Preface

The purpose of this monograph is to introduce and study a unifying object we call a generalized PROP, which includes the colored version of an operad, a PROP, a wheeled PROP, or any variant as a special case. Before we describe the topics discussed in this monograph, let us briefly review operads and PROPs.

Operads are an efficient machinery for organizing operations and the relations between them. Operads were introduced in homotopy theory by May [**May72**] to describe spaces with the weak homotopy types of iterated loop spaces. An earlier motivating example for the concept of an operad was Stasheff's A_∞-spaces [**Sta63**].

Briefly, an operad O has objects $\mathsf{O}(n)$ for $n \geq 0$ and structure maps

$$\gamma\colon \mathsf{O}(n) \otimes \mathsf{O}(k_1) \otimes \cdots \otimes \mathsf{O}(k_n) \longrightarrow \mathsf{O}(k_1 + \cdots + k_n)$$

that satisfy some associativity, equivariance, and unity conditions. The prototypical example of an operad is called the endomorphism operad of an object A with

$$\mathsf{E}_A(n) = \mathrm{Hom}(A^{\otimes n}, A).$$

Here the elements are normally called n-ary operations, and the structure map γ comes from using the outputs of choices of k_j-ary operations as the inputs of an n-ary operation, thereby producing a single operation with $\Sigma_j k_j$ inputs. As expected, an operad map $\mathsf{O} \longrightarrow \mathsf{E}_A$ then has entries $\mathsf{O}(n) \longrightarrow \mathsf{E}_A(n)$ compatible with the structure maps, and in this way $\mathsf{O}(n)$ can be used to parametrize n-ary operations on A. As a consequence, this special case of a map into the endomorphism operad of A earns the name of an O-algebra structure on A. In addition, relations described in terms of the structure map of O must also remain present among the families of operations in the image of such a map, due to the compatibility of a morphism $\mathsf{O} \longrightarrow \mathsf{E}_A$ with structure maps γ.

There are many important uses of operads in homotopy theory and algebra, so we will mention a few of them. Besides the study of iterated loop spaces, operads are used in the algebraic classification of homotopy types [**Man06, Smi82, Smi01**]. The singular cochain complex of a space is an E_∞-algebra, which is a homotopy version of a commutative algebra. For certain nice spaces, this E_∞-algebra determines the weak homotopy type. Another operadic link between topology and algebra is the solution of Deligne's Conjecture [**Kau07, MS02**]. It says that the Hochschild cochain complex of an associative algebra is an algebra over a suitable chain version of May's little 2-cubes operad. Furthermore, Stasheff's work on homotopy associative H-spaces can be generalized to other homotopy invariant structures. Given any reasonably nice operad O, Boardman and Vogt [**BV73**] constructed an operad $W\mathsf{O}$ that is weakly equivalent to O such that $W\mathsf{O}$-algebras are homotopy invariant. Other applications of operads are discussed in [**KM95, Mar08, MSS02, Smi01**].

In many algebraic situations, one encounters not only n-ary operations but also operations with multiple inputs and multiple outputs. The simplest example is a bialgebra, which has a multiplication and a comultiplication. PROPs are a machinery that can be used similarly to organize operations with multiple inputs and multiple outputs. PROPs were introduced by Mac Lane [**Mac63, Mac65**] to describe the structure on the iterated bar constructions on a commutative differential graded Hopf algebra. Briefly, a PROP P has objects

$$\mathsf{P}\binom{n}{m}$$

for $n, m \geq 0$ and structure maps

$$\mathsf{P}\binom{n}{m} \otimes \mathsf{P}\binom{q}{p} \longrightarrow \mathsf{P}\binom{n+q}{m+p} \quad \text{(horizontal composition)}$$

and

$$\mathsf{P}\binom{n}{m} \otimes \mathsf{P}\binom{m}{l} \longrightarrow \mathsf{P}\binom{n}{l} \quad \text{(vertical composition)}$$

that satisfy some associativity, bi-equivariance, unity, and compatibility conditions. The prototypical example here is the endomorphism PROP of an object A with

$$\mathsf{E}_A\binom{n}{m} = \operatorname{Hom}(A^{\otimes m}, A^{\otimes n}).$$

As above, the object $\mathsf{P}\binom{n}{m}$ parametrizes operations with m inputs and n outputs via an entry $\mathsf{P}\binom{n}{m} \longrightarrow \mathsf{E}_A\binom{n}{m}$ of a map $\mathsf{P} \longrightarrow \mathsf{E}_A$, so such a map is again called a P-algebra structure on A.

