow \\ \iota_{\mathbb{Q},*}\mathcal{O}^{top}_{\mathbb{Q}} & \longrightarrow & \left(\prod_p \iota_{p,*}\mathcal{O}^{top}_p\right)_{\mathbb{Q}} \end{array}$$

for a given choice of map $\alpha_{arith} : \iota_{\mathbb{Q},*}\mathcal{O}^{top}_{\mathbb{Q}} \to \left(\prod_p \iota_{p,*}\mathcal{O}^{top}_p\right)_{\mathbb{Q}}$.

Once $\mathcal{O}^{top}$ has been constructed, it will turn out that $\mathcal{O}^{top}_p$ is the p-completion of $\mathcal{O}^{top}$, and $\mathcal{O}^{top}_{\mathbb{Q}}$ is its rationalization, so that the above diagram is the arithmetic square for $\mathcal{O}^{top}$. This thus leaves one to construct each $\mathcal{O}^{top}_p$ and $\mathcal{O}^{top}_{\mathbb{Q}}$ and the gluing map α_{arith}. The sheaf $\mathcal{O}^{top}_{\mathbb{Q}}$ is not difficult to construct. Its value on an étale map $\mathrm{Spec}(R) \to (\mathcal{M}_{ell})_{\mathbb{Q}}$ is given by $\mathcal{O}^{top}_{\mathbb{Q}}(\mathrm{Spec}\, R) = H(R_*)$, the rational Eilenberg–MacLane spectrum associated to a certain evenly graded ring R_*. This ring is specified by $R_{2t} := \Gamma(\omega^{\otimes t}|_{\mathrm{Spec}\, R})$, where ω is the sheaf of invariant differentials.

The construction of $\mathcal{O}^{top}_p$ is more subtle. The first step in its construction is to employ a natural stratification of $(\overline{\mathcal{M}}_{ell})_p$. Each elliptic curve has an associated formal group which either has height equal to 1 if the curve is *ordinary*, or equal to 2 if the curve is *supersingular*. This gives a stratification of the moduli space with exactly two strata:

$$\mathcal{M}^{ord}_{ell} \xrightarrow{\iota_{ord}} (\overline{\mathcal{M}}_{ell})_p \xleftarrow{\iota_{ss}} \mathcal{M}^{ss}_{ell}.$$

The sheaf $\mathcal{O}^{top}_p$ is presented by a Hasse square, gluing together a sheaf $\mathcal{O}^{top}_{K(1)}$ on $\mathcal{M}^{ord}_{ell}$ and a sheaf $\mathcal{O}^{top}_{K(2)}$ on $\mathcal{M}^{ss}_{ell}$. (This notation is used because the sheaves $\mathcal{O}^{top}_{K(i)}$ are also the $K(i)$-localizations of $\mathcal{O}^{top}$, where $K(i)$ is the ith Morava K-theory at the prime p.)

- $\mathcal{O}^{top}_p$ is the limit

$$\begin{array}{ccc} \mathcal{O}^{top}_p & \longrightarrow & \iota_{ss,*}\mathcal{O}^{top}_{K(2)} \\ \downarrow & & \downarrow \\ \iota_{ord,*}\mathcal{O}^{top}_{K(1)} & \longrightarrow & \left(\iota_{ss,*}\mathcal{O}^{top}_{K(2)}\right)_{K(1)} \end{array}$$

  for a certain 'chromatic' attaching map

$$\alpha_{chrom} : \iota_{ord,*}\mathcal{O}^{top}_{K(1)} \longrightarrow \left(\iota_{ss,*}\mathcal{O}^{top}_{K(2)}\right)_{K(1)}.$$

The sheaf $\mathcal{O}^{top}_p$ is thus equivalent to the following triple of data: a sheaf $\mathcal{O}^{top}_{K(1)}$ on $\mathcal{M}^{ord}_{ell}$, a sheaf $\mathcal{O}^{top}_{K(2)}$ on $\mathcal{M}^{ss}_{ell}$, and a gluing map α_{chrom} as above. We have now arrived at the core of the construction of *tmf*: the construction of these three objects. This construction proceeds via Goerss–Hopkins obstruction theory.

