Monoidal categories

2.1. Definition of a monoidal category

A good way of thinking about category theory (which will be especially useful throughout this book) is that category theory is a refinement (or "categorification") of ordinary algebra. In other words, there exists a dictionary between these two subjects, such that usual algebraic structures are recovered from the corresponding categorical structures by passing to the set of isomorphism classes of objects.

For example, the notion of a category is a categorification of the notion of a set. Similarly, abelian categories are a categorification of abelian groups [1] (which justifies the terminology).

This dictionary goes surprisingly far, and many important constructions below will come from an attempt to enter into it a categorical "translation" of an algebraic notion.

In particular, the notion of a monoidal category is the categorification of the notion of a monoid.

Recall that a monoid may be defined as a set C with an associative multiplication operation $(x, y) \to x \cdot y$ (i.e., a semigroup), with an element 1 such that $1^2 = 1$ and the maps $x \mapsto 1 \cdot x$, $x \mapsto x \cdot 1 : C \to C$ are bijections. It is easy to show that in a semigroup, the last condition is equivalent to the usual unit axiom $1 \cdot x = x \cdot 1 = x$.[2]

As usual in category theory, to categorify the definition of a monoid, we should replace the equalities in the definition of a monoid (namely, the associativity equation $(xy)z = x(yz)$ and the equation $1^2 = 1$) by isomorphisms satisfying some consistency properties, and the word "bijection" by the word "equivalence" (of categories). This leads to the following definition.

DEFINITION 2.1.1. A *monoidal category* is a quintuple $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor called the *tensor product* bifunctor, $a : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ is a natural isomorphism:

$$(2.1) \qquad a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z), \qquad X, Y, Z \in \mathcal{C}$$

called the *associativity constraint* (or *associativity isomorphism*), $\mathbf{1} \in \mathcal{C}$ is an object of $\mathcal{C}$, and $\iota : \mathbf{1} \otimes \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ is an isomorphism, subject to the following two axioms.

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[1]To be more precise, the set of isomorphism classes of objects in an abelian category $\mathcal{C}$ is a commutative monoid, but one usually extends it to a group by considering "virtual objects" of the form $X - Y$, $X, Y \in \mathcal{C}$.

[2]Indeed, if left and right multiplication by 1 are bijections and $1^2 = 1$, then we have $1 \cdot 1 \cdot x = 1 \cdot x$, hence $1 \cdot x = x$, and similarly $x \cdot 1 = x$.

1. The pentagon axiom. The diagram

(2.2)
$$((W \otimes X) \otimes Y) \otimes Z$$

$$(W \otimes (X \otimes Y)) \otimes Z \xleftarrow{a_{W,X,Y} \otimes \mathsf{id}_Z} \qquad \xrightarrow{a_{W \otimes X,Y,Z}} (W \otimes X) \otimes (Y \otimes Z)$$

$$a_{W,X \otimes Y,Z} \downarrow \qquad \qquad \downarrow a_{W,X,Y \otimes Z}$$

$$W \otimes ((X \otimes Y) \otimes Z) \xrightarrow{\mathsf{id}_W \otimes a_{X,Y,Z}} W \otimes (X \otimes (Y \otimes Z))$$

is commutative for all objects W, X, Y, Z in $\mathcal{C}$.

2. The unit axiom. The functors

(2.3) $L_{\mathbf{1}}$: $X \mapsto \mathbf{1} \otimes X$ and

(2.4) $R_{\mathbf{1}}$: $X \mapsto X \otimes \mathbf{1}$

of left and right multiplication by $\mathbf{1}$ are autoequivalences of $\mathcal{C}$.

DEFINITION 2.1.2. The pair $(\mathbf{1}, \iota)$ is called the *unit object* of $\mathcal{C}$.[3]

REMARK 2.1.3. An alternative (and, perhaps, more traditional) definition of a monoidal category is given in the next section, see Definition 2.2.8.

We see that the set of isomorphism classes of objects in a monoidal category indeed has a natural structure of a monoid, with multiplication $\otimes$ and unit $\mathbf{1}$. Thus, in the categorical-algebraic dictionary, monoidal categories indeed correspond to monoids (which explains their name).

DEFINITION 2.1.4. A *monoidal subcategory* of a monoidal category $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ is a quintuple $(\mathcal{D}, \otimes, a, \mathbf{1}, \iota)$, where $\mathcal{D} \subset \mathcal{C}$ is a subcategory closed under the tensor product of objects and morphisms and containing $\mathbf{1}$ and ι.

Unless otherwise specified, we will always consider full monoidal subcategories.

DEFINITION 2.1.5. Let $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ be a monoidal category. The monoidal category $(\mathcal{C}^{\mathrm{op}}, \otimes^{\mathrm{op}}, \mathbf{1}, a^{\mathrm{op}}, \iota)$ *opposite* to $\mathcal{C}$ is defined as follows. As a category $\mathcal{C}^{\mathrm{op}} = \mathcal{C}$, its tensor product is given by $X \otimes^{\mathrm{op}} Y := Y \otimes X$ and the associativity constraint of $\mathcal{C}^{\mathrm{op}}$ is $a_{X,Y,Z}^{\mathrm{op}} := a_{Z,Y,X}^{-1}$.

REMARK 2.1.6. The notion of the opposite monoidal category is not to be confused with the usual notion of the *dual* category, which is the category $\mathcal{C}^{\vee}$ obtained from $\mathcal{C}$ by reversing arrows (for any category $\mathcal{C}$). Note that if $\mathcal{C}$ is monoidal, so is $\mathcal{C}^{\vee}$ (in a natural way), which makes it even easier to confuse the two notions.

2.2. Basic properties of unit objects

Let $(\mathcal{C}, \otimes, a, \mathbf{1}, \iota)$ be a monoidal category. Define natural isomorphisms

(2.5) $l_X : \mathbf{1} \otimes X \to X$ and $r_X : X \otimes \mathbf{1} \to X$

in such a way that $L_{\mathbf{1}}(l_X)$ and $R_{\mathbf{1}}(r_X)$ are equal, respectively, to the compositions

(2.6) $\mathbf{1} \otimes (\mathbf{1} \otimes X) \xrightarrow{a_{\mathbf{1},\mathbf{1},X}^{-1}} (\mathbf{1} \otimes \mathbf{1}) \otimes X \xrightarrow{\iota \otimes \mathsf{id}_X} \mathbf{1} \otimes X,$

(2.7) $(X \otimes \mathbf{1}) \otimes \mathbf{1} \xrightarrow{a_{X,\mathbf{1},\mathbf{1}}} X \otimes (\mathbf{1} \otimes \mathbf{1}) \xrightarrow{\mathsf{id}_X \otimes \iota} X \otimes \mathbf{1}.$

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[3]We note that there is no condition on the isomorphism ι, so it can be chosen arbitrarily.

DEFINITION 2.2.1. Isomorphisms (2.5) are called the left and right *unit constraints* or *unit isomorphisms*.

The unit constraints provide a categorical counterpart of the unit axiom $1X = X1 = X$ of a monoid in the same sense as the associativity isomorphism provides the categorical counterpart of the associativity equation.

PROPOSITION 2.2.2. *For any object X in $\mathcal{C}$ there are equalities*

$$(2.8) \qquad l_{\mathbf{1} \otimes X} = \mathsf{id}_{\mathbf{1}} \otimes l_X \quad and \quad r_{X \otimes \mathbf{1}} = r_X \otimes \mathsf{id}_{\mathbf{1}}.$$

PROOF. It follows from naturality of the left unit constraint l that the following diagram commutes

$$(2.9)$$

$$
\begin{array}{ccc}
\mathbf{1} \otimes (\mathbf{1} \otimes X) & \xrightarrow{\ \mathbf{1} \otimes l_X\ } & \mathbf{1} \otimes X \\
{\scriptstyle l_{\mathbf{1} \otimes X}} \downarrow & & \downarrow {\scriptstyle l_X} \\
\mathbf{1} \otimes X & \xrightarrow{\ l_X\ } & X.
\end{array}
$$

Since l_X is an isomorphism, the first identity follows. The second one follows similarly from naturality of r. $\qquad \square$

PROPOSITION 2.2.3. *The "triangle" diagram*

$$(2.10)$$

$$
\begin{array}{ccc}
(X \otimes \mathbf{1}) \otimes Y & \xrightarrow{\ a_{X,\mathbf{1},Y}\ } & X \otimes (\mathbf{1} \otimes Y) \\
& {\scriptstyle r_X \otimes \mathsf{id}_Y} \searrow \quad \swarrow {\scriptstyle \mathsf{id}_X \otimes l_Y} & \\
& X \otimes Y &
\end{array}
$$

is commutative for all $X, Y \in \mathcal{C}$.

PROOF. Consider the following diagram:

$$(2.11)$$

$$
\begin{array}{c}
((X \otimes \mathbf{1}) \otimes \mathbf{1}) \otimes Y \xrightarrow{\ \ a_{X,\mathbf{1},\mathbf{1}} \otimes \mathsf{id}_Y\ \ } (X \otimes (\mathbf{1} \otimes \mathbf{1})) \otimes Y
\end{array}
$$

with interior vertices $(X \otimes \mathbf{1}) \otimes Y$, $X \otimes (\mathbf{1} \otimes Y)$ and outer arrows labeled $r_X \otimes \mathsf{id}_{\mathbf{1}} \otimes \mathsf{id}_Y$, $(\mathsf{id}_X \otimes \iota) \otimes \mathsf{id}_Y$, $a_{X \otimes \mathbf{1}, \mathbf{1}, Y}$, $a_{X, \mathbf{1}, Y}$, $a_{X, \mathbf{1} \otimes \mathbf{1}, Y}$, and bottom vertices $(X \otimes \mathbf{1}) \otimes (\mathbf{1} \otimes Y)$, $X \otimes ((\mathbf{1} \otimes \mathbf{1}) \otimes Y)$, $X \otimes (\mathbf{1} \otimes (\mathbf{1} \otimes Y))$, with arrows $r_X \otimes \mathsf{id}_{\mathbf{1} \otimes Y}$, $\mathsf{id}_X \otimes l_{\mathbf{1} \otimes Y}$, $\mathsf{id}_X \otimes (\iota \otimes \mathsf{id}_Y)$, $a_{X, \mathbf{1}, \mathbf{1} \otimes Y}$, $\mathsf{id}_X \otimes a_{\mathbf{1}, \mathbf{1}, Y}$.

To prove the proposition, it suffices to establish the commutativity of the bottom left triangle (as any object of $\mathcal{C}$ is isomorphic to one of the form $\mathbf{1} \otimes Y$). Since the outside pentagon is commutative (by the pentagon axiom), it suffices to establish the commutativity of the other parts of the pentagon. Now, the two quadrangles are commutative due to the functoriality of the associativity isomorphisms, the

commutativity of the upper triangle is the definition of r, and the commutativity of the lower right triangle holds by Proposition 2.2.2. $\qquad\square$

PROPOSITION 2.2.4. *The following diagrams commute for all objects $X, Y \in \mathcal{C}$:*

$$(2.12)$$

$$(\mathbf{1} \otimes X) \otimes Y \xrightarrow{\ a_{\mathbf{1},X,Y}\ } \mathbf{1} \otimes (X \otimes Y)$$

$$l_X \otimes \mathrm{id}_Y \searrow \qquad \swarrow l_{X \otimes Y}$$

$$X \otimes Y$$

$$(2.13)$$

$$(X \otimes Y) \otimes \mathbf{1} \xrightarrow{\ a_{X,Y,\mathbf{1}}\ } X \otimes (Y \otimes \mathbf{1})$$

$$r_{X \otimes Y} \searrow \qquad \swarrow \mathrm{id}_X \otimes r_Y$$

$$X \otimes Y$$

PROOF. Consider the diagram

$$(2.14) \quad ((X \otimes \mathbf{1}) \otimes Y) \otimes Z \xrightarrow{\quad a_{X,\mathbf{1},Y} \otimes \mathrm{id}_Z \quad} (X \otimes (\mathbf{1} \otimes Y)) \otimes Z$$

$$(r_X \otimes \mathrm{id}_Y) \otimes \mathrm{id}_Z \qquad (\mathrm{id}_X \otimes l_Y) \otimes \mathrm{id}_Z$$

$$(X \otimes Y) \otimes Z$$

$$a_{X \otimes \mathbf{1}, Y, Z} \qquad a_{X,Y,Z} \qquad a_{X, \mathbf{1} \otimes Y, Z}$$

$$X \otimes (Y \otimes Z)$$

$$r_X \otimes \mathrm{id}_{Y \otimes Z} \qquad \mathrm{id}_X \otimes (l_Y \otimes \mathrm{id}_Z)$$

$$(X \otimes \mathbf{1}) \otimes (Y \otimes Z) \qquad \mathrm{id}_X \otimes l_{Y \otimes Z} \qquad X \otimes ((\mathbf{1} \otimes Y) \otimes Z)$$

$$a_{X,\mathbf{1},Y \otimes Z} \qquad \mathrm{id}_X \otimes a_{\mathbf{1},Y,Z}$$

$$X \otimes (\mathbf{1} \otimes (Y \otimes Z)),$$

where X, Y, Z are objects in $\mathcal{C}$. The outside pentagon commutes by the pentagon axiom (2.2). The functoriality of a implies the commutativity of the two middle quadrangles. The triangle axiom (2.10) implies the commutativity of the upper triangle and the lower left triangle. Consequently, the lower right triangle commutes as well. Setting $X = \mathbf{1}$ and applying the functor $L_{\mathbf{1}}^{-1}$ to the lower right triangle, we obtain commutativity of the triangle (2.12). The commutativity of the triangle (2.13) is proved similarly. $\qquad\square$