There are numerous applications in mathematics and physics of PROPs and variants, such as the smaller half-PROPs and properads and the bigger wheeled PROPs. For example, these objects are used prominently in deformation theory [**FMY09, MV09**], graph cohomology [**MV09, Mer09**], homotopy invariant structures [**JY09**], Batalin-Vilkovisky structures [**Mer10a**], the Master Equation [**MMS09**], deformation quantization [**Mer08, Mer10b**], Poisson structures [**Str10**], string topology [**Cha05, CG04, CV06**], and field theories [**JY09, Ion07, Seg01, Seg04**].

There are close relationships between operads and PROPs. Their definitions are formally similar to each other, and their algebras are both given by morphisms into the endomorphism objects. In fact, every PROP P has an underlying operad with

$$\mathsf{P}(m) = \mathsf{P}\binom{1}{m}.$$

Conversely, every operad O generates a PROP O' such that O-algebras are exactly O'-algebras [**BV73**]. There is a conceptual description of an operad as a monoid in the monoidal category of Σ-modules [**May97**]. There is a conceptually similar, but more complicated, description of a PROP as a 2-monoid in the category of Σ-bimodules [**JY09**]. Furthermore, the homotopy theory of PROPs is, in a precise sense, a homotopy refinement of the homotopy theory of operads [**JY09**].

In general, PROPs and wheeled PROPs are harder to deal with than operads, because they are much bigger. The operations in an operad are parametrized by level trees, which have nice combinatorial properties that allow one to do induction on the internal edges. For example, the Boardman-Vogt W-construction [**BV73, BM06, BM07, Vog03**] for an operad uses level trees in an essential way. On the other hand, the operations in a PROP are parametrized by directed cycle-free graphs, which may have multiple connected components. There are many more such graphs than level trees. Moreover, induction on the internal edges in directed cycle-free graphs is usually not possible because there can be many edges between two vertices, and one cannot generally shrink away an internal edge without the chance of producing a cycle in the resulting graph. Going even further, the operations in a wheeled PROP are parametrized by directed graphs, which may have multiple connected components along with directed cycles and loops.

As discussed by Markl in [**Mar08**], operads, PROPs, and wheeled PROPs can all be described using collections of graphs he called pasting schemes, in these cases consisting of the level trees, the directed cycle-free graphs, and the directed graphs, respectively. Since the definition of a pasting scheme was left ambiguous there, part of our aim in the first half of this monograph is to make precise this notion of pasting scheme. The main motivation is that there should be a variant of PROPs associated to any reasonable pasting scheme. By choosing the right pasting schemes, one can obtain colored versions of (wheeled) operads, (wheeled) properads, (wheeled) PROPs, dioperads, and half-PROPs, among others. Unfortunately, the related cyclic and modular operads would require more cumbersome versions of the underlying graph theory, so we have elected not to complicate our approach throughout in order to include those structures, which are mentioned only at the very end of the first chapter.

In this monograph, we introduce and study this unifying object, called a generalized PROP. This monograph is divided into two parts. The first part describes the theory of pasting schemes in careful detail, which requires a new definition of graph, a new description of graph substitution, a careful description of graph operations, a theory of generating sets for graph groupoids, and notions of intersections and free products of graph groupoids. This part is somewhat technical, but the point is to reduce the technical issues in the subsequent theory to the underlying questions about graphs by taking all aspects of the theory of pasting schemes seriously. The second part of this monograph contains categorical properties of generalized PROPs along with their algebras and modules. In this second part, we work over an arbitrary symmetric monoidal (closed) category with enough limits and colimits. In future work, we plan to investigate questions related to the homotopy theory of all of these objects, including constructive approaches to cofibrant replacements, where possible. The graph theory built up in the first part of this monograph and the equivalence established here between the biased and unbiased versions of (wheeled) properads are also used in [**HRY**] to develop a theory of higher (wheeled) properads.