That obstruction theory is an approach to solving the following problem: one wants to determine the space of all E_∞-ring spectra subject to some conditions, such as having prescribed homology. More specifically, for any generalized homology theory E_*, and any choice of E_*-algebra A in E_*E-comodules, one can calculate the homotopy type of the moduli space of E_∞-ring spectra with E_*-homology isomorphic to A. Goerss and Hopkins describe that moduli space as the homotopy limit of a sequence of spaces, where the homotopy fibers are certain André–Quillen cohomology spaces of A. As a consequence, there is a sequence of obstructions to specifying a point of the moduli space, i.e., an E_∞-ring spectrum whose E_*-homology is A. The obstructions lie in André–Quillen cohomology groups of E_*-algebras in E_*E-comodules. That obstruction theory is used to build the sheaf $\mathcal{O}^{top}_{K(2)}$.

There is also a 'global' version of this obstruction theory, where one tries to lift a whole diagram I of E_*-algebras in E_*E-comodules to the category of E_∞-ring spectra. Here, in general, the obstructions live in the Hochschild–Mitchell cohomology group of the diagram I with coefficients in André–Quillen cohomology. This diagrammatic enhancement of the obstruction theory is used to build the sheaf $\mathcal{O}^{top}_{K(1)}$.

Obstruction theory for $\mathcal{O}^{top}_{K(2)}$: The stack $\mathcal{M}^{ss}_{ell}$ is a 0-dimensional substack of $\overline{\mathcal{M}}_{ell}$. More precisely, it is the disjoint union of classifying stacks BG where G ranges over the automorphism groups of the various supersingular elliptic curves. The Serre–Tate theorem identifies the formal completion of these groups G with the automorphism groups of the associated formal group. Consequently, to construct the sheaf $\mathcal{O}^{top}_{K(2)}$ on the category of étale affines mapping to $\mathcal{M}^{ss}_{ell}$, it suffices to construct the stalks of the sheaf at each point of $\mathcal{M}^{ss}_{ell}$, together with the action of these automorphism groups. The spectrum associated to a stalk is a Morava E-theory, the uniqueness of which is the Goerss–Hopkins–Miller theorem: that theorem says that there is an essentially unique (unique up to a contractible space of choices) way to construct an E_∞-ring spectrum $E(k, \mathbb{G})$, from a pair $(k, \mathbb{G})$ of a formal group $\mathbb{G}$ of finite height over a perfect field k, whose underlying homology theory is the Landweber exact homology theory associated to $(k, \mathbb{G})$. Altogether then, given a formal affine scheme $\mathrm{Spf}(R)$, with maximal ideal $I \subset R$, and an étale map $\mathrm{Spf}(R) \to \mathcal{M}^{ss}_{ell}$ classifying an elliptic curve C over R, the value of the sheaf $\mathcal{O}^{top}_{K(2)}$ is

$$\mathcal{O}^{top}_{K(2)}(\mathrm{Spf}(R)) := \prod_i E(k_i, \widehat{C}^{(i)}_0).$$

In this formula, the product is indexed by the set i in the expression of the quotient $R/I = \prod_i k_i$ as a product of perfect fields, and $\widehat{C}^i_0$ is the formal group associated to the base change to k_i of the elliptic curve C_0 over R/I.

