Preface

Был один рыжий человек, у которого не было глаз и ушей. У него не было и волос, так что рыжим его называли условно. Говорить он не мог, так как у него не было рта. Носа тоже у него не было. У него не было даже рук и ног. И живота у него не было, и спины у него не было, и хребта у него не было, и никаких внутренностей у него не было. Ничего не было! Так что непонятно, о ком идет речь. Уж лучше мы о нем не будем больше говорить.

Д. Хармс, Голубая тетрадь но. 10, 1937

There was a red-haired man who had no eyes or ears. Neither did he have any hair, so he was called red-haired by convention. He couldn't speak, since he didn't have a mouth. Neither did he have a nose. He didn't even have any arms or legs. He had no stomach and he had no back and he had no spine and he had no innards whatsoever. He had nothing at all! Therefore there's no knowing whom we are even talking about. It's really better that we don't talk about him any more.

D. Harms, Blue notebook # 10, 1937

Tensor categories should be thought of as counterparts of rings in the world of categories.[1] They are ubiquitous in noncommutative algebra and representation theory, and also play an important role in many other areas of mathematics, such as algebraic geometry, algebraic topology, number theory, the theory of operator algebras, mathematical physics, and theoretical computer science (quantum computation).

The definition of a monoidal category first appeared in 1963 in the work of Mac Lane [**Mac1**], and later in his classical book [**Mac2**] (first published in 1971). Mac Lane proved two important general theorems about monoidal categories – the coherence theorem and the strictness theorem, and also defined symmetric and

[1]Philosophically, the theory of tensor categories may perhaps be thought of as a theory of vector spaces or group representations without vectors, similarly to how ordinary category theory may be thought of as a theory of sets without elements. As seen from the epigraph, this idea, as well as its dismissal as "abstract nonsense" common in early years of category theory, were discussed in Russian absurdist literature several years before the foundational papers of Mac Lane and Eilenberg on category theory (1942-1945).

braided monoidal categories. Later, Saavedra-Rivano in his thesis under the direction of Grothendieck [**Sa**], motivated by the needs of algebraic geometry and number theory (more specifically, the theory of motives), developed a theory of Tannakian categories, which studies symmetric monoidal structures on abelian categories (the prototypical example being the category of representations of an algebraic group). This theory was simplified and further developed by Deligne and Milne in their classical paper [**DelM**]. Shortly afterwards, the theory of tensor categories (i.e., monoidal abelian categories) became a vibrant subject, with spectacular connections to representation theory, quantum groups, infinite dimensional Lie algebras, conformal field theory and vertex algebras, operator algebras, invariants of knots and 3-manifolds, number theory, etc., which arose from the works of Drinfeld, Moore and Seiberg, Kazhdan and Lusztig, Jones, Witten, Reshetikhin and Turaev, and many others. Initially, in many of these works tensor categories were merely a tool for solving various concrete problems, but gradually a general theory of tensor categories started to emerge, and by now there are many deep results about properties and classification of tensor categories, and the theory of tensor categories has become fairly systematic. The goal of this book is to provide an accessible introduction to this theory.

We should mention another major source of inspiration for the theory of tensor categories, which is the theory of Hopf algebras. The notion of a Hopf algebra first appeared in topology (more specifically, in the work [**Ho**] of Hopf in 1941, as an algebraic notion capturing the structure of cohomology rings of H-spaces, in particular, Lie groups). Although classical topology gives rise only to cocommutative Hopf algebras, in the 1960s operator algebraists and ring theorists (notably G. Kac) became interested in noncommutative and noncocommutative Hopf algebras and obtained the first results about them. In 1969 Sweedler wrote a textbook on this subject [**Sw**], proving the first general results about Hopf algebras. In the 1970s and 1980s a number of fundamental general results were proved about finite dimensional Hopf algebras, notably Radford's formula for the 4th power of the antipode [**Ra2**], the theorems of Larson and Radford on semisimple Hopf algebras [**LaR1, LaR2**], the freeness theorem of Nichols and Zoeller [**NicZ**], and the work of Nichols on what is now called Nichols algebras (which give rise to pointed Hopf algebras). Also, at about the same time Drinfeld developed the theory of quantum groups and the quantum double construction, and a bit later Lusztig developed the theory of quantum groups at roots of unity, which provided many interesting new examples of Hopf algebras. Around that time, it was realized that Hopf algebras could be viewed as algebraic structures arising from tensor categories with a fiber functor, i.e., tensor functor to the category of vector spaces, through the so-called reconstruction theory (which takes its origins in [**Sa**] and [**DelM**]). Since then, the theory of Hopf algebras has increasingly been becoming a part of the theory of tensor categories, and in proving some of the more recent results on Hopf algebras (such as, e.g., the classification of semisimple Hopf algebras of prime power dimension, or the classification of triangular Hopf algebras) tensor categories play a fundamental role. In fact, this is the point of view on Hopf algebras that we want to emphasize in this book: many of the important results about Hopf algebras are better understood if viewed through the prism of tensor categories. Namely, we deduce many of the most important results about Hopf algebras, especially finite dimensional and semisimple ones (such as the Fundamental Theorem on Hopf modules and bimodules, Nichols-Zoeller theorem, Larson-Radford theorems, Radford's S^4 formula, Kac-Zhu theorem, and many others) as corollaries of the general theory of tensor categories.

