

Noncompact Gradient Ricci Solitons

There is no elite

Just take your place in the driver's seat.

– From “Driver’s Seat” by Sniff ‘n’ the Tears

Gradient Ricci solitons (**GRS**), which were introduced and used effectively in the Ricci flow by Hamilton, are generalizations of Einstein metrics. A motivation for studying GRS is that they arise in the analysis of singular solutions. By Myers’s theorem, there are no noncompact Einstein solutions to the Ricci flow with positive scalar curvature (which would homothetically shrink under the Ricci flow). In view of this, regarding *noncompact* GRS, one may expect to obtain the most information in the shrinking case. As we shall see in this chapter, this appears to be true.

A beautiful aspect of the study of GRS is the duality between the metric and the potential function (we use the term “duality” in a nontechnical way). On one hand, associated to the *metric* are geodesics and curvature. On the other hand, associated to the *potential function* are its gradient and Laplacian as well as its level sets and the integral curves of its gradient. In this chapter we shall see some of the interaction between quantities associated to the metric and to the potential function, which yields information about the geometry of GRS.

In Chapter 1 of Part I we constructed the Bryant soliton and we discussed some basic equations holding for GRS, leading to a no nontrivial steady or expanding compact breathers result. In the present chapter we focus on the qualitative aspects of the geometry of noncompact GRS.

In §1 we discuss a sharp lower bound for their scalar curvatures. In §2 we present estimates of the potential function and its gradient for GRS. In §3 we improve some lower bounds for the scalar curvatures of nontrivial GRS. In §4 we show that the volume growth of a shrinking GRS is at most Euclidean. If the scalar curvature has a positive lower bound, then one obtains a stronger estimate for the volume growth. In §5 we discuss the logarithmic Sobolev inequality on shrinking GRS. In §6 we prove that shrinking GRS with nonnegative Ricci curvature must have scalar curvature bounded below by a positive constant.

Although much is known about GRS, there is still quite a lot that is unknown. In this chapter we include some problems and conjectures (often standard or folklore) which speculate about certain geometric properties of GRS.

1. Basic properties of gradient Ricci solitons

The main topic of this section is a sharp lower bound for the scalar curvature of a complete GRS. As we shall see in §4 of this chapter, this is useful for the study the volume growth of GRS. The proof of the lower bound involves localizing the

application of the maximum principle to the elliptic equation satisfied by the scalar curvature of a GRS.

1.1. Normalized GRS structure.

A quadruple $\mathcal{G} = (\mathcal{M}^n, g, f, \varepsilon)$, consisting of a connected manifold, Riemannian metric, real-valued function, and real constant, is called a **gradient Ricci soliton (GRS) structure** if

$$(27.1) \quad \text{Rc} + \nabla^2 f + \frac{\varepsilon}{2}g = 0,$$

where Rc denotes the Ricci tensor and where $\nabla^2 f$ denotes the Hessian of the **potential function** f . We say that \mathcal{G} is **complete** if g is a complete metric. The GRS \mathcal{G} is called **shrinking**, **steady**, or **expanding** if $\varepsilon < 0$, $\varepsilon = 0$, or $\varepsilon > 0$, respectively.

Recall the following model cases:

(1) The **Gaussian soliton** on \mathbb{R}^n , which is given by $g = \sum_{i=1}^n dx^i \otimes dx^i$ and $f(x) = -\frac{\varepsilon}{4}|x|^2$, where $\varepsilon \in \mathbb{R}$.

(2) An **Einstein manifold** $(\mathcal{M}^n, g, 0, \varepsilon)$: $\text{Rc} + \frac{\varepsilon}{2}g = 0$.

(3) The **Bryant soliton**. This is the unique (up to homothety) complete rotationally symmetric steady GRS on \mathbb{R}^n for $n \geq 3$. It has positive curvature operator and its spherical sectional curvatures decay linearly, whereas its radial sectional curvatures decay quadratically.

For the Gaussian soliton the curvature is as trivial as possible, whereas for an Einstein manifold the potential function is as trivial as possible. For this reason, one may expect such solutions to represent borderline cases for various geometric invariants of GRS.

Let R denote the scalar curvature. We have the following standard formulas first derived by Hamilton (see Chapter 1 of Part I for example):

$$(27.2) \quad R + \Delta f = -\frac{n\varepsilon}{2},$$

$$(27.3) \quad 2\text{Rc}(\nabla f) = \nabla R,$$

$$(27.4) \quad \Delta R + 2|\text{Rc}|^2 + \varepsilon R = \langle \nabla R, \nabla f \rangle,$$

$$(27.5) \quad R + |\nabla f|^2 + \varepsilon f \equiv \text{const},$$

where the constant const depends on g and f .

If $\varepsilon \neq 0$, then we may take $\varepsilon = \pm 1$ and by adding a constant to the potential function f , we may assume that $\text{const} = 0$. If $\varepsilon = 0$ and g is nonflat, then by scaling the metric we may take $\text{const} = 1$. In these cases we say that \mathcal{G} is **normalized**:

$$(27.6a) \quad R + |\nabla f|^2 + \varepsilon f \equiv 0 \quad \text{for } \varepsilon = \pm 1,$$

$$(27.6b) \quad R + |\nabla f|^2 \equiv 1 \quad \text{for } \varepsilon = 0.$$

A complete normalized shrinking, steady, or expanding GRS \mathcal{G} is called a **shrinker**, a **steady**, or an **expander**, respectively.