Obstruction theory for $\mathcal{O}^{top}_{K(1)}$: We first explain the approach described in Chapter 11. Over the stack $\mathcal{M}^{ord}_{ell}$, there is a presheaf of homology theories given by the Landweber exact functor theorem. This presheaf assigns to an elliptic curve classified by an étale map $\mathrm{Spec}(R) \to \mathcal{M}^{ord}_{ell}$ the homology theory $X \mapsto BP_*(X) \otimes_{BP_*} R$. Ordinary elliptic curves have height 1, and so the representing spectrum is $K(1)$-local. In the setup of the Goerss–Hopkins obstruction theory for this situation, we take E_* to be p-adic K-theory, which has the structure of a *θ-algebra*. The moduli problem that we are trying to solve is that of determining the space of all lifts:

$$\begin{array}{ccc} & & E_\infty\text{-rings}_{K(1)} \\ & \nearrow & \big\downarrow {\scriptstyle K_p^\wedge} \\ I := \left(\mathit{Aff}/\mathcal{M}^{ord}_{ell}\right)_{\acute{e}t} & \longrightarrow & \mathit{Alg}_\theta. \end{array}$$

In the general obstruction theory, the obstructions live in certain Hochschild–Mitchell cohomology groups of the diagram I. For this particular diagram, the obstruction groups simplify, and are equivalent to just diagram cohomology of I with coefficients in André–Quillen cohomology. The diagram cohomology of I is in turn isomorphic to the étale cohomology of the stack $\mathcal{M}_{ell}^{ord}$. In the end, the essential calculation is

$$H^s(\mathcal{M}_{ell}^{ord}, \omega^{\otimes k}) = 0 \qquad \text{for} \quad s > 0,$$

where $k \in \mathbb{Z}$ and ω is the line bundle of invariant differentials on $\mathcal{M}_{ell}$. At odd primes, the obstruction groups vanish in the relevant degrees, thus proving the existence and uniqueness of $\mathcal{O}^{top}_{K(1)}$. Unfortunately, the higher homotopy groups of the space of lifts are not all zero, and so one doesn't get a contractible space of choices for the sheaf $\mathcal{O}^{top}_{K(1)}$. At the prime $p = 2$, one needs to use real instead of complex K-theory to get obstruction groups that vanish.

The same obstruction theory for E_∞-ring spectra also applies to E_∞-ring maps, such as the gluing maps α_{chrom} and α_{arith} for the Hasse square and the arithmetic square. For α_{chrom}, one considers the moduli space of all E_∞-ring maps $\iota_{ord,*}\mathcal{O}^{top}_{K(1)} \to \iota_{ss,*}(\mathcal{O}^{top}_{K(2)})_{K(1)}$ whose induced map of theta-algebras is prescribed, and one tries to compute the homotopy groups of this moduli space. The obstruction groups here vanish, and there is an essentially unique map. For α_{arith}, the map is of rational spectra and the analysis is much easier; the obstruction groups vanish, and again there is an essentially unique map.

The approach presented in Chapter 12 is a somewhat different way of constructing $\mathcal{O}^{top}_{K(1)}$. In this approach, one directly applies the $K(1)$-local obstruction theory to construct $L_{K(1)}tmf$, and then works backwards to construct $\mathcal{O}^{top}$. That approach allows one to avoid the obstruction theory for diagrams, but is more difficult in other steps—for instance, it requires use of level structures on the moduli stack $\overline{\mathcal{M}}_{ell}$ to resolve the obstructions.

<u>Chapter 13</u>: The homotopy groups of *tmf* and of its localizations. The homotopy groups of *tmf* are an elaborate amalgam of the classical ring of modular forms MF_* and certain pieces of the 2- and 3-primary part of the stable homotopy groups of spheres $\pi_*(\mathbb{S})$.

There are two homomorphisms

$$\pi_*(\mathbb{S}) \to \pi_*(tmf) \to MF_*.$$

The first map is the Hurewicz homomorphism, and it is an isomorphism on π_0 through π_6. Conjecturally, this map hits almost all of the interesting torsion classes in $\pi_*(tmf)$ and its image (except for the classes η, η^2, and ν) is periodic with period 576 (arising from a 192-fold periodicity at the prime 2 and a 72-fold periodicity at the prime 3). Among others, the map is nontrivial on the 3-primary stable homotopy classes $\alpha \in \pi_3(\mathbb{S})$ and $\beta \in \pi_{10}(\mathbb{S})$ and the 2-primary stable homotopy classes $\eta, \nu, \epsilon, \kappa, \overline{\kappa}, q \in \pi_*(\mathbb{S})$. The second map in the above display is the composite of the inclusion $\pi_* tmf \to \pi_* Tmf$ with the boundary homomorphism in the elliptic spectral sequence

$$H^s(\overline{\mathcal{M}}_{ell}; \pi_t\mathcal{O}^{\text{top}}) \;\Rightarrow\; \pi_{t-s}(Tmf).$$