COROLLARY 2.2.5. *In any monoidal category $l_{\mathbf{1}} = r_{\mathbf{1}} = \iota$.*

PROOF. Set $Y = Z = \mathbf{1}$ in (2.12). We have:

$$l_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = l_{\mathbf{1} \otimes \mathbf{1}} \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}} = (\mathrm{id}_{\mathbf{1}} \otimes l_{\mathbf{1}}) \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}}.$$

Next, setting $X = Y = \mathbf{1}$ in the triangle axiom (2.10) we obtain

$$r_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = (\mathrm{id}_{\mathbf{1}} \otimes l_{\mathbf{1}}) \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}}.$$

By the definition of the unit constraint $(\mathrm{id}_{\mathbf{1}} \otimes l_{\mathbf{1}}) \circ a_{\mathbf{1},\mathbf{1},\mathbf{1}} = \iota \otimes \mathrm{id}_{\mathbf{1}}$. Hence, $r_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = l_{\mathbf{1}} \otimes \mathrm{id}_{\mathbf{1}} = \iota \otimes \mathrm{id}_{\mathbf{1}}$ and $r_{\mathbf{1}} = l_{\mathbf{1}} = \iota$ since $R_{\mathbf{1}}$ is an equivalence. $\qquad\square$

PROPOSITION 2.2.6. *The unit object in a monoidal category is unique up to a unique isomorphism.*

PROOF. Let $(\mathbf{1}, \iota)$, $(\mathbf{1}', \iota')$ be two unit objects. Let (r, l), (r', l') be the corresponding unit constraints. Then we have the isomorphism $\eta := l_{\mathbf{1}'} \circ (r'_{\mathbf{1}})^{-1} : \mathbf{1} \xrightarrow{\sim} \mathbf{1}'$.

It is easy to show using the commutativity of the above triangle diagrams that η maps ι to ι'. It remains to show that η is the only isomorphism with this property. To do so, it suffices to show that if $b : \mathbf{1} \xrightarrow{\sim} \mathbf{1}$ is an isomorphism such that the diagram

$$(2.15) \qquad \begin{array}{ccc} \mathbf{1} \otimes \mathbf{1} & \xrightarrow{b \otimes b} & \mathbf{1} \otimes \mathbf{1} \\ {\scriptstyle \iota} \downarrow & & \downarrow {\scriptstyle \iota} \\ \mathbf{1} & \xrightarrow{b} & \mathbf{1} \end{array}$$

is commutative, then $b = \mathsf{id}_{\mathbf{1}}$. To see this, it suffices to note that for any morphism $c : \mathbf{1} \to \mathbf{1}$ the diagram

$$(2.16) \qquad \begin{array}{ccc} \mathbf{1} \otimes \mathbf{1} & \xrightarrow{c \otimes \mathsf{id}_{\mathbf{1}}} & \mathbf{1} \otimes \mathbf{1} \\ {\scriptstyle \iota} \downarrow & & \downarrow {\scriptstyle \iota} \\ \mathbf{1} & \xrightarrow{c} & \mathbf{1} \end{array}$$

is commutative (since $\iota = r_{\mathbf{1}}$ by Corollary 2.2.5), so $b \otimes b = b \otimes \mathsf{id}_{\mathbf{1}}$ and hence $b = \mathsf{id}_{\mathbf{1}}$. $\qquad \square$

EXERCISE 2.2.7. Verify the assertion in the proof of Proposition 2.2.6 that η maps ι to ι'.

Hint: use Propositions 2.2.3 and 2.2.4.

The results of this section show that a monoidal category can be alternatively defined as follows:

DEFINITION 2.2.8. A monoidal category is a sextuple $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ satisfying the pentagon axiom (2.2) and the triangle axiom (2.10).

REMARK 2.2.9. Definition 2.2.8 is perhaps more traditional than Definition 2.1.1, but Definition 2.1.1 is simpler. Besides, Proposition 2.2.6 implies that for a triple $(\mathcal{C}, \otimes, a)$ satisfying a pentagon axiom (which should perhaps be called a "*semigroup category*", as it categorifies the notion of a semigroup), being a monoidal category is a property and not a structure (similarly to how it is for semigroups and monoids).

Furthermore, one can show that the commutativity of the triangles implies that in a monoidal category one can safely identify $\mathbf{1} \otimes X$ and $X \otimes \mathbf{1}$ with X using the unit isomorphisms, and assume that the unit isomorphisms are the identities (which we will usually do from now on).[4]

In a sense, all this means that in constructions with monoidal categories, unit objects and isomorphisms always "go along for the ride", and one need not worry about them especially seriously. For this reason, below we will typically take less care dealing with them than we have done in this section.

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[4]We will return to this issue later when we discuss Mac Lane's coherence theorem is Section 2.9.

PROPOSITION 2.2.10. *Let $\mathcal{C}$ be a monoidal category. Then $\mathsf{End}_{\mathcal{C}}(\mathbf{1})$ is a commutative monoid under composition. Furthermore, $f \otimes g = \iota^{-1} \circ (f \circ g) \circ \iota$ for all $f, g \in \mathsf{End}_{\mathcal{C}}(\mathbf{1})$.*

PROOF. By naturality of unit constraints of $\mathcal{C}$ we have

$$f \otimes \mathsf{id}_{\mathbf{1}} = r_{\mathbf{1}}^{-1} \circ f \circ r_{\mathbf{1}} \quad \text{and} \quad \mathsf{id}_{\mathbf{1}} \otimes g = l_{\mathbf{1}}^{-1} \circ g \circ l_{\mathbf{1}}.$$

Combining this with the identity $r_{\mathbf{1}} = l_{\mathbf{1}} = \iota$ from Corollary 2.2.5 we obtain

$$
\begin{aligned}
f \otimes g &= (f \otimes \mathsf{id}_{\mathbf{1}}) \circ (\mathsf{id}_{\mathbf{1}} \otimes g) = \iota^{-1} \circ (f \circ g) \circ \iota, \\
g \otimes f &= (\mathsf{id}_{\mathbf{1}} \otimes f) \circ (g \otimes \mathsf{id}_{\mathbf{1}}) = \iota^{-1} \circ (f \circ g) \circ \iota,
\end{aligned}
$$

whence we obtain the result. □

2.3. First examples of monoidal categories

Monoidal categories are ubiquitous. You will see one whichever way you look. Here are some examples.

EXAMPLE 2.3.1. The category **Sets** of sets is a monoidal category, where the tensor product is the Cartesian product and the unit object is a one element set; the structure morphisms a, ι, l, r are obvious. The same holds for the subcategory of finite sets, which will be denoted by Sets [5]. This example can be widely generalized: one can take the category of sets with some structure, such as groups, topological spaces, etc.

EXAMPLE 2.3.2. Any additive category (see Definition 1.2.1) is monoidal, with $\otimes$ being the direct sum functor $\oplus$, and $\mathbf{1}$ being the zero object.

The remaining examples will be especially important below. Let $\Bbbk$ be any field.

EXAMPLE 2.3.3. The category $\Bbbk - \mathbf{Vec}$ of all $\Bbbk$-vector spaces is a monoidal category, where $\otimes = \otimes_{\Bbbk}$, $\mathbf{1} = \Bbbk$, and the morphisms a, ι, l, r are the obvious ones. The same is true about the category of finite dimensional vector spaces over $\Bbbk$, denoted by $\Bbbk - \mathsf{Vec}$. We will often drop $\Bbbk$ from the notation when no confusion is possible.

More generally, if R is a commutative unital ring, then replacing $\Bbbk$ by R we can define monoidal categories $R - \mathbf{mod}$ of R-modules and $R - \mathsf{mod}$ of R-modules of finite type.

EXAMPLE 2.3.4. Let G be a group. The category $\mathbf{Rep}_{\Bbbk}(G)$ of all representations of G over $\Bbbk$ is a monoidal category, with $\otimes$ being the tensor product of representations: if for a representation V one denotes by ρ_V the corresponding map $G \to GL(V)$, then

$$\rho_{V \otimes W}(g) := \rho_V(g) \otimes \rho_W(g).$$

The unit object in this category is the trivial representation $\mathbf{1} = \Bbbk$. A similar statement holds for the category $\mathsf{Rep}_{\Bbbk}(G)$ of finite dimensional representations of G. Again, we will drop the subscript $\Bbbk$ when no confusion is possible.

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[5]Here and below, the absence of a finiteness condition is indicated by the **boldface** font, while its presence is indicated by the Roman font.

EXAMPLE 2.3.5. Let G be an affine (pro)algebraic group[6] over $\Bbbk$.

The categories $\mathbf{Rep}(G)$ and $\mathsf{Rep}(G)$ of algebraic representations of G over $\Bbbk$ are monoidal categories (similarly to Example 2.3.4).

Similarly, if $\mathfrak{g}$ is a Lie algebra over $\Bbbk$, then the category of its representations $\mathbf{Rep}(\mathfrak{g})$ and the category of its finite dimensional representations $\mathsf{Rep}(\mathfrak{g})$ are monoidal categories: the tensor product is defined by

$$\rho_{V \otimes W}(a) = \rho_V(a) \otimes \mathsf{id}_W + \mathsf{id}_V \otimes \rho_W(a)$$

(where $\rho_Y : \mathfrak{g} \to \mathfrak{gl}(Y)$ is the homomorphism associated to a representation Y of $\mathfrak{g}$), and $\mathbf{1}$ is the 1-dimensional representation with the zero action of $\mathfrak{g}$.

EXAMPLE 2.3.6. Let G be a monoid (which we will usually take to be a group), and let A be an abelian group (with operation written multiplicatively). Let $\mathcal{C}_G = \mathcal{C}_G(A)$ be the category whose objects δ_g are labeled by elements of G (so there is only one object in each isomorphism class), $\mathsf{Hom}_{\mathcal{C}_G}(\delta_{g_1}, \delta_{g_2}) = \emptyset$ if $g_1 \neq g_2$, and $\mathsf{Hom}_{\mathcal{C}_G}(\delta_g, \delta_g) = A$, with the functor $\otimes$ defined by $\delta_g \otimes \delta_h = \delta_{gh}$, and the tensor product of morphisms defined by $a \otimes b = ab$. Then $\mathcal{C}_G$ is a monoidal category with the associativity isomorphism being the identity, and $\mathbf{1}$ being the unit element of G. This shows that in a monoidal category, $X \otimes Y$ need not be isomorphic to $Y \otimes X$ (indeed, it suffices to take a non-commutative monoid G).

This example has a "linear" version. Namely, let $\Bbbk$ be a field, and $\Bbbk - \mathbf{Vec}_G$ denote the category of G-graded vector spaces over $\Bbbk$, i.e., vector spaces V with a decomposition $V = \bigoplus_{g \in G} V_g$. Morphisms in this category are linear maps which preserve the grading. Define the tensor product on this category by the formula

$$(2.17) \qquad (V \otimes W)_g = \bigoplus_{x,y \in G : xy = g} V_x \otimes W_y,$$

and the unit object $\mathbf{1}$ by $\mathbf{1}_1 = \Bbbk$ and $\mathbf{1}_g = 0$ for $g \neq 1$. Then, defining a, ι in an obvious way, we equip $\Bbbk - \mathbf{Vec}_G$ with the structure of a monoidal category. Similarly one defines the monoidal category $\Bbbk - \mathsf{Vec}_G$ of finite dimensional G-graded $\Bbbk$-vector spaces.

In the category $\Bbbk - \mathbf{Vec}_G$, we have pairwise non-isomorphic objects δ_g, $g \in G$, defined by the formula $(\delta_g)_x = \Bbbk$ if $x = g$ and $(\delta_g)_x = 0$ otherwise. For these objects, we have $\delta_g \otimes \delta_h \cong \delta_{gh}$. Thus the category $\mathcal{C}_G(\Bbbk^\times)$ is a non-full monoidal subcategory of $\Bbbk - \mathbf{Vec}_G$ (since the zero morphisms are missing). This subcategory can be viewed as a "basis" of $\Bbbk - \mathbf{Vec}_G$ (and $\Bbbk - \mathbf{Vec}_G$ as "the linear span" of $\mathcal{C}_G(\Bbbk^\times)$), as any object of $\Bbbk - \mathbf{Vec}_G$ is isomorphic to a direct sum of objects δ_g with non-negative integer multiplicities.

