

Contents

Preface	xv
Organizational Graphics	xxii
Acknowledgments	xxiii
List of Notations	xxv
Part 1. Wheeled Graphs and Pasting Schemes	1
Chapter 1. Wheeled Graphs	3
1.1. Profiles	3
1.1.1. Colors	3
1.1.2. Profiles and Shuffles	3
1.1.3. Pairs of Profiles	5
1.2. Defining Wheeled Graphs	6
1.2.1. Basic Graphs	6
1.2.2. Structures on Graphs	9
1.3. Basic Examples of Wheeled Graphs	13
1.3.1. Corollas	13
1.3.2. Exceptional Edges	14
1.3.3. Exceptional Loops	14
1.3.4. Other Basic Graphs	15
1.4. Technical Variants of Wheeled Graphs	18
1.4.1. Pointed Graphs	18
1.4.2. Ordered Graphs	18
1.4.3. Multi-stage Graphs	19
1.4.4. Cyclic or Modular Graphs	19
Chapter 2. Special Sets of Graphs	21
2.1. Paths and Directed Paths	21
2.1.1. Paths	21
2.1.2. Directed Paths	22
2.2. Connected Graphs	24
2.2.1. Flag-Connected Graphs	24
2.2.2. Connected Graphs	24
2.3. Simply-Connected and Wheel-Free Graphs	26
2.3.1. Simply-Connected Graphs	26
2.3.2. Wheel-Free Graphs	26
2.4. Half-Graphs and Dioperadic Graphs	27
2.4.1. Half-Graphs	27
2.4.2. Basic Dioperadic Graphs	28

2.5. Trees	30
2.5.1. Simple Trees	30
2.5.2. Level Trees	30
2.5.3. Special Trees	31
2.5.4. Wheeled and Truncated Trees	32
Chapter 3. Basic Operations on Wheeled Graphs	35
3.1. Relabeling	36
3.1.1. Input and Output Relabeling	36
3.1.2. Permuted Corollas	37
3.2. Disjoint Union	37
3.2.1. Disjoint Union of Wheeled Graphs	37
3.2.2. Disjoint Union of Corollas	38
3.2.3. Disjoint Union Decompositions of Wheeled Graphs	39
3.3. Grafting	39
3.3.1. Grafting of Ordinary Wheeled Graphs	39
3.3.2. Grafted Corollas	40
3.3.3. Partial Grafting	41
3.3.4. The $\text{Comp-}i$ Operation	43
3.3.5. The $j\text{-comp-}i$ Operation	44
3.4. Contraction	44
3.4.1. Contraction of Ordinary Wheeled Graphs	44
3.4.2. Contracted Corolla	45
Chapter 4. Graph Groupoids	47
4.1. Strict Isomorphisms	47
4.1.1. Strict Isomorphisms and Automorphisms	47
4.1.2. Decomposing the Strict Automorphism Group	50
4.2. Weak Isomorphisms	52
4.3. Graph Groupoids	54
4.3.1. Graph Groupoids Defined by Strict and Weak Isomorphisms	55
4.3.2. Examples of Graph Groupoids	55
Chapter 5. Graph Substitution	57
5.1. Graph Substitution in the Ordinary Case	57
5.2. Pre-Graphs and Pre-Substitution	59
5.2.1. Pre-Graphs	61
5.2.2. Pre-Substitution	61
5.2.3. Properties of Pre-substitution	63
5.3. Ambiguous Paths and Associated Graphs	64
5.3.1. Ambiguous Paths	64
5.3.2. The Associated Graph Construction	66
5.3.3. Compatibility of Associated Graphs with Pre-substitution	67
5.4. General Graph Substitution	70
5.4.1. Technical Variants of Graph Substitution	72
Chapter 6. Properties of Graph Substitution	75
6.1. Operations Coming from Graph Substitution	75
6.1.1. Characterizing Representable Graph Operations	76
6.1.2. Altering the Listing by Graph Substitution	76