This map $\pi_*(tmf) \to MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - (12)^3\Delta)$ is an isomorphism after inverting the primes 2 and 3. The kernel of this map is exactly the torsion in $\pi_*(tmf)$ and the cokernel is a cyclic group of order dividing 24 in degrees divisible by 24, along with some number of cyclic groups of order 2 in degrees congruent to 4 mod 8. In particular, the map from $\pi_*(tmf)$ hits the modular forms c_4, $2c_6$, and 24Δ, but c_6 and Δ themselves are not in the image. The localization $\pi_*(tmf)_{(p)}$ at any prime larger than 3 is isomorphic to $(MF_*)_{(p)} \cong \mathbb{Z}_{(p)}[c_4, c_6]$.

The homotopy of *tmf* can be computed directly using the Adams spectral sequence. Alternatively, one can use the elliptic spectral sequence to compute the homotopy of *Tmf*. The Adams spectral sequence has the form

$$E_2 = \mathrm{Ext}_{A_p^{tmf}}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \pi_*(tmf)^{\hat{}}_p,$$

where $A_p^{tmf} := \hom_{tmf-\mathrm{modules}}(H\mathbb{F}_p, H\mathbb{F}_p)$ is a *tmf*-analog of the Steenrod algebra. At the prime 2, the map $A_2^{tmf} \to A \equiv A_2$ to the classical Steenrod algebra is injective, and the *tmf*-module Adams spectral sequence can be identified with the classical Adams spectral sequence

$$E_2 = \mathrm{Ext}_A(H^*(tmf), \mathbb{F}_2) = \mathrm{Ext}_A(A/\!/A(2), \mathbb{F}_2) = \mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*(tmf)^{\hat{}}_2.$$

The elliptic spectral sequence has the form $H^s(\overline{\mathcal{M}}_{ell}, \pi_t\mathcal{O}^{\mathrm{top}}) \Rightarrow \pi_{t-s}(Tmf)$. The homotopy $\pi_t\mathcal{O}^{\mathrm{top}}$ is concentrated in even degrees and is the $t/2$-th power of a line bundle ω; the spectral sequence thus has the form $H^q(\overline{\mathcal{M}}_{ell}; \omega^{\otimes p}) \Rightarrow \pi_{2p-q}(Tmf)$.

Part II

The manuscripts. The book concludes with three of the original, previously unpublished, manuscripts on *tmf*: “Elliptic curves and stable homotopy I” (1996) by Hopkins and Miller, “From elliptic curves to homotopy theory” (1998) by Hopkins and Mahowald, and “$K(1)$-local E_∞ ring spectra” (1998) by Hopkins. The first focuses primarily on the construction of the sheaf of (associative) ring spectra on the moduli stack of elliptic curves, the second on the computation of the homotopy of the resulting spectrum of sections around the supersingular elliptic curve at the prime 2, and the third on a direct cellular construction of the $K(1)$-localization of *tmf*. These documents have been left, for the most part, in their original draft form; they retain the attendant roughness and sometimes substantive loose ends, but also the dense, heady insight of their original composition. The preceding chapters of this book can be viewed as a communal exposition, more than fifteen years on, of aspects of these and other primary sources about *tmf*.

4. Reader’s guide

This is not a textbook. Though the contents spans all the way from classical aspects of elliptic cohomology to the construction of *tmf*, there are substantive gaps of both exposition and content, and an attempt to use this book for a lecture, seminar, or reading course will require thoughtful supplementation.