When no confusion is possible, we will denote the categories $\Bbbk - \mathsf{Vec}_G$, $\Bbbk - \mathbf{Vec}_G$ simply by Vec_G, $\mathbf{Vec}_G$.

EXERCISE 2.3.7. Let G be a group, and A an abelian group with an action $\rho : G \to \mathsf{Aut}(A)$. Define the category $\mathcal{C}_G(A, \rho)$ in the same way as $\mathcal{C}_G(A)$, except that the tensor product of morphisms is defined as follows: if $a : \delta_g \to \delta_g$ and $b : \delta_h \to \delta_h$ then $a \otimes b = a g(b)$, where $g(b) := \rho(g)b$. Show that $\mathcal{C}_G(A, \rho)$ is a monoidal category.

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[6]Recall that an *affine algebraic group* over $\Bbbk$ is an affine algebraic variety with a group structure, such that the multiplication and inversion maps are regular, and that an *affine proalgebraic group* is an inverse limit of affine algebraic groups. A typical example of an affine proalgebraic group which is not an algebraic group is the group $G(\Bbbk[[t]])$ of formal series valued points of an affine algebraic group G defined over $\Bbbk$.

EXAMPLE 2.3.8. Here is a generalization of Example 2.3.6, which shows that the associativity isomorphism is not always "the obvious one".

Let G be a group, let A be an abelian group, and let ω be a 3-cocycle of G with values in A. This means that $\omega : G \times G \times G \to A$ is a function satisfying the equation

$$(2.18) \qquad \omega(g_1 g_2, g_3, g_4)\omega(g_1, g_2, g_3 g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2 g_3, g_4)\omega(g_2, g_3, g_4),$$

for all $g_1, g_2, g_3, g_4 \in G$.

Let us define the monoidal category $\mathcal{C}_G^\omega = \mathcal{C}_G^\omega(A)$ as follows. As a category, it is the same as the category $\mathcal{C}_G$ defined in Example 2.3.6. The bifunctor $\otimes$ and the unit object $(\mathbf{1}, \iota)$ in this category are also the same as those in $\mathcal{C}_G$. The only difference is in the new associativity isomorphism a^ω, which is not the identity as in $\mathcal{C}_G$, but instead it is defined by the formula

$$(2.19) \qquad a^\omega_{\delta_g, \delta_h, \delta_m} = \omega(g, h, m)\,\mathsf{id}_{\delta_{ghm}} : (\delta_g \otimes \delta_h) \otimes \delta_m \to \delta_g \otimes (\delta_h \otimes \delta_m),$$

where $g, h, m \in G$.

The fact that $\mathcal{C}_G^\omega$ with these structures is indeed a monoidal category follows from the properties of ω. Namely, the pentagon axiom (2.2) follows from equation (2.18), and the unit axiom is obvious.

Similarly, for a field $\Bbbk$ and a 3-cocycle ω with values in $\Bbbk^\times$ one can define the category $\Bbbk-\mathbf{Vec}_G^\omega$, which differs from $\mathbf{Vec}_G$ just by the associativity isomorphism. This is done by extending the associativity isomorphism of $\mathcal{C}_G^\omega$ by additivity to arbitrary direct sums of objects δ_g. This category contains a monoidal subcategory Vec_G^ω of finite dimensional G-graded vector spaces with associativity defined by ω.

EXERCISE 2.3.9. Verify that the unit morphisms l and r in $\mathbf{Vec}_G^\omega$ are given on 1-dimensional spaces by the formulas

$$l_{\delta_g} = \omega(1,\, 1,\, g)^{-1}\,\mathsf{id}_{\delta_g},\ \ r_{\delta_g} = \omega(g,\, 1,\, 1)\,\mathsf{id}_{\delta_g},$$

and the triangle axiom says that $\omega(g,\, 1,\, h) = \omega(g,\, 1,\, 1)\omega(1,\, 1,\, h)$. Thus, we have $l_X = r_X = \mathsf{id}_X$ for all X if and only if

$$(2.20) \qquad\qquad\qquad \omega(g,\, 1,\, 1) = \omega(1,\, 1,\, g) = 1,$$

for any $g \in G$ or, equivalently,

$$(2.21) \qquad\qquad\qquad \omega(g,\, 1,\, h) = 1, \quad g,\, h \in G.$$

A cocycle satisfying this condition is said to be *normalized*.

REMARK 2.3.10. We will show in Proposition 2.6.1 that cohomologically equivalent ω's give rise to equivalent monoidal categories.

REMARK 2.3.11. In Section 2.11 we will consider monoidal categories generalizing Examples 2.3.6 and 2.3.8 and Exercise 2.3.7 – the so-called *Gr-categories*, or *categorical groups*.

EXAMPLE 2.3.12. Let $\mathcal{C}$ be a category. Then the category $\mathsf{End}(\mathcal{C})$ of all functors from $\mathcal{C}$ to itself is a monoidal category, where $\otimes$ is given by composition of functors. The associativity isomorphism in this category is the identity. The unit object is the identity functor, and the structure morphisms are obvious. If $\mathcal{C}$ is an abelian category, then the categories of additive, left exact, right exact, and exact endofunctors of $\mathcal{C}$ are monoidal.

EXAMPLE 2.3.13. Let A be an associative ring with unit. Then the category $A-\textbf{bimod}$ of bimodules over A is a monoidal category, with $\otimes$ being the tensor product $\otimes_A$ over A. The unit object in this category is the ring A itself (regarded as an A-bimodule).

If A is commutative, this category has a full monoidal subcategory $A-\textbf{mod}$, consisting of A-modules, regarded as bimodules in which the left and right actions of A coincide. More generally, if X is a scheme, one can define the monoidal category $\textbf{QCoh}(X)$ of quasicoherent sheaves on X; if X is affine and $A = \mathcal{O}_X$, then $\textbf{QCoh}(X) = A-\textbf{mod}$.

Similarly, if A is a finite dimensional algebra, we can define the monoidal category $A-\textsf{bimod}$ of finite dimensional A-bimodules. Other similar examples which often arise in geometry are the category $\mathrm{Coh}(X)$ of coherent sheaves on a Noetherian scheme X, its subcategory $\mathrm{VB}(X)$ of vector bundles (i.e., locally free coherent sheaves) on X, and the category $\mathrm{Loc}(X)$ of locally constant sheaves of finite dimensional $\Bbbk$-vector spaces (also called local systems) on any topological space X. All of these are monoidal categories in a natural way.

EXAMPLE 2.3.14. **The category of tangles.** Let $S_{m,n}$ be the disjoint union of m circles $\mathbb{R}/\mathbb{Z}$ and n intervals $[0,1]$. A *tangle* is a smooth embedding $f : S_{m,n} \to \mathbb{R}^2 \times [0,1]$ such that the boundary maps to the boundary and the interior to the interior. We will abuse the terminology by also using the term "tangle" for the image of f.

Let x, y, z be the Cartesian coordinates on $\mathbb{R}^2 \times [0,1]$. Any tangle has inputs (points of the image of f with $z = 0$) and outputs (points of the image of f with $z = 1$). For any integers $p, q \geq 0$, let $\widetilde{T}_{p,q}$ be the set of all tangles which have p inputs and q outputs, all having a vanishing y-coordinate. Let $T_{p,q}$ be the set of isotopy classes of elements of $\widetilde{T}_{p,q}$; thus, during an isotopy, the inputs and outputs are allowed to move (preserving the condition $y = 0$), but cannot meet each other. We can define a canonical composition map $T_{p,q} \times T_{q,r} \to T_{p,r}$, induced by the concatenation of tangles. Namely, if $s \in T_{p,q}$ and $t \in T_{q,r}$, we pick representatives $\widetilde{s} \in \widetilde{T}_{p,q}, \widetilde{t} \in \widetilde{T}_{q,r}$ such that the inputs of $\widetilde{t}$ coincide with the outputs of $\widetilde{s}$, concatenate them, perform an appropriate reparametrization, and rescale $z \to z/2$. The obtained tangle represents the desired composition ts.

We will now define a monoidal category $\mathcal{T}$ called the category of tangles. The objects of this category are non-negative integers, and the morphisms are defined by $\mathsf{Hom}_\mathcal{T}(p, q) = T_{p,q}$, with composition as above. The identity morphisms are the elements $\mathsf{id}_p \in T_{p,p}$ represented by p vertical intervals and no circles (in particular, if $p = 0$, the identity morphism id_p is the empty tangle).

Now let us define the monoidal structure on the category $\mathcal{T}$. The tensor product of objects is defined by $m \otimes n = m + n$. However, we also need to define the tensor product of morphisms. This tensor product is induced by union of tangles. Namely, if $t_1 \in T_{p_1,q_1}$ and $t_2 \in T_{p_2,q_2}$, we pick representatives $\widetilde{t_1} \in \widetilde{T}_{p_1,q_1}, \widetilde{t_2} \in \widetilde{T}_{p_2,q_2}$ in such a way that any point of $\widetilde{t_1}$ is to the left of any point of $\widetilde{t_2}$ (i.e., has a smaller x-coordinate). Then $t_1 \otimes t_2$ is represented by the tangle $\widetilde{t_1} \cup \widetilde{t_2}$.

EXERCISE 2.3.15. Check the following:

(1) The tensor product $t_1 \otimes t_2$ is well defined, and its definition makes $\otimes$ a bifunctor.

(2) There is an obvious associativity isomorphism for $\otimes$, which turns $\mathcal{T}$ into a monoidal category (with unit object being the empty tangle).

2.4. Monoidal functors and their morphisms

As we have explained, the notion of a monoidal category is a categorification of the notion of a monoid. Now we pass to categorification of morphisms between monoids, namely, to monoidal functors.

DEFINITION 2.4.1. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}^l, \otimes^l, \mathbf{1}^l, a^l, \iota^l)$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}^l$ is a pair (F, J), where $F : \mathcal{C} \to \mathcal{C}^l$ is a functor, and

$$\text{(2.22)} \qquad J_{X,Y} : F(X) \otimes^l F(Y) \xrightarrow{\sim} F(X \otimes Y)$$

is a natural isomorphism, such that $F(\mathbf{1})$ is isomorphic to $\mathbf{1}^l$ and the diagram

$$\text{(2.23)} \qquad
\begin{array}{ccc}
(F(X) \otimes^l F(Y)) \otimes^l F(Z) & \xrightarrow{a^l_{F(X),F(Y),F(Z)}} & F(X) \otimes^l (F(Y) \otimes^l F(Z)) \\
{\scriptstyle J_{X,Y} \otimes^l \mathrm{id}_{F(Z)}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{F(X)} \otimes^l J_{Y,Z}} \\
F(X \otimes Y) \otimes^l F(Z) & & F(X) \otimes^l F(Y \otimes Z) \\
{\scriptstyle J_{X \otimes Y, Z}} \downarrow & & \downarrow {\scriptstyle J_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z))
\end{array}$$

is commutative for all X, Y, $Z \in \mathcal{C}$ ("the monoidal structure axiom").

A monoidal functor F is said to be an *equivalence of monoidal categories* if it is an equivalence of ordinary categories.

REMARK 2.4.2. It is important to stress that, as seen from this definition, a monoidal functor is not just a functor between monoidal categories, but a functor with an additional structure (the isomorphism J) satisfying a certain equation (the monoidal structure axiom). As we will see in Section 2.5, this equation may have more than one solution or no solutions at all, so the same functor can be equipped with different monoidal structures or not admit any monoidal structure at all.

It turns out that if F is a monoidal functor, then there is a canonical isomorphism $\varphi : \mathbf{1}^l \to F(\mathbf{1})$. This isomorphism is defined by the commutative diagram

$$\text{(2.24)} \qquad
\begin{array}{ccc}
\mathbf{1}^l \otimes^l F(\mathbf{1}) & \xrightarrow{l^l_{F(\mathbf{1})}} & F(\mathbf{1}) \\
{\scriptstyle \varphi \otimes^l \mathrm{id}_{F(\mathbf{1})}} \downarrow & & \downarrow {\scriptstyle F(l_{\mathbf{1}})^{-1}} \\
F(\mathbf{1}) \otimes^l F(\mathbf{1}) & \xrightarrow{J_{\mathbf{1},\mathbf{1}}} & F(\mathbf{1} \otimes \mathbf{1})
\end{array}$$

where l, r, l^l, r^l are the unit isomorphisms for $\mathcal{C}$ and $\mathcal{C}^l$ defined in (2.5).