Let us now summarize the contents of the book, chapter by chapter.

In Chapter 1 we discuss the basics about abelian categories (mostly focusing on locally finite, or artinian, categories over a field, which is the kind of categories we will work with throughout the book). Many results here are presented without proofs, as they are well known (in fact, we view the basic theory of abelian categories as a prerequisite for reading this book). We do, however, give a more detailed discussion of the theory of locally finite categories, coalgebras, the Coend construction, and the reconstruction theory for coalgebras, for which it is harder to find an in-depth exposition in the literature. In accordance with the general philosophy of this book, we present the basic results about coalgebras (such as the Taft-Wilson theorem) as essentially categorical statements.

In Chapter 2 we develop the basic theory of monoidal categories. Here we give the detailed background on monoidal categories and functors, using a formalism that allows one not to worry much about the units, and give short proofs of the Mac Lane coherence and strictness theorems. We also develop a formalism of rigid monoidal categories, and give a number of basic examples. Finally, we briefly discuss 2-categories.

In Chapter 3 we discuss the combinatorics needed to study tensor categories, namely, the theory of $\mathbb{Z}_+$-rings (i.e., rings with a basis in which the structure constants are nonnegative integers). Such rings serve as Grothendieck rings of tensor categories, and it turns out that many properties of tensor categories actually have combinatorial origin, i.e., come from certain properties of the Grothendieck ring. In particular, this chapter contains the theory of Frobenius-Perron dimension, which plays a fundamental role in studying finite tensor categories (in particular, fusion categories), and also study $\mathbb{Z}_+$-modules over $\mathbb{Z}_+$-rings, which arise in the study of module categories. All the results in this chapter are purely combinatorial or ring-theoretical, and do not rely on category theory.

In Chapter 4 we develop the general theory of multitensor categories. Here we prove the basic results about multitensor and tensor categories, such as the exactness of tensor product and semisimplicity of the unit object, introduce a few key notions and constructions, such as pivotal and spherical structures, Frobenius-Perron dimensions of objects and categories, categorification, etc. We also provide many examples.

In Chapter 5 we consider tensor categories with a fiber functor, i.e., a tensor functor to the category of vector spaces. This leads to the notion of a Hopf algebra; we develop reconstruction theory, which establishes an equivalence between the notion of a Hopf algebra and the notion of a tensor category with a fiber functor. Then we proceed to develop the basic theory of Hopf algebras, and consider categories of modules and comodules over them. We also give a number of examples of Hopf algebras, such as Nichols Hopf algebras of dimension 2^{n+1}, Taft algebras, small quantum groups, etc., and discuss their representation theory. Then we prove a few classical theorems about Hopf algebras (such as the Cartier-Gabriel-Kostant theorem), discuss pointed Hopf algebras, quasi-Hopf algebras, and twisting.

In Chapter 6 we develop the theory of finite tensor categories, i.e., tensor categories which are equivalent, as an abelian category, to the category of representations of a finite dimensional algebra (a prototypical example being the category of representations of a finite dimensional Hopf algebra). In particular, we study the behavior of projectives in such a category, and prove a categorical version of the Nichols-Zoeller theorem (stating that a finite dimensional Hopf algebra is free over a Hopf subalgebra). We also introduce the distinguished invertible object of a finite tensor category, which is the categorical counterpart of the distinguished grouplike element (or character) of a finite dimensional Hopf algebra. Finally, we develop the theory of integrals of finite dimensional Hopf algebras.

In Chapter 7 we develop the theory of module categories over tensor and multitensor categories. Similarly to how understanding the structure of modules over a ring is necessary to understand the structure of the ring itself, the theory of module categories is now an indispensable tool in studying tensor categories. We first develop the basic theory of module categories over monoidal categories, and then pass to the abelian setting, introducing the key notion of an exact module category over a finite tensor category (somewhat analogous to the notion of a projective module over a ring). We also show that module categories arise as categories of modules over algebras in tensor categories (a categorical analog of module and comodule algebras over Hopf algebras). This makes algebras in tensor categories the main technical tool of studying module categories. We then proceed to studying the category of module functors between two module categories, and the Drinfeld center construction as an important special case of that. For Hopf algebras, this gives rise to the famous Drinfeld double construction and Yetter-Drinfeld modules. We then discuss dual categories and categorical Morita equivalence of tensor categories, prove the Fundamental Theorem for Hopf modules and bimodules over a Hopf algebra, and prove the categorical version of Radford's S^4 formula. Finally, we develop the theory of categorical dimensions of fusion categories and of Davydov-Yetter cohomology and deformations of tensor categories. At the end of the chapter, we discuss weak Hopf algebras, which are generalizations of Hopf algebras arising from semisimple module categories via reconstruction theory.

