Preface

Keys to ignition, use at your discretion.

– From "Starin' Through My Rear View" by Tupac Shakur

This is Part IV (a.k.a. $R^{\#}_{ijk\ell}$), the sequel to Volume One ([**75**]; a.k.a. g_{ij}) and Parts I, II, III ([**69**], [**70**], [**71**]; a.k.a. $R_{ijk\ell}$, $\frac{\partial}{\partial t}R_{ijk\ell}$, $\Delta R_{ijk\ell}$, respectively) of Volume Two on techniques and applications of the Ricci flow. For the reader's convenience, we have included the titles of each chapter on the pages that follow.

In this part we mainly discuss aspects of the *long-time behavior* of solutions to the Ricci flow, including the geometry of noncompact gradient Ricci solitons, ancient solutions, Hamilton's classification of 3-dimensional nonsingular solutions, and the stability of the Ricci flow. Any theory about singularities of the Ricci flow requires an understanding of ancient solutions and, in particular, gradient Ricci solitons. Building on the success in dimensions at most 3, the study of higher-dimensional Ricci solitons is currently an active field; we discuss some of the progress in this direction. We also present recent progress on (1) the classification of ancient 2-dimensional solutions without the κ-noncollapsing hypothesis and (2) Type I ancient solutions and singularities. In a direction complementary to the study of singularities, we discuss 3-dimensional nonsingular solutions. These solutions underlie the Ricci flow approach to the geometrization conjecture; Hamilton's work on this is a precursor to Perelman's more general theory of immortal solutions to the Ricci flow with surgery. Finally, a largely unexplored direction in the Ricci flow concerns the sensitivity of solutions to their initial data; the study of stability of solutions represents an aspect of this.

The choice of topics is based on our familiarity and taste. Due to the diversity of the field of Ricci flow, we have inevitably omitted many important works. We have also omitted some topics originally slated for this part, such as the linearized Ricci flow and the space-time formulation of the Ricci flow. We now give detailed descriptions of the chapter contents.

Chapter 27. This chapter is a continuation of Chapter 1 of Part I. Here we discuss some recent progress on the geometry of noncompact gradient Ricci solitons (**GRS**), including some qualitatively sharp estimates for the volume growth, potential functions, and scalar curvatures of GRS. We also discuss the logarithmic Sobolev inequality for shrinking GRS as well as shrinking GRS with nonnegative Ricci curvature.

Chapter 28. This chapter complements the discussion in Part III on Perelman's theory of 3-dimensional ancient κ-solutions. The topics discussed are a local lower bound for the scalar curvature under Ricci flow, some geometric properties of 3-dimensional singularity models, noncompact 2-dimensional ancient solutions without the κ-noncollapsed condition, and classifying certain ancient solutions with positive curvature.

Chapter 29. In this chapter we present the results of Daskalopoulos, Hamilton, and Sesum that any simply-connected ancient solution to the Ricci flow on a closed surface must be either a round shrinking 2-sphere or the rotationally symmetric King–Rosenau solution. The proof involves an eclectic collection of geometric and analytic methods. Monotonicity formulas that rely on being in dimension 2 are used.

Chapter 30. This chapter is focused on the general study of Type I singularities and Type I ancient solutions. We study properties and applications of Perelman's reduced distance and reduced volume based at the singular time for Type I singular solutions. We also discuss the result that Type I singular solutions have unbounded scalar curvature.

Chapter 31. In the study of nonsingular solutions to the Ricci flow on closed 3-manifolds in the subsequent chapters, of vital importance are finite-volume hyperbolic limits. In this chapter we present some prerequisite knowledge on the geometry and topology of hyperbolic 3-manifolds. Key topics are the Margulis lemma (including the ends of finite-volume hyperbolic manifolds) and the Mostow rigidity theorem.

Chapter 32. Hamilton's celebrated result says that for solutions to the normalized Ricci flow on closed 3-manifolds which exist for all forward time and have uniformly bounded curvature, the underlying differentiable 3-manifold admits a geometric decomposition in the sense of Thurston. The proof of the main result requires an understanding of the asymptotic behavior of the solution as time tends to infinity. If collapse occurs in the sense of Cheeger and Gromov, then the underlying differentiable 3-manifold admits an F-structure and in particular admits a geometric decomposition. Otherwise, one may extract limits of noncollapsing sequences by the uniformly bounded curvature assumption. In the cases where these limits have nonnegative sectional curvature, we can topologically classify the original 3-manifolds.

Chapter 33. In the cases where the limits do not have nonnegative sectional curvature, they must be hyperbolic 3-manifolds with finite volume, which may be either compact or noncompact. If these hyperbolic limits are compact, then they are diffeomorphic to the original 3-manifold. On the other hand, if these hyperbolic limits are noncompact, then the difficult result is that their truncated embeddings in the original 3-manifold are such that the boundary tori are incompressible in the complements. To establish this, one proves the stability of hyperbolic limits by the use of harmonic maps and Mostow rigidity. Then, assuming the compressibility of any boundary tori, one applies a minimal surface argument to obtain a contradiction.

Chapter 34. The purpose of this chapter is to prove, by the implicit function theorem, two results used in the previous chapter. We first show that almost hyperbolic cusps are swept out by constant mean curvature tori. Second, for any metric g on a compact manifold with negative Ricci curvature and concave boundary and for any metric $\tilde{g}$ sufficiently close to g, we prove the existence of a harmonic diffeomorphism from g to $\tilde{g}$ near the identity map.

Chapter 35. A potentially useful direction in Ricci flow is to study the perturbational aspects of the flow, in particular, stability of solutions, dependence on initial data, and properties of generic solutions and 1-parameter families of solutions. In this chapter we discuss the stability of solutions. The analysis of stability is partly dependent on understanding the Ricci flow coupled to the Lichnerowicz Laplacian heat equation for symmetric 2-tensors.

Chapter 36. In this chapter we survey a numerical approach, due to Garfinkle and one of the authors, to modeling rotationally symmetric degenerate neckpinches including the reflectionally symmetric case of two Bryant solitons simultaneously forming as limits. We also survey the matched asymptotic analysis of rotationally symmetric degenerate neckpinches and the related Ważewski retraction method.

Appendix K. In this appendix we recall some concepts and results about the analysis on manifolds that are used in various places in the book. In particular, we discuss the implicit function theorem, Hölder and Sobolev spaces of sections of bundles, formulas for harmonic maps, and the eigenvalues of the Hodge–de Rham Laplacian acting on differential forms on the round sphere.