PROPOSITION 2.4.3. *For any monoidal functor* $(F, J) : \mathcal{C} \to \mathcal{C}^l$, *the diagrams*

(2.25)
$$
\begin{array}{ccc}
\mathbf{1}^l \otimes^l F(X) & \xrightarrow{\ l^l_{F(X)}\ } & F(X) \\
{\scriptstyle \varphi \otimes^l \mathrm{id}_{F(X)}} \downarrow & & \downarrow {\scriptstyle F(l_X)^{-1}} \\
F(\mathbf{1}) \otimes^l F(X) & \xrightarrow{\ J_{\mathbf{1},X}\ } & F(\mathbf{1} \otimes X)
\end{array}
$$

and

(2.26)
$$
\begin{array}{ccc}
F(X) \otimes^l \mathbf{1}^l & \xrightarrow{\ r^l_{F(X)}\ } & F(X) \\
{\scriptstyle \mathrm{id}_{F(X)} \otimes^l \varphi} \downarrow & & \downarrow {\scriptstyle F(r_X)^{-1}} \\
F(X) \otimes^l F(\mathbf{1}) & \xrightarrow{\ J_{X,\mathbf{1}}\ } & F(X \otimes \mathbf{1})
\end{array}
$$

are commutative for all $X \in \mathcal{C}$.

EXERCISE 2.4.4. Prove Proposition 2.4.3.

Proposition 2.4.3 implies that a monoidal functor can be equivalently defined as follows.

DEFINITION 2.4.5. A monoidal functor $\mathcal{C} \to \mathcal{C}^l$ is a triple (F, J, φ) which satisfies the monoidal structure axiom and Proposition 2.4.3.

Definition 2.4.5 is a more traditional definition of a monoidal functor.

REMARK 2.4.6. It can be seen from the above that for any monoidal functor (F, J) one can safely identify $\mathbf{1}^l$ with $F(\mathbf{1})$ using the isomorphism φ, and assume that $F(\mathbf{1}) = \mathbf{1}^l$ and $\varphi = \mathrm{id}_{\mathbf{1}^l}$ (similarly to how we have identified $\mathbf{1} \otimes X$ and $X \otimes \mathbf{1}$ with X and assumed that $l_X = r_X = \mathrm{id}_X$). We will usually do so from now on. Proposition 2.4.3 implies that with these conventions, one has

(2.27)
$$
J_{\mathbf{1},X} = J_{X,\mathbf{1}} = \mathrm{id}_X .
$$

REMARK 2.4.7. It is clear that the composition of monoidal functors is a monoidal functor. Also, the identity functor has a natural structure of a monoidal functor.

Monoidal functors between two monoidal categories themselves form a category. Namely, one has the following notion of a morphism (or natural transformation) between two monoidal functors.

DEFINITION 2.4.8. Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ and $(\mathcal{C}^l, \otimes^l, \mathbf{1}^l, a^l, \iota^l)$ be two monoidal categories, and let (F^1, J^1) and (F^2, J^2) be two monoidal functors from $\mathcal{C}$ to $\mathcal{C}^l$. A *morphism* (or a *natural transformation*) of monoidal functors $\eta : (F^1, J^1) \to (F^2, J^2)$ is a natural transformation $\eta : F^1 \to F^2$ such that $\eta_{\mathbf{1}}$ is an isomorphism, and the diagram

(2.28)
$$
\begin{array}{ccc}
F^1(X) \otimes^l F^1(Y) & \xrightarrow{\ J^1_{X,Y}\ } & F^1(X \otimes Y) \\
{\scriptstyle \eta_X \otimes^l \eta_Y} \downarrow & & \downarrow {\scriptstyle \eta_{X \otimes Y}} \\
F^2(X) \otimes^l F^2(Y) & \xrightarrow{\ J^2_{X,Y}\ } & F^2(X \otimes Y)
\end{array}
$$

is commutative for all $X, Y \in \mathcal{C}$.

REMARK 2.4.9. It is easy to show that if $\varphi_i : \mathbf{1}^l \xrightarrow{\sim} F^i(\mathbf{1})$, $i = 1, 2$, are isomorphisms defined by (2.24) then $\eta_{\mathbf{1}} \circ \varphi_1 = \varphi_2$, so if one makes the convention that $\varphi_1 = \varphi_2 = \mathsf{id}_{\mathbf{1}^l}$, one has $\eta_{\mathbf{1}} = \mathsf{id}_{\mathbf{1}^l}$.

REMARK 2.4.10. It is easy to show that if $F : \mathcal{C} \to \mathcal{C}^l$ is an equivalence of monoidal categories, then there exists a monoidal equivalence $F^{-1} : \mathcal{C}^l \to \mathcal{C}$ such that the functors $F \circ F^{-1}$ and $F^{-1} \circ F$ are isomorphic to the identity functor as monoidal functors. Thus, for any monoidal category $\mathcal{C}$, the monoidal auto-equivalences of $\mathcal{C}$ up to isomorphism form a group with respect to composition.

2.5. Examples of monoidal functors

Let us now give some examples of monoidal functors and natural transformations.

EXAMPLE 2.5.1. An important class of examples of monoidal functors is *forgetful functors* (e.g., functors of "forgetting the structure", from the categories of groups, topological spaces, etc., to the category of sets). Such functors have an obvious monoidal structure. An example especially important in this book is the forgetful functor $\mathbf{Rep}(G) \to \mathbf{Vec}$ from the representation category of a group to the category of vector spaces. More generally, if $H \subset G$ is a subgroup, then we have a forgetful (or restriction) functor $\mathbf{Rep}(G) \to \mathbf{Rep}(H)$. Still more generally, if $f : H \to G$ is a group homomorphism, then we have the pullback functor $f^* : \mathbf{Rep}(G) \to \mathbf{Rep}(H)$. All these functors are monoidal.

EXAMPLE 2.5.2. Let $f : H \to G$ be a homomorphism of groups. Then any H-graded vector space is naturally G-graded (by pushforward of grading). Thus we have a natural monoidal functor $f_* : \mathbf{Vec}_H \to \mathbf{Vec}_G$. If G is the trivial group, then f_* is just the forgetful functor $\mathbf{Vec}_H \to \mathbf{Vec}$.

EXAMPLE 2.5.3. Let $\Bbbk$ be a field, let A be a $\Bbbk$-algebra with unit, and let $\mathcal{C} = A-\mathbf{mod}$ be the category of left A-modules. Then we have a functor

$$(2.29) \qquad\qquad F : M \mapsto (M \otimes_A -) : A-\mathbf{bimod} \to \mathsf{End}(\mathcal{C}).$$

This functor is naturally monoidal. A similar functor $F : A-\mathsf{bimod} \to \mathsf{End}(\mathcal{C})$ can be defined if A is a finite dimensional $\Bbbk$-algebra, and $\mathcal{C} = A-\mathsf{mod}$ is the category of finite dimensional left A-modules.

PROPOSITION 2.5.4. *The functor* (2.29) *takes values in the full monoidal subcategory* $\mathsf{End}_{re}(\mathcal{C})$ *of right exact endofunctors of* $\mathcal{C}$, *and defines an equivalence between the monoidal categories* $A-\mathsf{bimod}$ *and* $\mathsf{End}_{re}(\mathcal{C})$.

PROOF. The first statement is clear, since the tensor product functor is right exact. To prove the second statement, let us construct the quasi-inverse functor F^{-1}. Let $G \in \mathsf{End}_{re}(\mathcal{C})$. Define $F^{-1}(G)$ by the formula $F^{-1}(G) = G(A)$; this is clearly an A-bimodule, since it is a left A-module with a commuting action of $\mathrm{End}_A(A) = A^{\mathrm{op}}$ (the opposite algebra). We leave it to the reader to check that the functor F^{-1} is indeed a quasi-inverse to F (cf. Proposition 1.8.10). $\square$

REMARK 2.5.5. A similar statement is valid without the finite dimensionality assumption, if one adds the condition that the right exact functors must commute with inductive limits.

EXAMPLE 2.5.6. Let S be a monoid, and let $\mathcal{C} = \mathsf{Vec}_S$ (see Example 2.3.6). Let us view $\mathsf{id}_{\mathcal{C}}$, the identity functor of $\mathcal{C}$, as a monoidal functor. It is easy to see that morphisms $\eta : \mathsf{id}_{\mathcal{C}} \to \mathsf{id}_{\mathcal{C}}$ as monoidal functors correspond to homomorphisms of monoids: $\eta : S \to \Bbbk$ (where $\Bbbk$ is equipped with the multiplication operation). In particular, $\eta(s)$ may be 0 for some s, so η does not have to be an isomorphism.

2.6. Monoidal functors between categories of graded vector spaces

Let G_1, G_2 be groups, let A be an abelian group, and let $\omega_i \in Z^3(G_i, A)$, $i = 1, 2$, be 3-cocycles (the actions of G_1, G_2 on A are assumed to be trivial). Let $\mathcal{C}_i = \mathcal{C}_{G_i}^{\omega_i}$, $i = 1, 2$, be the monoidal categories of graded vector spaces introduced in Example 2.3.8.

Any monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defines, by restriction to simple objects, a group homomorphism $f : G_1 \to G_2$. Using axiom (2.23) of a monoidal functor, we see that a monoidal structure on F is given by

$$(2.30) \qquad J_{g,h} = \mu(g, h) \, \mathsf{id}_{\delta_{f(gh)}} : F(\delta_g) \otimes F(\delta_h) \xrightarrow{\sim} F(\delta_{gh}), \ g, h \in G_1,$$

where $\mu : G_1 \times G_1 \to A$ is a function such that

$$\omega_1(g, h, l)\mu(gh, l)\mu(g, h) = \mu(g, hl)\mu(h, l)\omega_2(f(g), f(h), f(l)),$$

for all $g, h, l \in G_1$. That is,

$$(2.31) \qquad \omega_1 = f^* \omega_2 \cdot d_3(\mu),$$

i.e., ω_1 and $f^* \omega_2$ are cohomologous in $Z^3(G_1, A)$.

Conversely, given a group homomorphism $f : G_1 \to G_2$, any function

$$\mu : G_1 \times G_1 \to A$$

satisfying (2.31) gives rise to a monoidal functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ defined by $F(\delta_g) = \delta_{f(g)}$ with the monoidal structure given by formula (2.30). This functor is an equivalence if and only if f is an isomorphism.

To summarize, monoidal functors $\mathcal{C}_{G_1}^{\omega_1} \to \mathcal{C}_{G_2}^{\omega_2}$ correspond to pairs (f, μ), where $f : G_1 \to G_2$ is a group homomorphism such that ω_1 and $f^* \omega_2$ are cohomologous, and μ is a function satisfying (2.31) (such functions are in a (non-canonical) bijection with A-valued 2-cocycles on G_1). Let $F_{f,\mu}$ denote the corresponding functor.

Let us determine natural monoidal transformations between $F_{f,\mu}$ and $F_{f',\mu'}$. Clearly, such a transformation exists if and only if $f = f'$, is always an isomorphism, and is determined by a collection of morphisms $\eta_g : \delta_{f(g)} \to \delta_{f(g)}$ (i.e., $\eta_g \in A$), satisfying the equation

$$(2.32) \qquad \mu'(g, h)(\eta_g \otimes \eta_h) = \eta_{gh}\mu(g, h)$$

for all $g, h \in G_1$, i.e.,

$$(2.33) \qquad \mu = \mu' \cdot d_2(\eta).$$

Conversely, every function $\eta : G_1 \to A$ satisfying (2.33) gives rise to a morphism of monoidal functors $\eta : F_{f,\mu} \to F_{f,\mu'}$ defined as above. Therefore, monoidal functors $F_{f,\mu}$ and $F_{f',\mu'}$ are isomorphic if and only if $f = f'$ and μ is cohomologous to μ'.

Thus, we have obtained the following proposition.

PROPOSITION 2.6.1. (i) *The set of isomorphisms between monoidal functors $F_{f,\mu}$, $F_{f,\mu'} : \mathcal{C}_{G_1}^{\omega_1} \to \mathcal{C}_{G_2}^{\omega_2}$ is a torsor over the group $H^1(G_1, A) = \mathsf{Hom}(G_1, A)$.*

(ii) *For a fixed homomorphism $f : G_1 \to G_2$, the set of μ parameterizing isomorphism classes of monoidal functors $F_{f,\mu}$ is a torsor over $H^2(G_1, A)$.*

(iii) *Equivalence classes of monoidal categories $\mathcal{C}_G^\omega$ are parametrized by the set $H^3(G, A)/\mathsf{Out}(G)$, where $\mathsf{Out}(G)$ denotes the group of outer automorphisms of G.* [7]

REMARK 2.6.2. The same results, including Proposition 2.6.1, are valid if we specialize to the case when $A = \Bbbk^\times$, where $\Bbbk$ is a field, replace the categories $\mathcal{C}_G^\omega$ by their "linear spans" Vec_G^ω, and require that the monoidal functors we consider are additive. To see this, it is enough to note that by definition, for any morphism η of monoidal functors, $\eta_1 \neq 0$, so equation (2.32) (with $h = g^{-1}$) implies that all η_g must be nonzero. Thus, if a morphism $\eta : F_{f,\mu} \to F_{f',\mu'}$ exists, then it is an isomorphism, and we must have $f = f'$.