Reading straight through the book would require, among much else, some familiarity and comfort with *commutative ring spectra*, *stacks*, and *spectral sequences*. Many of the chapters, though, presume knowledge of none of these topics; instead of thinking of them as prerequisites, we suggest one simply starts reading, and as appropriate or necessary selects from among the following as companion sources:

Commutative ring spectra:

- May, J. Peter. *E_∞ ring spaces and E_∞ ring spectra.* With contributions by Frank Quinn, Nigel Ray, and Jorgen Tornehave. Lecture Notes in Mathematics, Vol. 577. Springer-Verlag, Berlin-New York, 1977.

- Rezk, Charles. *Notes on the Hopkins–Miller theorem.* Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), 313–366, Contemp. Math., 220, Amer. Math. Soc., Providence, RI, 1998.
- Schwede, Stefan. *Book project about symmetric spectra.* Book preprint. Available at http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf

Stacks:

- *Complex oriented cohomology theories and the language of stacks.* Course notes for 18.917, taught by Mike Hopkins (1999), available at http://www.math.rochester.edu/people/faculty/doug/otherpapers/coctalos.pdf
- Naumann, Niko. *The stack of formal groups in stable homotopy theory.* Adv. Math. 215 (2007), no. 2, 569–600.
- *The stacks project.* Open source textbook, available at http://stacks.math.columbia.edu
- Vistoli, Angelo. *Grothendieck topologies, fibered categories and descent theory.* Fundamental algebraic geometry, 1–104, Math. Surveys Monogr., 123, Amer. Math. Soc., Providence, RI, 2005.

Spectral sequences:

- Hatcher, Allen. *Spectral sequences in algebraic topology.* Book preprint. Available at http://www.math.cornell.edu/~hatcher/SSAT/SSATpage.html
- McCleary, John. *A user's guide to spectral sequences.* Second edition. Cambridge Studies in Advanced Mathematics, 58. Cambridge University Press, Cambridge, 2001. xvi+561 pp. ISBN: 0-521-56759-9
- Weibel, Charles A. *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics, 38. Cambridge University Press, Cambridge, 1994. xiv+450 pp. ISBN: 0-521-43500-5; 0-521-55987-1

The contents of this book span four levels: the first five chapters (elliptic cohomology, elliptic curves, the moduli stack, the exact functor theorem, sheaves in homotopy theory) are more elementary, classical, and expository and we hope will be tractable for all readers and instructive or at least entertaining for all but the experts; the next three chapters (the Hasse square, the local structure of the moduli stack, obstruction theory) are somewhat more sophisticated in both content and tone, and especially for novice and intermediate readers will require more determination, patience, and willingness to repeatedly pause and read other references before proceeding; the last five chapters (from spectra to stacks, string orientation, the sheaf of ring spectra, the construction, the homotopy groups) are distinctly yet more advanced, with Mike Hopkins' reflective account of and perspective on the subject, followed by an extensive technical treatment of the construction and homotopy of *tmf*; finally the three classic manuscripts (Hopkins–Miller, Hopkins–Mahowald, Hopkins) illuminate the original viewpoint on *tmf*—a careful reading of them will require serious dedication even from experts.

In addition to the references listed above, we encourage the reader to consult the following sources about *tmf* more broadly:

- Goerss, Paul. *Topological modular forms [after Hopkins, Miller, and Lurie].* Séminaire Bourbaki (2008/2009). Astérisque No. 332 (2010), 221-255.
- Hopkins, Michael. *Topological modular forms, the Witten genus, and the theorem of the cube.* Proceedings of the International Congress of Mathematicians (Zurich 1994), 554-565, Birkhäuser, Basel, 1995.
- Hopkins, Michael. *Algebraic topology and modular forms.* Proceedings of the International Congress of Mathematicians (Beijing 2002), 291–317, Higher Ed. Press, Beijing, 2002.
- Rezk, Charles. *Supplementary notes for Math 512.* Available at http://www.math.uiuc.edu/~rezk/512-spr2001-notes.pdf