6.1.3.	Disjoint Union from Graph Substitution	77
6.1.4.	Grafting from Graph Substitution	77
6.1.5.	Contraction from Graph Substitution	78
6.1.6.	Shrinking Internal Edges Is Not a Representable Graph Operation	78
6.1.7.	Input and Output Extensions	79
6.2.	Properties Preserved by Graph Substitution	80
6.2.1.	Connected Graphs	80
6.2.2.	Wheel-free Graphs	81
6.2.3.	Simply-Connected Graphs	82
6.3.	Substitution Properties of Basic Operations	82
6.3.1.	Iterated Relabelings	83
6.3.2.	Union	83
6.3.3.	Contraction	84
6.3.4.	Grafting	87
6.4.	Substitution Properties of Partial Grafting	89
6.4.1.	Partial Grafting	89
6.4.2.	Partial Grafting and Contraction	96
6.5.	Substitution Properties for Trees and Basic Dioperadic Graphs	99
6.5.1.	Simple Trees	100
6.5.2.	Special Trees	101
6.5.3.	Truncated and Contracted Trees	103
6.5.4.	Basic Dioperadic Graphs	104
Chapter 7.	Generators for Graphs	107
7.1.	Graph Simplices and Generating Sets	109
7.1.1.	Equivalent Graph Simplices	109
7.1.2.	Relaxed Moves	110
7.1.3.	Pointed Graph Simplices	111
7.1.4.	Generating Set for a Graph Groupoid	111
7.2.	Strong Generating Set for Wheeled Graphs	112
7.3.	Strong Generating Set for Wheel-Free Graphs	115
7.4.	Strong Generating Set for Level Trees	117
7.5.	Strong Generating Set for Unital Trees	119
7.6.	Strong Generating Set for Wheeled Trees	120
7.6.1.	Building from Level Trees to Wheeled Trees	120
7.6.2.	Strong Generating Set for Wheeled Trees	122
7.7.	Strong Generating Set for Simply-Connected Graphs	123
7.8.	Strong Generating Set for Half-Graphs	124
7.9.	Strong Generating Sets for Connected Wheel-Free Graphs	125
7.9.1.	An Alternative Strong Generating Set for Connected Wheel-Free Graphs	128
7.10.	Strong Generating Set for Connected Wheeled Graphs	129
Chapter 8.	Pasting Schemes	133
8.1.	Definitions and First Examples	133
8.1.1.	Definition of a Pasting Scheme	134
8.1.2.	Partial Ordering on Pasting Schemes	134
8.1.3.	Restriction on S	135
8.1.4.	First Examples of Pasting Schemes	135

8.1.5.	Unital Pasting Schemes	136
8.1.6.	Key Examples of Pasting Schemes	136
8.1.7.	Summary of Relationships Between Pasting Schemes	138
8.1.8.	Virtual Pasting Schemes	138
8.1.9.	Non- Σ Pasting Schemes	138
8.2.	Free Product Decompositions of Pasting Schemes	139
8.2.1.	Free Product	139
8.2.2.	Generating Sets for Free Products	141
8.2.3.	Separating Generating Sets for Free Products	141
8.3.	Monogenic Pasting Schemes	144
Chapter 9. Well-Matched Pasting Schemes		147
9.1.	Kontsevich Groupoid	147
9.1.1.	Intersection	148
9.1.2.	Orthogonality	148
9.1.3.	Prime Graph Groupoids	149
9.1.4.	Kontsevich Groupoid	149
9.2.	Well-Matched Pasting Schemes	152
9.2.1.	Well-Matched Examples	153
9.2.2.	Examples That Are Not Well-Matched	154
Part 2. Generalized PROPs, Algebras, and Modules		157
Chapter 10. Generalized PROPs		159
10.1.	Categorical Preliminaries	160
10.1.1.	Monoidal Categories	160
10.1.2.	Symmetric Monoidal Categories	162
10.1.3.	Ordered and Unordered Tensor Products	164
10.1.4.	Monads and Their Algebras	165
10.2.	Pointed Extensions of Monads	169
10.2.1.	Monad Replacement for Pointed Extensions of a Monad	171
10.3.	Colored Objects, Bimodules, and Decorated Graphs	176
10.3.1.	Diagram Categories	176
10.3.2.	Colored Objects and Bimodules	177
10.3.3.	Decorated Graphs	179
10.4.	Generalized PROPs as Monadic Algebras	181
10.4.1.	The Monad Associated to a Pasting Scheme	181
10.4.2.	Generalized PROPs	184
10.4.3.	Bi-equivariant Structure	185
10.4.4.	Generalized PROPs for Unital Pasting Schemes	186
10.5.	First Examples of Generalized PROPs	187
10.5.1.	Unital Linear PROPs	188
10.5.2.	Contraction PROPs	188
10.5.3.	Horizontal and Vertical PROPs	188
10.5.4.	Generalized PROPs over Monogenic Pasting Schemes	189
Chapter 11. Biased Characterizations of Generalized PROPs		191
11.1.	Biased Definition Theorem	192
11.2.	Biased Morphism Theorem	194