REMARK 2.6.3. The above discussion implies that in the definition of the categories $\mathcal{C}_G^\omega$ and Vec_G^ω, it may be assumed without loss of generality that the cocycle ω is normalized, i.e., $\omega(g, 1, h) = 1$, and thus $l_{\delta_g} = r_{\delta_g} = \mathsf{id}_{\delta_g}$ (which is convenient in computations). Indeed, we claim that any 3-cocycle ω is cohomologous to a normalized one. To see this, it is enough to alter ω by dividing it by $d_2(\mu)$, where μ is any 2-cochain such that $\mu(g, 1) = \omega(g, 1, 1)$, and $\mu(1, h) = \omega(1, 1, h)^{-1}$.

EXAMPLE 2.6.4. Let $G = \mathbb{Z}/n\mathbb{Z}$, where $n > 1$ is an integer, and $\Bbbk = \mathbb{C}$. Consider the cohomology of $\mathbb{Z}/n\mathbb{Z}$.

Since $H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}) = 0$ for all $i > 0$, writing the long exact sequence of cohomology for the short exact sequence of coefficient groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}^\times = \mathbb{C}/\mathbb{Z} \longrightarrow 0,$$

we obtain a natural isomorphism $H^i(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \cong H^{i+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$.

As we saw in Exercise 1.7.5, the graded ring $H^*(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z})$ is

$$H^*(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}[x]/(nx) = \mathbb{Z} \oplus x(\mathbb{Z}/n\mathbb{Z})[x],$$

where x is a generator in degree 2. Moreover, as a module over $\mathsf{Aut}(\mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^\times$, we have $H^2(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = (\mathbb{Z}/n\mathbb{Z})^\vee$. Therefore, using the graded ring structure, we find that

$$H^{2m-1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) \cong H^{2m}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}) \cong ((\mathbb{Z}/n\mathbb{Z})^\vee)^{\otimes m}$$

as an $\mathsf{Aut}(\mathbb{Z}/n\mathbb{Z})$-module. In particular, $H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times) = ((\mathbb{Z}/n\mathbb{Z})^\vee)^{\otimes 2}$.

Let us give an explicit formula for the 3-cocycles on $\mathbb{Z}/n\mathbb{Z}$. Modulo coboundaries, these cocycles are given by

(2.34) $$\phi(i, j, k) = \varepsilon^{\frac{si(j+k-(j+k)')}{n}},$$

where ε is a primitive nth root of unity, $s \in \mathbb{Z}/n\mathbb{Z}$, and for an integer m we denote by m' the remainder of division of m by n.

EXERCISE 2.6.5. Show that when s runs over $\mathbb{Z}/n\mathbb{Z}$, this formula defines cocycles representing all the cohomology classes in $H^3(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times)$.

---

[7]Recall that the group $\mathsf{Inn}(G)$ of inner automorphisms of a group G acts trivially on $H^*(G, A)$ (for any coefficient group A), and thus the action of the group $\mathsf{Aut}(G)$ on $H^*(G, A)$ factors through $\mathsf{Out}(G)$.

EXERCISE 2.6.6. Derive an explicit formula for cocycles representing cohomology classes in $H^{2m+1}(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times)$ for $m \geq 1$.

Hint: use that a generator of this group has the form $x^m \otimes y$, where y is a generator of $H^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{C}^\times)$.

EXERCISE 2.6.7. Describe $H^*(\mathbb{Z}/n\mathbb{Z}, \Bbbk^\times)$, where $\Bbbk$ is an algebraically closed field of any characteristic p.

Hint: Show that this cohomology coincides with the cohomology with coefficients in the group of roots of unity in $\Bbbk$; consider separately the case when p divides n and when p does not divide n.

2.7. Group actions on categories and equivariantization

Let $\mathcal{C}$ be a category. Consider the category $\mathsf{Aut}(\mathcal{C})$, whose objects are autoequivalences of $\mathcal{C}$ and whose morphisms are isomorphisms of functors. It is a monoidal subcategory of the monoidal category $\mathsf{End}(\mathcal{C})$ from Example 2.3.12.

If $\mathcal{C}$ is a monoidal category, we consider the category $\mathsf{Aut}_\otimes(\mathcal{C})$ of monoidal autoequivalences of $\mathcal{C}$.

For a group G let $\mathrm{Cat}(G)$ denote the monoidal category whose objects are elements of G, the only morphisms are the identities, and the tensor product is given by multiplication in G. In the notation of Example 2.3.6 we have $\mathrm{Cat}(G) = \mathcal{C}_G(1)$.

DEFINITION 2.7.1. Let G be a group.

(i) An *action* of G on a category $\mathcal{C}$ is a monoidal functor

(2.35) $$T : \mathrm{Cat}(G) \to \mathsf{Aut}(\mathcal{C}).$$

(ii) An *action* of G on a monoidal category $\mathcal{C}$ is a monoidal functor

(2.36) $$T : \mathrm{Cat}(G) \to \mathsf{Aut}_\otimes(\mathcal{C}).$$

In these situations we also say that G *acts* on $\mathcal{C}$.

Let G be a group acting on a category $\mathcal{C}$. Let $g \mapsto T_g$ denote the corresponding action (2.35). For any $g \in G$ let $T_g \in \mathsf{Aut}(\mathcal{C})$ be the corresponding functor, and for any $g, h \in G$ let $\gamma_{g,h}$ be the isomorphism $T_g \circ T_h \simeq T_{gh}$ that defines the monoidal structure on the functor $\mathrm{Cat}(G) \to \mathsf{Aut}(\mathcal{C})$.

DEFINITION 2.7.2. A *G-equivariant object* in $\mathcal{C}$ is a pair (X, u) consisting of an object X of $\mathcal{C}$ and a family of isomorphisms $u = \{u_g : T_g(X) \xrightarrow{\sim} X \mid g \in G\}$, such that the diagram

$$
\begin{array}{ccc}
T_g(T_h(X)) & \xrightarrow{T_g(u_h)} & T_g(X) \\
{\scriptstyle \gamma_{g,h}(X)} \downarrow & & \downarrow {\scriptstyle u_g} \\
T_{gh}(X) & \xrightarrow{u_{gh}} & X
\end{array}
$$

commutes for all $g, h \in G$. One defines morphisms of equivariant objects to be morphisms in $\mathcal{C}$ commuting with u_g, $g \in G$.

The *category of G-equivariant objects of $\mathcal{C}$*, or the *G-equivariantization* of $\mathcal{C}$, will be denoted by $\mathcal{C}^G$. There is an obvious forgetful functor

(2.37) $$\mathrm{Forg} : \mathcal{C}^G \to \mathcal{C}.$$

A similar definition can be made for monoidal categories $\mathcal{C}$, replacing $\mathsf{Aut}(\mathcal{C})$ with $\mathsf{Aut}_\otimes(\mathcal{C})$. When $\mathcal{C}$ is a monoidal category, the category $\mathcal{C}^G$ inherits a structure of a monoidal category such that the functor (2.37) is a monoidal functor.

EXERCISE 2.7.3. Show that actions of a group G on the category Vec viewed as an *abelian* category correspond to elements of $H^2(G, \Bbbk^\times)$, while any action of G on Vec viewed as a *monoidal* category is trivial.

2.8. The Mac Lane strictness theorem

As we have seen above, it is simpler to work with monoidal categories in which the associativity and unit constrains are the identity maps.

DEFINITION 2.8.1. A monoidal category $\mathcal{C}$ is *strict* if for all objects X, Y, Z in $\mathcal{C}$ one has equalities $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $X \otimes \mathbf{1} = X = \mathbf{1} \otimes X$, and the associativity and unit constraints are the identity maps.

EXAMPLE 2.8.2. The category $\mathsf{End}(\mathcal{C})$ of endofunctors of a category $\mathcal{C}$ (see Example 2.3.12) is strict.

EXAMPLE 2.8.3. Let $\overline{\mathsf{Sets}}$ be the category whose objects are non-negative integers, and $\mathsf{Hom}_{\overline{\mathsf{Sets}}}(m, n)$ is the set of maps from $\{0, ..., m-1\}$ to $\{0, ..., n-1\}$. Define the tensor product functor on objects by $m \otimes n = mn$, and for

$$f_1 : m_1 \to n_1 \quad \text{and} \quad f_2 : m_2 \to n_2$$

define $f_1 \otimes f_2 : m_1 m_2 \to n_1 n_2$ by

$$(f_1 \otimes f_2)(m_2 x + y) = n_2 f_1(x) + f_2(y), 0 \leq x \leq m_1 - 1, \ 0 \leq y \leq m_2 - 1.$$

Then $\overline{\mathsf{Sets}}$ is a strict monoidal category. Moreover, we have a natural inclusion $\overline{\mathsf{Sets}} \hookrightarrow \mathsf{Sets}$ (recall that Sets stands for the category of finite sets), which is obviously a monoidal equivalence.

EXAMPLE 2.8.4. This is a linear version of the previous example. Let $\Bbbk$ be a field. Let $\Bbbk - \overline{\mathsf{Vec}}$ be the category whose objects are non-negative integers, and $\mathsf{Hom}_{\Bbbk - \overline{\mathsf{Vec}}}(m, n)$ is the set of matrices with m columns and n rows over $\Bbbk$ (and the composition of morphisms is the product of matrices). Define the tensor product functor on objects by $m \otimes n = mn$, and for $f_1 : m_1 \to n_1, f_2 : m_2 \to n_2$, define $f_1 \otimes f_2 : m_1 m_2 \to n_1 n_2$ to be the Kronecker product of f_1 and f_2. Then $\Bbbk - \overline{\mathsf{Vec}}$ is a strict monoidal category. Moreover, we have a natural inclusion $\Bbbk - \overline{\mathsf{Vec}} \hookrightarrow \Bbbk - \mathsf{Vec}$, which is obviously a monoidal equivalence.

Similarly, for any group G one can define a strict monoidal category $\Bbbk - \overline{\mathsf{Vec}}_G$, whose objects are $\mathbb{Z}_+$-valued functions on G with finitely many nonzero values, and which is monoidally equivalent to $\Bbbk - \mathsf{Vec}_G$. We leave this definition to the reader.

On the other hand, some of the most important monoidal categories, such as $\mathbf{Sets}, \mathbf{Vec}, \mathbf{Vec}_G$, Sets, Vec, Vec_G, should be regarded as non-strict (at least if one defines them in the usual way). It is even more indisputable that the categories $\mathbf{Vec}_G^\omega$, Vec_G^ω for cohomologically nontrivial ω are not strict.

However, the following remarkable theorem of Mac Lane implies that in practice, one may always assume that a monoidal category is strict.

THEOREM 2.8.5. *Any monoidal category is monoidally equivalent to a strict monoidal category.*

PROOF. We will establish an equivalence between $\mathcal{C}$ and the monoidal category of right $\mathcal{C}$-module endofunctors of $\mathcal{C}$, which we will discuss in more detail in Chapter 7. The non-categorical counterpart of this result is the fact that every monoid M is isomorphic to the monoid consisting of maps from M to itself commuting with the right multiplication.

For a monoidal category $\mathcal{C}$ let $\mathcal{C}^l$ be the monoidal category defined as follows. The objects of $\mathcal{C}^l$ are pairs (F, c) where $F : \mathcal{C} \to \mathcal{C}$ is a functor and

$$c_{X,Y} : F(X) \otimes Y \xrightarrow{\sim} F(X \otimes Y)$$

is a natural isomorphism such that the following diagram is commutative for all objects X, Y, Z in $\mathcal{C}$:

(2.38)

$$
\begin{array}{ccc}
& (F(X) \otimes Y) \otimes Z & \\
\swarrow^{c_{X,Y} \otimes \mathrm{id}_Z} & & \searrow^{a_{F(X),Y,Z}} \\
F(X \otimes Y) \otimes Z & & F(X) \otimes (Y \otimes Z) \\
\downarrow^{c_{X \otimes Y, Z}} & & \downarrow^{c_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow{F(a_{X,Y,Z})} & F(X \otimes (Y \otimes Z)).
\end{array}
$$

A morphism $\theta : (F^1, c^1) \to (F^2, c^2)$ in $\mathcal{C}^l$ is a natural transformation $\theta : F^1 \to F^2$ such that the following square commutes for all objects X, Y in $\mathcal{C}$:

(2.39)

$$
\begin{array}{ccc}
F^1(X) \otimes Y & \xrightarrow{c^1_{X,Y}} & F^1(X \otimes Y) \\
\downarrow^{\theta_X \otimes \mathrm{id}_Y} & & \downarrow^{\theta_{X \otimes Y}} \\
F^2(X) \otimes Y & \xrightarrow{c^2_{X,Y}} & F^2(X \otimes Y),
\end{array}
$$

Composition of morphisms is the vertical composition of natural transformations. The tensor product of objects is given by $(F^1, c^1) \otimes (F^2, c^2) = (F^1 F^2, c)$ where c is given by a composition

(2.40) $\qquad F^1 F^2(X) \otimes Y \xrightarrow{c^1_{F^2(X),Y}} F^1(F^2(X) \otimes Y) \xrightarrow{F^1(c^2_{X,Y})} F^1 F^2(X \otimes Y)$

for all $X, Y \in \mathcal{C}$, and the tensor product of morphisms is the composition of natural transformations. Thus $\mathcal{C}^l$ is a strict tensor category (the unit object is the identity functor).