Several other projects have worked to provide a unifying view of a variety of operational structures, (e.g., [**Get09**], [**KW**], or [**BM14**]), while [**BB**, Subsec. 15.4] even does so using a version of graphs which they show to be equivalent to our presentation here. We hope the present monograph will serve as a fully detailed reference for at least one such unifying approach. A brief description of each chapter follows.

The first chapter introduces the new definition of a wheeled graph. First, a basic graph consists of a partitioned finite set of flags equipped with an involution, so the partitions correspond to vertices, the flags correspond to half-edges, and the involution pairs two half-edges together to form edges. Flags fixed by the involution are called legs and represent either inputs or outputs of the whole graph. Unfortunately, some additional structure must also be included to deal properly with exceptional graphs, which contain no vertices, and so any graph can have an exceptional part. A wheeled graph is then a basic graph together with three extra pieces of structure, called a direction, a coloring, and a listing. The listing is a new feature, introduced so the inputs or outputs of any vertex, or of the full graph, may be expressed as a (finite) ordered sequence of colors, which is vital to defining graph substitution in full detail later. A series of small examples is included to clarify the many definitions.

The second chapter is devoted to understanding the various technical properties which form the distinctions between the pasting schemes of interest. Connected and simply-connected graphs are defined without recourse to any geometric realization, exploiting a careful presentation of the notion of paths in a directed graph. Wheel-free graphs, half-graphs, dioperadic graphs, and several variants of trees, including level trees, simple trees, special trees, and wheeled trees are also discussed, including some pictures.

Some basic graph operations are the topic of the third chapter. In Chapter 6, a more general viewpoint is taken to characterize all graph operations compatible with the fundamental operation of graph substitution, but a few key operations must be introduced much earlier. These include relabeling operations, which shuffle the ordered sequences of inputs and outputs, and a disjoint union operation. There is also a grafting operation where one matches the inputs of one graph with the outputs of another, creating a series of new internal edges as a result. Also included is a partial grafting variant, that can be used to describe the comp-i operations of Gerstenhaber and the related j-comp-i operations. Finally, a contraction operation, which connects two former legs to construct a new internal edge is described, followed by a discussion of invertible graph operations. Along with each of these operations, an example involving a graph with one or two vertices is included, and these graphs will be shown in Chapter 6 to generate the associated operations via the fundamental operation of graph substitution.

In Chapter 4, we present two different notions of isomorphism and describe the main examples of graph groupoids. Since our notion of listing is new, in some instances we want to insist it is preserved by isomorphism, so we define what we call strict isomorphisms, and in other instances we want to relax this constraint, so we define weak isomorphisms. A variety of results concerning both strict automorphism groups and weak automorphism groups of graphs is included, and the strict automorphisms are shown to be quite rigid. For example, the strict automorphism group of any simply-connected graph is trivial.

Chapter 5 is devoted to a careful construction of the fundamental operation of graph substitution, as well as verifying that it is unital, associative, and natural with respect to both types of isomorphisms. The basic idea of substitution is to cut a small hole around any vertex and to insert a shrunken copy of another graph with the same ordered sequence of inputs and outputs as the vertex removed. Unfortunately, the exceptional parts of the graphs inserted cause a variety of technical problems. Thus, we introduce a new object called a pre-graph which is discussed only in this chapter, and building the associated graph of a pre-graph becomes one key technical complication. Forming the substitution pre-graph is relatively straightforward, and we show the process is associative and nearly unital. However, there are several choices for where to apply the associated graph construction, and verifying they all produce the same eventual substitution graph is also a key technical issue, which is necessary in order to verify the associativity of the full graph substitution operation.