11.3.	Markl Non-Unital Operads as Tree -PROPs	195
11.3.1.	Defining a Markl Non-Unital Operad	195
11.3.2.	Interpreting the Axioms for a Markl Non-Unital Operad	196
11.3.3.	Markl Non-Unital Operads Are Tree -PROPs	196
11.4.	May Operads as UTree -PROPs	196
11.4.1.	Defining a May Operad	196
11.4.2.	Interpreting the Axioms for a May Operad	198
11.4.3.	May Operads are UTree -PROPs	198
11.5.	Dioperads as $\mathbf{Gr}_{\text{di}}^{\uparrow}$ -PROPs	199
11.5.1.	Defining a Dioperad	199
11.5.2.	Interpreting the Axioms for a Dioperad	201
11.5.3.	Dioperads are $\mathbf{Gr}_{\text{di}}^{\uparrow}$ -PROPs	201
11.6.	Half-PROPs as $\mathbf{Gr}_{\frac{1}{2}}$ -PROPs	202
11.6.1.	Defining a Half-PROP	202
11.6.2.	Interpreting the Axioms for a Half-PROP	203
11.6.3.	Half-PROPs are $\mathbf{Gr}_{\frac{1}{2}}$ -PROPs	203
11.7.	Properads as \mathbf{Gr}_c^{\uparrow} -PROPs	203
11.7.1.	Defining a Properad	203
11.7.2.	An Alternate Definition of a Properad	207
11.7.3.	Interpreting the Axioms for a Properad	208
11.7.4.	Interpreting the Axioms for an Alternate Properad	208
11.7.5.	Properads and Alternate Properads are \mathbf{Gr}_c^{\uparrow} -PROPs	208
11.8.	PROPs as \mathbf{Gr}^{\uparrow} -PROPs	209
11.8.1.	Defining a PROP	209
11.8.2.	Interpreting the Axioms for a PROP	212
11.8.3.	PROPs are \mathbf{Gr}^{\uparrow} -PROPs	212
11.9.	Wheeled PROPs as $\mathbf{Gr}_w^{\mathcal{Q}}$ -PROPs	213
11.9.1.	Defining a Wheeled PROP	213
11.9.2.	Interpreting the Axioms for a Wheeled PROP	215
11.9.3.	Wheeled PROPs are $\mathbf{Gr}_w^{\mathcal{Q}}$ -PROPs	215
11.10.	Wheeled Properads as $\mathbf{Gr}_c^{\mathcal{Q}}$ -PROPs	215
11.10.1.	Defining a Wheeled Properad	215
11.10.2.	Interpreting the Axioms for Wheeled Properads	217
11.10.3.	Wheeled Properads are $\mathbf{Gr}_c^{\mathcal{Q}}$ -PROPs	217
11.11.	Wheeled Operads as $\mathbf{Tree}^{\mathcal{Q}}$ -PROPs	218
11.11.1.	Defining a Wheeled Operad	218
11.11.2.	Interpreting the Axioms for a Wheeled Operad	220
11.11.3.	Wheeled Operads are $\mathbf{Tree}^{\mathcal{Q}}$ -PROPs	221
Chapter 12.	Functors of Generalized PROPs	223
12.1.	Adjunction Induced by an Inclusion of Pasting Schemes	223
12.1.1.	The Right Adjoint	224
12.1.2.	Adjunction for a Well-Matched Pair	224
12.1.3.	The General Left Adjoint	227
12.1.4.	Examples of the Left Adjoint	231
12.2.	Generalized PROPs under a Change of Base Category	233
12.2.1.	Transferring Generalized PROPs	233
12.2.2.	Changing Pasting Scheme and Base Category	235