Consider a functor of left multiplication $L : \mathcal{C} \to \mathcal{C}^l$ given by

$$L(X) = (X \otimes -, a_{X,-,-}), \quad L(f) = (f \otimes -).$$

Note that the diagram (2.38) for L is nothing but the pentagon diagram (2.2).

We will show that this functor L is a monoidal equivalence. First of all, note that any (F, c) in $\mathcal{C}^l$ is isomorphic to $L(F(\mathbf{1}))$.

Let us now show that L is fully faithful. Let $\theta : L(X) \to L(Y)$ be a morphism in $\mathcal{C}^l$. Define $f : X \to Y$ to be the composition

(2.41) $\qquad\qquad\qquad X \xrightarrow{r_X^{-1}} X \otimes \mathbf{1} \xrightarrow{\theta_{\mathbf{1}}} Y \otimes \mathbf{1} \xrightarrow{r_Y} Y,$

where r is the unit constraint. We claim that for all Z in $\mathcal{C}$ one has $\theta_Z = f \otimes \mathsf{id}_Z$ (so that $\theta = L(f)$ and L is full). Indeed, this follows from the commutativity of the following diagram

$$(2.42) \quad \begin{array}{ccccccc} X \otimes Z & \xrightarrow{r_X^{-1} \otimes \mathsf{id}_Z} & (X \otimes \mathbf{1}) \otimes Z & \xrightarrow{a_{X,1,Z}} & X \otimes (\mathbf{1} \otimes Z) & \xrightarrow{X \otimes l_Z} & X \otimes Z \\ {\scriptstyle f \otimes \mathsf{id}_Z} \downarrow & & {\scriptstyle \theta_{\mathbf{1}} \otimes \mathsf{id}_Z} \downarrow & & {\scriptstyle \theta_{\mathbf{1} \otimes Z}} \downarrow & & {\scriptstyle \theta_Z} \downarrow \\ Y \otimes Z & \xrightarrow[r_Y^{-1} \otimes \mathsf{id}_Z]{} & (Y \otimes \mathbf{1}) \otimes Z & \xrightarrow[a_{Y,1,Z}]{} & Y \otimes (\mathbf{1} \otimes Z) & \xrightarrow[Y \otimes l_Z]{} & Y \otimes Z, \end{array}$$

where the rows are the identity morphisms by the triangle axiom (2.10), the left square commutes by the definition of f, the right square commutes by the naturality of θ, and the central square commutes since θ is a morphism in $\mathcal{C}^l$.

Next, if $L(f) = L(g)$ for some morphisms f, g in $\mathcal{C}$ then $f \otimes \mathsf{id}_\mathbf{1} = g \otimes \mathsf{id}_\mathbf{1}$ so that $f = g$ by the naturality of r. So L is faithful. Thus, it is an equivalence.

Finally, we define a monoidal functor structure

$$\phi : \mathbf{1}_{\mathcal{C}^l} \xrightarrow{\sim} L(\mathbf{1}_{\mathcal{C}}) \quad \text{and} \quad J_{X,Y} : L(X) \circ L(Y) \xrightarrow{\sim} L(X \otimes Y)$$

on L by $\phi = l^{-1} : (\mathsf{id}_{\mathcal{C}}, \mathsf{id}) \xrightarrow{\sim} (\mathbf{1} \otimes -, a_{\mathbf{1},-,-})$ and

$$J_{X,Y} = a_{X,Y,-}^{-1} : ((X \otimes (Y \otimes -)), (\mathsf{id}_X \otimes a_{Y,-,-} \circ a_{X,Y\otimes-,-}))$$
$$\xrightarrow{\sim} ((X \otimes Y) \otimes -, a_{X\otimes Y,-,-}).$$

The diagram (2.39) for the latter natural isomorphism is the pentagon diagram in $\mathcal{C}$. For the functor L the hexagon diagram (2.23) in the definition of a monoidal functor also reduces to the pentagon diagram in $\mathcal{C}$. The square diagrams (2.25) and (2.26) reduce to triangles, one of which is the triangle axiom (2.10) for $\mathcal{C}$ and another is (2.13). $\qquad \square$

REMARK 2.8.6. The nontrivial nature of Mac Lane's strictness theorem is demonstrated by the following instructive example, which shows that even though a monoidal category is always **equivalent** to a strict category, it need **not** be **isomorphic** to one. (By definition, an isomorphism of monoidal categories is a monoidal equivalence which is an isomorphism of categories).

Namely, let $\mathcal{C}$ be the category $\mathcal{C}_G^\omega(A)$. If ω is cohomologically nontrivial, this category is clearly not isomorphic to a strict one. However, by Mac Lane's strictness theorem, it is equivalent to a strict category $\mathcal{C}^l$.

In fact, in this example a strict category $\mathcal{C}^l$ monoidally equivalent to $\mathcal{C}$ can be constructed quite explicitly, as follows. Let $\widetilde{G}$ be another group with a surjective homomorphism $f : \widetilde{G} \to G$ such that the 3-cocycle $f^*\omega$ is cohomologically trivial. Such $\widetilde{G}$ always exists, e.g., a free group (since the cohomology of a free group in degrees higher than 1 is trivial). Let $\mathcal{C}^l$ be the category whose objects δ_g are labeled by elements of $\widetilde{G}$, $\mathsf{Hom}(\delta_g, \delta_h) = A$ if g, h have the same image in G, and $\mathsf{Hom}(\delta_g, \delta_h) = \emptyset$ otherwise. This category has an obvious tensor product, and a monoidal structure defined by the 3-cocycle $f^*\omega$. We have an obvious monoidal functor $F : \mathcal{C}^l \to \mathcal{C}$ defined by the homomorphism $f : \widetilde{G} \to G$, and it is an equivalence, even though not an isomorphism. However, since the cocycle $f^*\omega$ is cohomologically trivial, the category $\mathcal{C}^l$ is isomorphic to the same category with the trivial associativity isomorphism, which is strict.

REMARK 2.8.7. A category is called *skeletal* if it has only one object in each isomorphism class. The axiom of choice implies that any category is equivalent to a skeletal one. Also, by Mac Lane's strictness theorem, any monoidal category is monoidally equivalent to a strict one. However, Remark 2.8.6 shows that a monoidal category need not be monoidally equivalent to a category which is skeletal and strict at the same time. Indeed, as we have seen, to make a monoidal category strict, it may be necessary to add new objects to it (which are isomorphic, but not equal to already existing ones). In fact, the desire to avoid adding such objects is the reason why we sometimes use nontrivial associativity isomorphisms, even though Mac Lane's strictness theorem tells us we do not have to. This also makes precise the sense in which the categories Sets, Vec, Vec_G, are "more strict" than the category Vec_G^ω for cohomologically nontrivial ω. Namely, the first three categories are monoidally equivalent to strict skeletal categories $\overline{\mathsf{Sets}}$, $\overline{\mathsf{Vec}}$, $\overline{\mathsf{Vec}}_G$, while the category Vec_G^ω is not monoidally equivalent to a strict skeletal category.

EXERCISE 2.8.8. Show that any monoidal category $\mathcal{C}$ is monoidally equivalent to a skeletal monoidal category $\overline{\mathcal{C}}$. Moreover, $\overline{\mathcal{C}}$ can be chosen in such a way that $l_X, r_X = \mathsf{id}_X$ for all objects $X \in \overline{\mathcal{C}}$.

Hint: without loss of generality one can assume that $\mathbf{1} \otimes X = X \otimes \mathbf{1} = X$ and $l_X, r_X = \mathsf{id}_X$ for all objects $X \in \mathcal{C}$. Now in every isomorphism class i of objects of $\mathcal{C}$ fix a representative X_i, so that $X_1 = \mathbf{1}$, and for any two classes i, j fix an isomorphism $\mu_{ij} : X_i \otimes X_j \to X_{i \cdot j}$, so that $\mu_{i1} = \mu_{1i} = \mathsf{id}_{X_i}$. Let $\overline{\mathcal{C}}$ be the full subcategory of $\mathcal{C}$ consisting of the objects X_i, with tensor product defined by $X_i \overline{\otimes} X_j = X_{i \cdot j}$, and with all the structure transported using the isomorphisms μ_{ij}. Then $\overline{\mathcal{C}}$ is the required skeletal category, monoidally equivalent to $\mathcal{C}$.

2.9. The coherence theorem

In a monoidal category, one can form n-fold tensor products of any ordered sequence of objects $X_1, ..., X_n$. Namely, such a product can be attached to any parenthesizing of the expression $X_1 \otimes ... \otimes X_n$, and such products are, in general, distinct objects of $\mathcal{C}$.

However, for $n = 3$, the associativity isomorphism gives a canonical identification of the two possible parenthesizings, $(X_1 \otimes X_2) \otimes X_3$ and $X_1 \otimes (X_2 \otimes X_3)$. An easy combinatorial argument then shows that one can identify any two parenthesized products of $X_1, ..., X_n$, $n \geq 3$, using a chain of associativity isomorphisms.

EXERCISE 2.9.1. Show that the number of ways in which an n-fold product can be parenthesized is given by the *Catalan number* $\frac{1}{n+1}\binom{2n}{n}$.

We would like to say that for this reason we can completely ignore parentheses in computations in any monoidal category, identifying all possible parenthesized products with each other. But this runs into the following problem: for $n \geq 4$ there may be two or more different chains of associativity isomorphisms connecting two different parenthesizings, and a priori it is not clear that they provide the same identification.

Luckily, for $n = 4$, this is settled by the pentagon axiom, which states exactly that the two possible identifications are the same. But what about $n > 4$?

This problem is solved by the following theorem of Mac Lane, which is the first important result in the theory of monoidal categories.

THEOREM 2.9.2. *(Coherence Theorem) Let $X_1, \ldots, X_n \in \mathcal{C}$. Let P_1, P_2 be any two parenthesized products of $X_1, \ldots, X_n$ (in this order) with arbitrary insertions of the unit object $\mathbf{1}$. Let $f, g : P_1 \to P_2$ be two isomorphisms, obtained by composing associativity and unit isomorphisms and their inverses possibly tensored with identity morphisms. Then $f = g$.*

PROOF. We derive this theorem as a corollary of the Mac Lane's strictness Theorem 2.8.5. Let $L : \mathcal{C} \to \mathcal{C}'$ be a monoidal equivalence between $\mathcal{C}$ and a strict monoidal category $\mathcal{C}'$. Consider a diagram in $\mathcal{C}$ representing f and g and apply L to it. Over each arrow of the resulting diagram representing an associativity isomorphism, let us build a rectangle as in (2.23), and do similarly for the unit morphisms. This way we obtain a prism one of whose faces consists of identity maps (associativity and unit isomorphisms in $\mathcal{C}'$) and whose sides are commutative. Hence, the other face is commutative as well, i.e., $f = g$. $\qquad\square$

REMARK 2.9.3. As we mentioned, Theorem 2.9.2 implies that any two parenthesized products of $X_1, \ldots, X_n$ with insertions of unit objects are indeed canonically isomorphic, and thus one can safely identify all of them with each other and ignore bracketings in calculations in a monoidal category. We will do so from now on, unless confusion is possible.

2.10. Rigid monoidal categories

Let $(\mathcal{C}, \otimes, \mathbf{1}, a, \iota)$ be a monoidal category, and let X be an object of $\mathcal{C}$. In what follows, we suppress the unit constraints l and r.

DEFINITION 2.10.1. An object X^* in $\mathcal{C}$ is said to be a *left dual* of X if there exist morphisms $\mathsf{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathsf{coev}_X : \mathbf{1} \to X \otimes X^*$, called the *evaluation* and *coevaluation*, such that the compositions

$$(2.43) \qquad X \xrightarrow{\mathsf{coev}_X \otimes \mathsf{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathsf{id}_X \otimes \mathsf{ev}_X} X,$$

$$(2.44) \quad X^* \xrightarrow{\mathsf{id}_{X^*} \otimes \mathsf{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathsf{ev}_X \otimes \mathsf{id}_{X^*}} X^*$$

are the identity morphisms.

DEFINITION 2.10.2. An object $^*X$ in $\mathcal{C}$ is said to be a *right dual* of X if there exist morphisms $\mathsf{ev}'_X : X \otimes {}^*X \to \mathbf{1}$ and $\mathsf{coev}'_X : \mathbf{1} \to {}^*X \otimes X$ such that the compositions

$$(2.45) \qquad X \xrightarrow{\mathsf{id}_X \otimes \mathsf{coev}'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a^{-1}_{X,{}^*X,X}} (X \otimes {}^*X) \otimes X \xrightarrow{\mathsf{ev}'_X \otimes \mathsf{id}_X} X,$$

$$(2.46) \quad {}^*X \xrightarrow{\mathsf{coev}'_X \otimes \mathsf{id}_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a_{{}^*X,X,{}^*X}} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{\mathsf{id}_{{}^*X} \otimes \mathsf{ev}'_X} {}^*X$$

are the identity morphisms.