In Chapter 8 we develop the theory of braided categories, which is perhaps the most important part of the theory of tensor categories. We discuss pointed braided categories (corresponding to quadratic forms on abelian groups), quasitriangular Hopf algebras (arising from braided categories through reconstruction theory), and show that the center of a tensor category is a braided category. We develop a theory of commutative algebras in braided categories, and show that modules over such an algebra form a tensor category. We also develop the theory of factorizable, ribbon and modular categories, the S-matrix, Gauss sums, and prove the Verlinde formula and the existence of an $SL_2(\mathbb{Z})$-action. We prove the Anderson-Moore-Vafa theorem (saying that the central charge and twists of a modular category are roots of unity). Finally, we develop the theory of centralizers and projective centralizers in braided categories, de-equivariantization of braided categories, and braided G-crossed categories.

In Chapter 9 we mostly discuss results about fusion categories. This chapter is mainly concerned with applications, where we bring various tools from the previous chapters to bear on concrete problems about fusion categories. In particular, in this chapter we prove Ocneanu rigidity (the statement that fusion categories in characteristic zero have no deformations), develop the theory of dual categories, pseudo-unitary categories (showing that they have a canonical spherical structure – the categorical analog of the statement that a semisimple Hopf algebra in characteristic zero is involutive), study integral, weakly integral, group-theoretical, and weakly group-theoretical fusion categories. Next, we discuss symmetric and Tannakian fusion categories, prove Deligne's theorem on classification of such categories and discuss its nonfusion generalization stating that a symmetric tensor category of subexponential growth is the representation category of a supergroup (with a parity condition). We also give examples of symmetric categories with faster growth – Deligne's categories $\mathsf{Rep}(S_t)$, $\mathsf{Rep}(GL_t)$, $\mathsf{Rep}(O_t)$, and $\mathsf{Rep}(Sp_{2t})$. Next, we give a criterion of group-theoreticity of a fusion category, and show that any integral fusion category of prime power dimension is group theoretical. For categories of dimension p and p^2, this gives a very explicit classification (they are representation categories of an abelian group with a 3-cocycle). We introduce the notion of a solvable fusion category, and prove a categorical analog of Burnside's theorem, stating that a fusion category of dimension $p^a q^b$, where p, q are primes, is solvable. Finally, we discuss lifting theory for fusion categories (from characteristic p to characteristic zero).

Thus, Chapters 7–9 form the main part of the book.

A few disclaimers. First, we do not provide a detailed history of the subject, or a full list of references, containing all the noteworthy works; this was not our intention, and we do not even come close. Second, this book does not aim to be an exhaustive monograph on tensor categories; the subject is so vast that it would have taken a much longer text, perhaps in several volumes, to touch upon all the relevant topics. The reader will notice that we have included very little material (or none at all) on some of the key applications of tensor categories and their connections with other fields (representation theory, quantum groups, knot invariants, homotopy theory, vertex algebras, subfactors, etc.) In fact, our goal was not to give a comprehensive treatment of the entire subject, but rather to provide a background that will allow the reader to proceed to more advanced and specialized works. Also, we tried to focus on topics which are not described in detail in books or expository texts, and left aside many of those which are well addressed in the literature. Finally, as authors of any textbook, we had a preference for subjects that we understand better!

To increase the amount of material that we are able to discuss and to enable active reading, we have presented many examples and applications in the form of exercises. The more difficult exercises are provided with detailed hints, which should allow the reader to solve them without consulting other sources. At the end of Chapters 5, 7, 8, and 9 we provide a summary of some noteworthy results related to the material of the chapter that we could not discuss, and provide references. Finally, we end each chapter with references for the material discussed in the chapter.

Finally, let us discuss how this book might be used for a semester-long graduate course. Clearly, it is not feasible to go through the entire book in one semester and some omissions are necessary. The challenge is to get to Chapters 7–9 (which contain the main results) as quickly as possible. We recommend that the course start from Chapter 2, and auxiliary material from Chapters 1 and 3 be covered as needed. More specifically, we suggest the following possible course outline.

(1) Chapter 2: Sections 2.1–2.10 (monoidal categories and functors, the MacLane strictness theorem, rigidity).
(2) Chapter 4: Sections 4.1–4.9 (basic properties of tensor categories, Grothendieck rings, Frobenius-Perron dimensions).
(3) Chapter 5: Sections 5.1–5.6 (fiber functors and basic examples of Hopf algebras).
(4) Chapter 6: Sections 6.1–6.3 (properties of injective and surjective tensor functors).
(5) Chapter 7: Sections 7.1–7.12 (exact module categories, categorical Morita theory).
(6) Chapter 8: Sections 8.1–8.14 (examples from metric groups, Drinfeld centers, modular categories).
(7) Chapter 9: Sections 9.1–9.9 and 9.12 (absence of deformations, integral fusion categories, symmetric categories).

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