12.2.3. An Application to Homology	236
12.3. Notes	237
Chapter 13. Algebras over Generalized PROPs	239
13.1. Endomorphism Objects	240
13.1.1. Hom-Tensor Adjunction	240
13.1.2. Endomorphism Objects	241
13.1.3. Pasting Scheme Admitting an Endomorphism Object	242
13.1.4. Examples of Endomorphism Objects	243
13.1.5. Relative Endomorphism Object	245
13.2. Unbiased Algebras	249
13.2.1. Unbiased Definition of Algebras	250
13.2.2. Alternative Descriptions of Algebras	251
13.3. Algebras under Change of Pasting Scheme or Base Category	252
13.3.1. Change of Pasting Scheme	253
13.3.2. Change of \mathcal{G} -PROP	254
13.3.3. Change of Base Category	254
13.4. Biased Algebras	257
13.4.1. Biased Algebra Theorem	257
13.4.2. Algebras over a Markl Non-Unital Operad	257
13.4.3. Algebras over a May Operad	258
13.4.4. Algebras over a Dioperad	258
13.4.5. Algebras over a Half-PROP	259
13.4.6. Algebras over a Properad	260
13.4.7. Algebras over a PROP	260
13.4.8. Algebras over a Wheeled PROP	261
13.4.9. Algebras over a Wheeled Properad	261
13.4.10. Algebras over a Wheeled Operad	261
13.5. Notes	262
Chapter 14. Alternative Descriptions of Generalized PROPs	263
14.1. Generalized PROPs as Operadic Algebras	263
14.1.1. The Operad Associated to a Pasting Scheme	264
14.1.2. The Colored Operad of \mathcal{G} -PROPs	265
14.2. Generalized PROPs as Multicategorical Functors	266
14.2.1. Defining an Enriched Multicategory	266
14.2.2. Functors of Enriched Multicategories	268
14.2.3. Corepresenting \mathcal{G} -PROPs	269
14.3. Notes	272
Chapter 15. Modules over Generalized PROPs	273
15.1. Pointed Decorated Graphs and a Monad Variation	274
15.1.1. Pointed Decorated Graphs	274
15.1.2. A Pointed Extension of the Monad $F_{\mathcal{G}}$	274
15.2. Unbiased Modules over a \mathcal{G} -PROP	276
15.2.1. Unbiased Definition of Modules over a \mathcal{G} -PROP	276
15.2.2. Bi-equivariant Structure of a Module	278
15.2.3. Graphical Interpretation	278

15.3.	Modules under Change of Pasting Scheme, \mathcal{G} -PROP, or Base Category	280
15.3.1.	Change of Pasting Scheme	280
15.3.2.	Change of \mathcal{G} -PROP	280
15.3.3.	Change of Base Category	281
15.4.	Biased Characterizations of Modules	282
15.4.1.	Biased Module Theorem	282
15.4.2.	Modules over a Markl Non-Unital Operad	284
15.4.3.	Modules over a May Operad	285
15.4.4.	Modules over a Dioperad	285
15.4.5.	Modules over a Half-PROP	286
15.4.6.	Modules over a Properad	287
15.4.7.	Modules over a PROP	287
15.4.8.	Modules over a Wheeled PROP	288
15.4.9.	Modules over a Wheeled Properad	289
15.4.10.	Modules over a Wheeled Operad	290
Chapter 16.	May Modules over Algebras over Operads	291
16.1.	Preliminaries on Modules over an Algebra over an Operad	291
16.1.1.	Groupoid-Indexed Colimit	292
16.1.2.	A Pointed Monad Extension for Modules over an Algebra over a Colored Operad	293
16.2.	May Modules over an Operadic Algebra	297
16.2.1.	Definition of a May Module	297
16.2.2.	Alternative Description of a May Module	299
16.2.3.	May Modules are Monadic Algebras	300
16.2.4.	Modules Over Generalized PROPs are May Modules	300
	Bibliography	303
	Index	307