REMARK 2.10.3. It is obvious that if X^* is a left dual of an object X then X is a right dual of X^* with $\mathsf{ev}'_{X^*} = \mathsf{ev}_X$ and $\mathsf{coev}'_{X^*} = \mathsf{coev}_X$, and vice versa. Therefore, $^*(X^*) \cong X \cong ({}^*X)^*$ for any object X admitting left and right duals. Also, in any monoidal category, $\mathbf{1}^* = {}^*\mathbf{1} = \mathbf{1}$ with the evaluation and coevaluation morphisms ι and ι^{-1}. Also note that changing the order of tensor product switches left duals and right duals, so to any statement about right duals there corresponds a symmetric statement about left duals.

EXERCISE 2.10.4. Let $\mathcal{C}$ be a category and $\mathsf{End}(\mathcal{C})$ be the monoidal category of endofunctors of $\mathcal{C}$. Show that a left (respectively, right) dual of $F \in \mathsf{End}(\mathcal{C})$ is the same thing as a functor left (respectively, right) adjoint to F. This justifies our terminology.

PROPOSITION 2.10.5. *If $X \in \mathcal{C}$ has a left (respectively, right) dual object, then it is unique up to a unique isomorphism.*

PROOF. Let X_1^*, X_2^* be two left duals of X. Denote by e_1, c_1, e_2, c_2 the corresponding evaluation and coevaluation morphisms. Then we have a morphism $\alpha : X_1^* \to X_2^*$ defined as the composition

$$X_1^* \xrightarrow{\;\mathsf{id}_{X_1^*} \otimes c_2\;} X_1^* \otimes (X \otimes X_2^*) \xrightarrow{\;a^{-1}_{X_1^*, X, X_2^*}\;} (X_1^* \otimes X) \otimes X_2^* \xrightarrow{\;e_1 \otimes \mathsf{id}_{X_2^*}\;} X_2^*.$$

Similarly, one defines a morphism $\beta : X_2^* \to X_1^*$. We claim that $\beta \circ \alpha$ and $\alpha \circ \beta$ are the identity morphisms, so α is an isomorphism. Indeed, consider the following diagram:

Here we suppress the associativity constraints. It is clear that the three small squares commute. The triangle in the upper right corner commutes by axiom (2.43) applied to X_2^*. Hence, the perimeter of the diagram commutes. The composition through the top row is the identity by (2.44) applied to X_1^*. The composition through the bottom row is $\beta \circ \alpha$ and so $\beta \circ \alpha = \mathsf{id}_{X_1^*}$. The proof of $\alpha \circ \beta = \mathsf{id}_{X_2^*}$ is completely similar.

Moreover, it is easy to check that $\alpha : X_1^* \to X_2^*$ is the only isomorphism which preserves the evaluation and coevaluation morphisms. This proves the proposition for left duals. The proof for right duals is similar. $\qquad\square$

If X, Y are objects in $\mathcal{C}$ which have left duals X^*, Y^* and $f : X \to Y$ is a morphism, one defines the *left dual $f^* : Y^* \to X^*$* of f by

$$(2.47) \qquad f^* := Y^* \xrightarrow{\;\mathsf{id}_{Y^*} \otimes \mathsf{coev}_X\;} Y^* \otimes (X \otimes X^*) \xrightarrow{\;a^{-1}_{Y^*, X, X^*}\;} (Y^* \otimes X) \otimes X^*$$
$$\xrightarrow{\;(\mathsf{id}_{Y^*} \otimes f) \otimes \mathsf{id}_{X^*}\;} (Y^* \otimes Y) \otimes X^* \xrightarrow{\;\mathsf{ev}_Y \otimes \mathsf{id}_{X^*}\;} X^*.$$

Similarly, if X, Y are objects in $\mathcal{C}$ which have right duals $^*X, {}^*Y$ and $f : X \to Y$ is a morphism one defines the *right dual $^*f : {}^*Y \to {}^*X$* of f by

$$(2.48) \qquad {}^*f := {}^*Y \xrightarrow{\;\mathsf{coev}'_X \otimes \mathsf{id}_{{}^*Y}\;} ({}^*X \otimes X) \otimes {}^*Y \xrightarrow{\;a_{{}^*X, X, {}^*Y}\;} {}^*X \otimes (X \otimes {}^*Y)$$
$$\xrightarrow{\;\mathsf{id}_{{}^*X} \otimes (f \otimes \mathsf{id}_{{}^*Y})\;} {}^*X \otimes (Y \otimes {}^*Y) \xrightarrow{\;\mathsf{id}_{{}^*X} \otimes \mathsf{ev}'_Y\;} {}^*X.$$

EXERCISE 2.10.6. Let $\mathcal{C}, \mathcal{D}$ be monoidal categories. Suppose $F = (F, J, \varphi)$ is a monoidal functor, $F : \mathcal{C} \to \mathcal{D}$. Let X be an object in $\mathcal{C}$ with a left dual X^*. Prove that $F(X^*)$ is a left dual of $F(X)$ with the evaluation and coevaluation given by

$$\mathsf{ev}_{F(X)} \quad : \quad F(X^*) \otimes F(X) \xrightarrow{J_{X^*, X}} F(X^* \otimes X) \xrightarrow{F(\mathsf{ev}_X)} F(\mathbf{1}) \xrightarrow{\varphi^{-1}} \mathbf{1},$$

$$\mathsf{coev}_{F(X)} \quad : \quad \mathbf{1} \xrightarrow{\varphi} F(\mathbf{1}) \xrightarrow{F(\mathsf{coev}_X)} F(X \otimes X^*) \xrightarrow{J_{X, X^*}^{-1}} F(X) \otimes F(X^*).$$

State and prove a similar result for right duals.

EXERCISE 2.10.7. Let $\mathcal{C}$ be a monoidal category, let U, V, W be objects in $\mathcal{C}$, and let $f : V \to W$, $g : U \to V$ be morphisms in $\mathcal{C}$. Prove that
   (a) If U, V, W have left (respectively, right) duals then $(f \circ g)^* = g^* \circ f^*$ (respectively, $^*(f \circ g) = {}^*g \circ {}^*f$).
   (b) If U, V have left (respectively, right) duals then $U \otimes V$ has a left dual $V^* \otimes U^*$ (respectively, right dual $^*V \otimes {}^*U$).

PROPOSITION 2.10.8. *Let $\mathcal{C}$ be a monoidal category and let V be an object in $\mathcal{C}$.*

   (a) *If V has a left dual V^* then there are natural adjunction isomorphisms*

(2.49) $\mathsf{Hom}_{\mathcal{C}}(U \otimes V, W) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(U, W \otimes V^*),$

(2.50) $\mathsf{Hom}_{\mathcal{C}}(V^* \otimes U, W) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(U, V \otimes W).$

   (b) *If V has a right dual $^*V$ then there are natural adjunction isomorphisms*

(2.51) $\mathsf{Hom}_{\mathcal{C}}(U \otimes {}^*V, W) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(U, W \otimes V),$

(2.52) $\mathsf{Hom}_{\mathcal{C}}(V \otimes U, W) \xrightarrow{\sim} \mathsf{Hom}_{\mathcal{C}}(U, {}^*V \otimes W).$

PROOF. An isomorphism in (2.49) is given by $f \mapsto (f \otimes \mathsf{id}_{V^*}) \circ (\mathsf{id}_U \otimes \mathsf{coev}_V)$ and has the inverse $g \mapsto (\mathsf{id}_W \otimes \mathsf{ev}_V) \circ (g \otimes \mathsf{id}_V)$. Other isomorphisms are similar and are left to the reader as an exercise. $\square$

REMARK 2.10.9. Proposition 2.10.8 says, in particular, that when a left (respectively, right) dual of V exists, then the functor $V^* \otimes -$ is the left adjoint of $V \otimes -$ (respectively, $- \otimes V^*$ is the right adjoint of $- \otimes V$).

REMARK 2.10.10. Proposition 2.10.8 provides another proof of Proposition 2.10.5. Namely, setting $U = \mathbf{1}$ and $V = X$ in (2.50), we obtain a natural isomorphism $\mathsf{Hom}_{\mathcal{C}}(X^*, W) \cong \mathsf{Hom}_{\mathcal{C}}(\mathbf{1}, X \otimes W)$ for any left dual X^* of X. Hence, if Y_1, Y_2 are two such duals then there is a natural isomorphism $\mathsf{Hom}_{\mathcal{C}}(Y_1, W) \cong \mathsf{Hom}_{\mathcal{C}}(Y_2, W)$, whence there is a canonical isomorphism $Y_1 \cong Y_2$ by the Yoneda Lemma. The proof for right duals is similar.

DEFINITION 2.10.11. An object in a monoidal category is called *rigid* if it has left and right duals. A monoidal category $\mathcal{C}$ is called *rigid* if every object of $\mathcal{C}$ is rigid.

EXAMPLE 2.10.12. The category Vec of finite dimensional $\Bbbk$-vector spaces is rigid: the right and left dual to a finite dimensional vector space V are its dual space V^*, with the evaluation map $\mathsf{ev}_V : V^* \otimes V \to \Bbbk$ being the contraction, and the coevaluation map $\mathsf{coev}_V : k \to V \otimes V^*$ being the usual embedding. On the other hand, the category $\mathbf{Vec}$ of all $\Bbbk$-vector spaces is not rigid, since for infinite dimensional spaces there is no coevaluation maps (indeed, suppose that

$c : \Bbbk \to V \otimes Y$ is a coevaluation map, and consider the subspace V' of V spanned by the first component of $c(1)$; this subspace is finite dimensional, and yet the composition $V \to V \otimes Y \otimes V \to V$, which is supposed to be the identity map, lands in V' - a contradiction).

EXAMPLE 2.10.13. Let $\Bbbk$ be a field. The category $\mathsf{Rep}(G)$ of finite dimensional representations of a group G over $\Bbbk$ is rigid: for a finite dimensional representation V, the (left or right) dual representation V^* is the usual dual space (with the evaluation and coevaluation maps as in Example 2.10.12), and with the G-action given by $\rho_{V^*}(g) = (\rho_V(g)^{-1})^*$. Similarly, the category $\mathsf{Rep}(\mathfrak{g})$ of finite dimensional representations of a Lie algebra $\mathfrak{g}$ is rigid, with $\rho_{V^*}(a) = -\rho_V(a)^*$.

EXAMPLE 2.10.14. The category Vec_G (see Example 2.3.6) is rigid if and only if the monoid G is a group; namely, $\delta_g^* = {}^*\delta_g = \delta_{g^{-1}}$ (with the obvious structure maps). More generally, for any group G and 3-cocycle $\omega \in Z^3(G, \Bbbk^\times)$, the category Vec_G^ω is rigid. Namely, assume for simplicity that the cocycle ω is normalized (as we know, we can do so without loss of generality). Then we can define duality as above, and normalize the coevaluation morphisms of δ_g to be the identities. The evaluation morphisms will then be defined by the formula $\mathsf{ev}_{\delta_g} = \omega(g, g^{-1}, g)\,\mathsf{id}_\mathbf{1}$.

It follows from Proposition 2.10.5 that in a monoidal category $\mathcal{C}$ with left (respectively, right) duals, one can define a contravariant left (respectively, right) *duality functor* by
$$X \mapsto X^*, \, f \mapsto f^* : \mathcal{C} \to \mathcal{C}$$
(respectively, by $X \mapsto {}^*X$, $f \mapsto {}^*f$) for every object X and morphism f in $\mathcal{C}$. By Exercise 2.10.7(ii), these are monoidal functors $\mathcal{C}^\vee \to \mathcal{C}^{\mathrm{op}}$, where the monoidal structure of the opposite category $\mathcal{C}^{\mathrm{op}}$ is given in Definition 2.1.5. Hence, the functors $X \mapsto X^{**}$, $X \mapsto {}^{**}X$ are monoidal. Also, it follows from Proposition 2.10.8(a) that the functors of left and right duality, when they are defined, are fully faithful.

Moreover, it follows from Remark 2.10.3 that in a rigid monoidal category, the functors of left and right duality are mutually quasi-inverse monoidal equivalences of categories $\mathcal{C}^\vee \xrightarrow{\sim} \mathcal{C}^{\mathrm{op}}$ (so for rigid categories, the notions of dual and opposite category are the same up to equivalence). This implies that the functors $X \mapsto X^{**}$ and $X \mapsto {}^{**}X$ are mutually quasi-inverse monoidal autoequivalences. We will see later in Example 7.19.5 that these autoequivalences may be nontrivial; in particular, it is possible that objects V^* and ${}^*V$ are not isomorphic.

EXERCISE 2.10.15. Show that if $\mathcal{C}, \mathcal{D}$ are rigid monoidal categories, $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ are monoidal functors, and $\eta : F_1 \to F_2$ is a morphism of monoidal functors, then η is an isomorphism (as we have seen in Remark 2.5.6, this is false for non-rigid categories).