With the formal properties of graph substitution now established, the sixth chapter starts by verifying the consequence that any operation compatible with graph substitution can be described as graph substitution into a fixed graph. This includes all of the major operations of Chapter 3, which establishes graph substitution as a strong unifying principle. For example, grafting can be viewed as graph substitution into a graph with two vertices, where the inputs and outputs of these two vertices match those of the two graphs to be grafted. Contraction becomes graph substitution into a one vertex graph with a directed loop, and so on. Also included is verification that graph substitution preserves the key technical properties of being either (simply-)connected or wheel-free. Since these properties are easily established for many of our representing graphs, it follows immediately that many operations preserve these properties as well. The chapter ends with an extensive series of technical lemmas, which we refer to as the calculus of graph substitution. In all cases, there is a result involving graph substitutions with relatively few vertices, and in most cases this is paired with a corresponding statement about the interaction of two graph operations. It is convenient for various purposes later to have these results collected somewhere, and at this point in the presentation, they also serve as a list of examples of graph substitutions computed explicitly. Finally, they set the stage for the Reidemeister theory of graphs and strong generating sets introduced in Chapter 7, as well as the examples of compatible pairs of strong generating sets in Chapter 8.

Chapter 7 is dedicated to determining strong generating sets for all of the major examples of graph groupoids. The consequence later will be that an abstract definition in terms of an algebra over a certain monad associated to a pasting scheme will be reduced to instead requiring a series of structure maps and prescribed relations among them. This material, while also somewhat technical, will apply not only to generalized PROPs but also to the algebras and modules over them. As a consequence, each strong generating set will be used three times in Part 2, and the variety inherent in these methods of decomposing graphs while performing only operations whose generating graphs are included within the pasting scheme itself is quite attractive. The main idea is to introduce an analog of Reidemeister moves from knot theory, so a strong generating set is required to satisfy an analog of Reidemeister's Theorem, connecting any two finite strings of possible graph substitutions with the same composite by a finite string of relaxed moves. We establish strong generating sets for wheeled graphs, wheel-free graphs, level trees, unital trees, wheeled trees, simply-connected graphs, and half-graphs, in addition to the connected graphs and connected wheel-free graphs which appear to be more of a surprise.

Chapter 8 presents the definition of a pasting scheme, implied but never stated by Markl, as a graph groupoid closed under graph substitution and containing the units thereof. A large number of examples is now available, with strong generating sets established earlier in most cases. Next is a discussion of free products of pasting schemes, and the technical conditions necessary to say the union of strong generating sets remains a strong generating set for the free product of pasting schemes. The chapter also includes a discussion of the pasting schemes where each graph contains a single vertex, which implies the strict isomorphism classes of graphs can be used to define the morphisms in a category where graph substitution defines the composition law.

The ninth chapter introduces the notion of orthogonal pasting schemes, the Kontsevich groupoid associated to a pair of pasting schemes, and the notion of well-matched pasting schemes. The point here is to understand the free construction, or induction functor, for moving from a small pasting scheme up to a larger pasting scheme. Once again, this technical theory will be applied in the context of generalized PROPs as well as that of algebras and modules. The Kontsevich groupoid generalizes an idea underlying Kontsevich's suggestion of using half-PROPs to establish results about dioperads and PROPs, and over a dozen examples are computed explicitly. The idea behind well-matched pasting schemes, studied in some form in [**MV09**], is that the free functor has a particularly convenient presentation. In fact, one of our important non-examples is established in [**MV09**], but we also provide two more non-examples, to contrast with a variety of examples. This discussion ends the foundational work of Part 1.

Part 2 begins with a review of relevant categorical background, including symmetric monoidal categories, unordered tensor products, as well as monads and their algebras. Then a notion of a pointed extension of a monad is presented, which can be used to describe analogs of classical modules over a ring, where algebras over a monad play the role of the ring. Here the technical issue is that we are able to produce a new monad whose algebras will be the analog of modules in question, which will allow us later to produce free module constructions as well as (co)limits of modules over a generalized PROP. The next step is to proceed with the definition of generalized PROPs associated to a pasting scheme. Given a pasting scheme $\mathcal{G}$, there is a monad $F_{\mathcal{G}}$ on the category of appropriately colored objects in a symmetric monoidal $\mathcal{E}$ whose monadic multiplication is induced by graph substitution. A $\mathcal{G}$-PROP is then defined simply as an $F_{\mathcal{G}}$-algebra, and a few simple examples are included to illustrate the basic theory.