EXERCISE 2.10.16. Let A be an algebra. Show that $M \in A-\mathbf{bimod}$ has a left (respectively, right) dual if and only if it is finitely generated projective when considered as a left (respectively, right) A-module. Similarly, if A is commutative, $M \in A-\mathbf{mod}$ has left and right duals if and only if it is finitely generated projective.

2.11. Invertible objects and Gr-categories

Let $\mathcal{C}$ be a rigid monoidal category.

DEFINITION 2.11.1. An object X in $\mathcal{C}$ is *invertible* if $\mathsf{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathsf{coev}_X : \mathbf{1} \to X \otimes X^*$ are isomorphisms.

Clearly, this notion categorifies the notion of an invertible element in a monoid.

EXAMPLE 2.11.2. Let G be a group.

(1) The objects δ_g in Vec_G^ω (see Example 2.3.8) are invertible.
(2) The invertible objects in $\mathsf{Rep}(G)$ (see Example 2.3.4) are precisely the 1-dimensional representations of G.

PROPOSITION 2.11.3. *Let X be an invertible object in $\mathcal{C}$. Then*

(i) *$^*X \cong X^*$ and X^* is invertible;*
(ii) *if Y is another invertible object then $X \otimes Y$ is invertible.*

PROOF. Dualizing coev_X and ev_X we get isomorphisms $X \otimes {}^*X \cong \mathbf{1}$ and $^*X \otimes X \cong \mathbf{1}$. Hence $^*X \cong {}^*X \otimes X \otimes X^* \cong X^*$. In any rigid category the evaluation and coevaluation morphisms for $^*X$ can be defined by $\mathsf{ev}_{^*X} := {}^*\mathsf{coev}_X$ and $\mathsf{coev}_{^*X} := {}^*\mathsf{ev}_X$, so $^*X$ is invertible. The second statement follows from the fact that $\mathsf{ev}_{X \otimes Y}$ can be defined as a composition of ev_X and ev_Y, and similarly $\mathsf{coev}_{X \otimes Y}$ can be defined as a composition of coev_Y and coev_X. $\qquad\square$

Proposition 2.11.3 implies that invertible objects of $\mathcal{C}$ form a monoidal subcategory $\mathrm{Inv}(\mathcal{C})$ of $\mathcal{C}$.

DEFINITION 2.11.4. A *Gr-category*, or a *categorical group*, is a rigid monoidal category in which every object is invertible and all morphisms are isomorphisms.

The second condition of Definition 2.11.4 means that a Gr-category is a groupoid. In fact, it is precisely a group object in the category of groupoids.

The next theorem provides a classification of Gr-categories.

THEOREM 2.11.5. *Monoidal equivalence classes of Gr-categories are in bijection with triples (G, A, ω), where G is a group, A is a G-module, and ω is an orbit in $H^3(G, A)$ under the action of $\mathsf{Out}(G)$.*

PROOF. We may assume that a Gr-category $\mathcal{C}$ is skeletal, i.e., there is only one object in each isomorphism class, and objects form a group G. Also, by Proposition 2.2.10, $\mathsf{End}_{\mathcal{C}}(\mathbf{1})$ is an abelian group; let us denote it by A. Then for any $g \in G$ we can identify $\mathsf{End}_{\mathcal{C}}(g)$ with A, by sending $f \in \mathsf{End}_{\mathcal{C}}(g)$ to $f \otimes \mathsf{id}_{g^{-1}} \in \mathsf{End}_{\mathcal{C}}(\mathbf{1}) = A$. Then we have an action of G on A by

$$a \in \mathsf{End}_{\mathcal{C}}(\mathbf{1}) \mapsto g(a) := \mathsf{id}_g \otimes a \in \mathsf{End}_{\mathcal{C}}(g).$$

Let us now consider the associativity isomorphism. It is defined by a function $\omega : G \times G \times G \to A$. The pentagon relation gives

$$(2.53) \quad \omega(g_1 g_2, g_3, g_4)\omega(g_1, g_2, g_3 g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2 g_3, g_4)g_1(\omega(g_2, g_3, g_4)),$$

for all $g_1, g_2, g_3, g_4 \in G$, which means that ω is a 3-cocycle of G with coefficients in the (in general, nontrivial) G-module A. We see that any such 3-cocycle defines a rigid monoidal category, which we call $\mathcal{C}_G^\omega(A)$. The analysis of monoidal equivalences between such categories is similar to the case when A is a trivial G-module and yields that for a given group G and G-module A, equivalence classes of $\mathcal{C}_G^\omega(A)$ are parametrized by $H^3(G, A)/\mathsf{Out}(G)$. $\qquad\square$

2.12. 2-categories

The notion of a 2-category extends the notion of a category, in the sense that in a 2-category one has in addition to objects and morphisms between them, also "morphisms between morphisms". Here is the formal definition.

DEFINITION 2.12.1. A *strict 2-category* $\underline{\mathcal{C}}$ consists of objects A, B, ..., 1-morphisms between objects $f : A \to B, \dots$ and 2-morphisms $\alpha : f \Rightarrow g, \dots$ between 1-morphisms $f, g : A \to B$ such that the following axioms are satisfied:

(1) The objects together with the 1-morphisms form a category $\mathcal{C}$. The composition of 1-morphisms is denoted by $\circ$.

(2) For any fixed pair of objects (A, B), the 1-morphisms from A to B together with the 2-morphisms between them form a category $\underline{\mathcal{C}}(A, B)$. The unital associative composition (along 1-morphisms) $\beta \cdot \alpha : f \Rightarrow h$ of two 2-morphisms $\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$ is called the *vertical composition*

$$A \overset{f}{\underset{h}{\Longrightarrow}} B \ .$$

The identities are denoted by $\mathsf{id}_f : f \Rightarrow f$; $A \overset{f}{\underset{f}{\Downarrow \mathsf{id}_f}} B$.
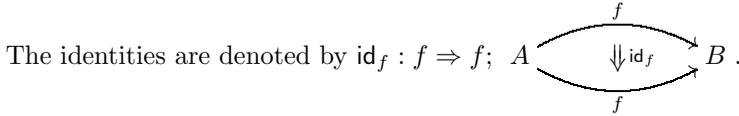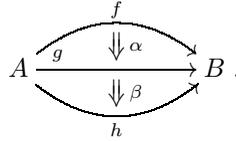
(3) There is a unital associative *horizontal composition* (along objects) $\beta \circ \alpha : h \circ f \Rightarrow i \circ g$ of 2-morphisms $\alpha : f \Rightarrow g$, $\beta : h \Rightarrow i$, where $f, g : A \to B$ and $h, i : B \to C$:

$$A \overset{f}{\underset{g}{\Downarrow \alpha}} B \overset{h}{\underset{i}{\Downarrow \beta}} C = A \overset{h \circ f}{\underset{i \circ g}{\Downarrow \beta \circ \alpha}} C \ .$$

The identities are $\mathsf{id}_{\mathsf{id}_A} : \mathsf{id}_A \Rightarrow \mathsf{id}_A$; $A \overset{\mathsf{id}_A}{\underset{\mathsf{id}_A}{\Downarrow \mathsf{id}_{\mathsf{id}_A}}} A$.
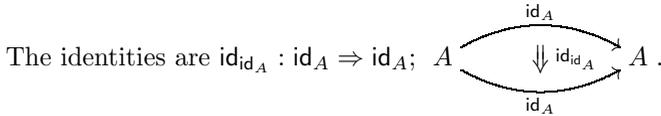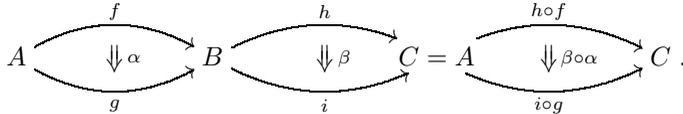
(4) For any triple of objects (A, B, C), 1-morphisms $f, g, h : A \to B$ and $i, j, k : B \to C$, and 2-morphisms $\alpha : f \Rightarrow g$, $\beta : g \Rightarrow h$, $\gamma : i \Rightarrow j$, $\delta : j \Rightarrow k$, we have $(\delta \circ \beta) \cdot (\gamma \circ \alpha) = (\delta \cdot \gamma) \circ (\beta \cdot \alpha)$ ("interchange law").

(5) The horizontal composition preserves vertical units, i.e., for any objects A, B, C, and 1-morphisms $f : A \to B$, $i : B \to C$, we have $\mathsf{id}_i \circ \mathsf{id}_f = \mathsf{id}_{i \circ f}$.

DEFINITION 2.12.2. A *2-category* $\underline{\mathcal{C}}$ consists of the same data as a strict 2-category $\underline{\mathcal{C}}$, except that the composition of 1-morphisms is required to be unital associative only up to *associativity and unital constraints*. Namely, there exist natural families of invertible 2-morphisms

$$\alpha_{f,g,h} : h \circ (g \circ f) \Rightarrow (h \circ g) \circ f, \ \lambda_f : f \circ \mathsf{id}_A \Rightarrow f, \ \rho_f : \mathsf{id}_B \circ f \Rightarrow f,$$

satisfying the *pentagon axiom*

$$\alpha_{f,g,i\circ h} \cdot \alpha_{g\circ f,h,i} = (\mathrm{id}_i \circ \alpha_{h,g,f}) \cdot \alpha_{f,h\circ g,i} \cdot (\alpha_{g,h,i} \circ \mathrm{id}_f)$$

and the *triangle axiom*

$$(\mathrm{id}_{\mathrm{id}_A} \circ \lambda_f) \cdot \alpha_{f,\mathrm{id}_B,g} = \rho_r \circ \mathrm{id}_f \,.$$

REMARK 2.12.3. The objects, 1-morphisms and 2-morphisms are called in some texts, 0-cells, 1-cells and 2-cells, respectively.

EXAMPLE 2.12.4. (*The* 2-*category of categories*) The objects are categories, the 1-morphisms are functors between categories, and the 2-morphisms are natural transformations between functors. The horizontal composition of natural transformations is the so called *Godement product.*

EXAMPLE 2.12.5. (*Rings and bimodules*) Rings are the objects, bimodules are the 1-morphisms, and homomorphisms between bimodules are the 2-morphisms.

EXAMPLE 2.12.6. Recall that the notion of a monoid is a special case of the notion of a category; namely, a monoid is the same thing as a category with one object (the morphisms of this category are the elements of the corresponding monoid). Similarly, the notion of a monoidal category is a special case of the notion of a 2-category [**Mac2**]: a monoidal category is the same thing as a 2-category with one object. Namely, the 1-morphisms and 2-morphisms of such a 2-category are the objects and morphisms of the corresponding monoidal category, and composition of 1-morphisms is the tensor product functor.

Below we will see other examples of 2-categories: a multitensor category (see Remark 4.3.7) and the 2-category of module categories over a multitensor category (see Remark 7.12.15).

2.13. Bibliographical notes

2.1.-2.2. Monoidal categories were introduced by Bénabou [**Ben1**] as "categories with multiplication". The pentagon and triangle axioms (2.2) and (2.10) were introduced by Mac Lane [**Mac1**]. Definition 2.1.1 (with unit being an idempotent that can be cancelled) and Proposition 2.2.10 appeared in the paper [**Sa**] by Saavedra Rivano. Note that Proposition 2.2.10 is a categorical version of the famous Eckmann-Hilton argument. Proposition 2.2.4 is due to Kelly [**Ke**].

2.3. Categories of tangles considered in Example 2.3.14 were introduced by Turaev [**Tu2**] and Yetter [**Ye2**]. These categories are important for applications to low-dimensional topology. A detailed description of such categories can be found in books by Kassel, Turaev, and Bakalov-Kirillov [**Kas, Tu4, BakK**].

2.4-2.6. For a discussion of monoidal functors, see e.g. the papers by Joyal and Street [**JoyS5**] and Müger [**Mu6**].

2.7. For a discussion of equivariantization, see e.g. [**DrGNO2**].

2.8-2.9. The proof of the Mac Lane strictness Theorem 2.8.5 presented here is given by Joyal and Street [**JoyS5**]. The coherence Theorem 2.9.2 is due to Mac Lane [**Mac1**]. Remark 2.8.7 is due to Kuperberg [**Ku**].

2.10. The notions of duality and of a rigid monoidal category appeared independently in many classical works, in particular in the papers by Saavedra Rivano [**Sa**] and Kelly [**Ke**]. A convenient way to do computations involving duality, e.g.,

in the proof of Proposition 2.10.8, is via the graphical calculus (see, e.g., Kassel's book [**Kas**, Chapter XIV]).

2.11. Definition 2.11.4 of a Gr-category goes back to Sính [**Sin**]. Gr-categories are also known as "2-groups". A theorem of Verdier establishes an equivalence between the category of Gr-categories (with monoidal functors as morphisms) and the category of *crossed modules*, see the paper by Barrett and Mackaay [**BarM**] for definition and discussion.

2.12. Good references for the theory of 2-categories are the books of Mac Lane [**Mac2**] and Lenster [**Le**].