Chapter 11 begins with the Biased Definition Theorem and Biased Morphism Theorem, which are the first main results exploiting our strong generating sets from Chapter 7. Essentially, this is a blueprint for how to turn a strong generating set into a characterization of the $\mathcal{G}$-PROPs formally defined as algebras over a monad instead using a collection of structure maps and relations among them. The remainder of the chapter consists of an extensive list of major examples, including Markl Non-Unital Operads, May Operads, Dioperads, Half-PROPs, Properads, PROPs, Wheeled PROPs, Wheeled Properads, and Wheeled Operads. In each case, a complete definition of the relevant structure is carefully stated, which in several cases seems to be new in the literature, followed by a description of how the strong generating set translates into the indicated structure maps and required commutative diagrams, before stating the characterizations formally.

The discussion in Chapter 12 centers around the relationships between the generalized PROPs associated to different pasting schemes and base categories. Given two pasting schemes with one contained in the other, we give a detailed construction of the free-forgetful adjunction between their categories of generalized PROPs and study the left adjoint. This left adjoint contains all such free functors in the operad/PROP literature as special cases. For example, the free wheeled PROP generated by a wheeled operad and the free PROP generated by a Σ-bimodule are both special cases of this construction. In the situation that the pasting schemes are well-matched, we have a simpler construction reminiscent of the monad associated to a pasting scheme, but in this case depending heavily upon the Kontsevich groupoid. We also discuss the change-of-base functors on the category of generalized PROPs for a fixed pasting scheme, for example, to understand when functors or adjoint pairs extend to $\mathcal{G}$-PROP categories.

In Chapter 13 we introduce and discuss algebras over a generalized PROP. This begins by defining endomorphism objects E_A associated to a colored object A for a given pasting scheme, which is subtle whenever contractions might be involved. As usual, we define an algebra over a $\mathcal{G}$-PROP P as a morphism $\mathsf{P} \longrightarrow \mathsf{E}_A$. Then we use the relative endomorphism object associated to a morphism of colored objects, which involves a pullback construction, to characterize a morphism of algebras. There are also various results about the category of algebras under a change of pasting scheme, $\mathcal{G}$-PROP P or base category. Finally, the chapter closes with presentations of P-algebras in terms of each of our strong generating sets, for the same list of examples considered in detail for generalized PROPs.

The focus in Chapter 14 is on two conceptual characterizations of generalized PROPs in terms of more familiar objects. First, we associate to any pasting scheme $\mathcal{G}$ a colored operad whose algebras are exactly the $\mathcal{G}$-PROPs. The essence of this construction is that graph substitution can be repackaged as a colored operad composition, or equivalently as the composition in a multicategory. Generalized PROPs associated to a pasting scheme $\mathcal{G}$ are also characterized as enriched multicategorical functors from a fixed small enriched multicategory into the base category.

Chapter 15 is devoted to the study of modules over a generalized PROP, which is a new definition in this generality. This builds upon the theory of pointed extensions of a monad developed early in Chapter 10 for precisely this purpose. The fundamental viewpoint is that a classical bimodule over a ring can be viewed as involving multiplications of a number of factors where a single factor is the module and all others are the ring. The key technical mechanism here involves pointed graphs, which is why we worked out the implications of strong generating sets for pointed graphs and their graph substitutions. In essence, we take the construction of the monad $F_{\mathcal{G}}$ associated to a pasting scheme and insert pointed graphs in appropriate places to construct a pointed extension, which leads to the definition of modules. The theory from Chapter 10 still provides a new monad with these as its algebras, so a free object construction and the existence of (co)limits. We then provide a series of results about changes of base category, pasting scheme, or $\mathcal{G}$-PROP P. Finally, the chapter ends with characterizations of modules in concrete terms for each of our usual examples, once again exploiting our strong generating sets, and verifying that our definition coincides with others in the literature even if it is presented differently.

Finally, Chapter 16 is devoted to the study of May modules over an algebra over an operad. In this case, the initial definition is given in concrete terms, but then shown to agree with a notion of module associated to a pointed extension of a monad once again. When using the material of Chapter 14 to view a $\mathcal{G}$-PROP P as the algebras over the specific colored operad indicated there, the result is to recover the modules over P in the sense of Chapter 15 as May modules. This point of view fits in well with deformation theory (e.g., [**FMY09**]) and provides closer connections to a variety of previous work.

Organizational Graphics

Diagram of the inclusions of pasting schemes:

(0.1)