Let $d\mu_g$ denote the Riemannian measure of g . Given a **metric measure space** $(\mathcal{M}^n, g, e^{-f}d\mu_g)$, define the **f -Laplacian** to be

$$(27.7) \quad \Delta_f \doteq \Delta - \nabla f \cdot \nabla.$$

Define the f -**Ricci tensor** to be $\text{Rc}_f \doteq \text{Rc} + \nabla^2 f$; this is also known as the **Bakry–Emery Ricci tensor**. Define the f -**scalar curvature** to be $R_f = R + 2\Delta f - |\nabla f|^2$.

Given a measurable set $X \subset \mathcal{M}$, define its f -**volume** to be

$$(27.8) \quad \text{Vol}_f(X) \doteq \int_X e^{-f} d\mu.$$

The operator Δ_f is self-adjoint with respect to the L^2 -inner product of functions using the measure $e^{-f} d\mu$.

EXERCISE 27.1. Show that for any $\varphi \in C^\infty(\mathcal{M})$ we have that

$$\left(\Delta_f - \frac{1}{4} R_f \right) \varphi = e^{f/2} \left(\Delta - \frac{1}{4} R \right) (e^{-f/2} \varphi).$$

Equations for normalized GRS may be conveniently expressed using Δ_f , Rc_f , and R_f . First of all,

$$\text{Rc}_f + \frac{\varepsilon}{2} g = 0$$

by (27.1) and we also have

$$\begin{aligned} R_f &= \varepsilon(f - n), \quad \text{for } \varepsilon = \pm 1, \\ R_f &= -1, \quad \text{for } \varepsilon = 0. \end{aligned}$$

We may rewrite the difference of (27.2) and (27.6) as

$$(27.9a) \quad \Delta_f f = \varepsilon \left(f - \frac{n}{2} \right) \quad \text{for } \varepsilon = \pm 1,$$

$$(27.9b) \quad \Delta_f f = -1 \quad \text{for } \varepsilon = 0$$

and we may rewrite (27.4) as

$$(27.10) \quad \Delta_f R = -2|\text{Rc}|^2 - \varepsilon R \leq -\frac{2}{n} R^2 - \varepsilon R.$$

1.2. Scalar curvature lower bound for a GRS structure.

The following result of B.-L. Chen does not require any curvature bound in its hypothesis. In the case where \mathcal{M} is compact, this follows directly from applying the maximum principle to equation (27.4).

THEOREM 27.2 (Lower bound for the scalar curvature of GRS). *Let $(\mathcal{M}^n, g, f, \varepsilon)$, where $\varepsilon = -1, 0$, or 1 , be a complete GRS structure.*

- (1) *If the GRS is shrinking or steady, then $R \geq 0$.*
- (2) *If the GRS is expanding, then $R \geq -\frac{n}{2}$.*

REMARK 27.3. Note that the Gaussian soliton shows that part (1) is sharp, whereas the Einstein solutions with $\text{Rc} = -\frac{1}{2}g$ show that part (2) is sharp.

PROOF OF THEOREM 27.2. We may assume that \mathcal{M} is noncompact. The proof consists of localizing equation (27.4).

STEP 1. *The Laplacian of R times a cutoff function η .* Fix a point $\tilde{O} \in \mathcal{M}$ and let $r(x) \doteq d(x, \tilde{O})$. Let $b \in [2, \infty)$ and define $\eta : [0, \infty) \rightarrow [0, 1]$ to be a C^∞ nonincreasing cutoff function with

$$(27.11) \quad \eta(u) = \begin{cases} 1 & \text{for } u \in [0, 1], \\ 0 & \text{for } u \in [1 + b, \infty) \end{cases} \quad \text{and} \quad \eta'' - 2\frac{(\eta')^2}{\eta} \geq -\text{const}$$

for some universal const $< \infty$. We define $\Phi_c : \mathcal{M} \rightarrow \mathbb{R}$ by

$$(27.12) \quad \Phi_c(x) = \eta \left(\frac{r(x)}{c} \right) R(x).$$

Throughout the proof, $c \in [2, \infty)$; eventually we let $c \rightarrow \infty$.

Taking the f -Laplacian of (27.12), we have

$$\Delta_f \Phi_c = \left(\eta \circ \frac{r}{c} \right) \Delta_f R + \frac{2}{c} \left(\eta' \circ \frac{r}{c} \right) \langle \nabla r, \nabla R \rangle + \left(\frac{\eta' \circ \frac{r}{c}}{c} \Delta_f r + \frac{\eta'' \circ \frac{r}{c}}{c^2} |\nabla r|^2 \right) R.$$

Applying (27.10) and $|\nabla r|^2 = 1$ to this, while dropping “ $\circ \frac{r}{c}$ ” in our notation, we have

$$(27.13) \quad \begin{aligned} \Delta_f \Phi_c &= \eta(-2|\text{Rc}|^2 - \varepsilon R) + \frac{2\eta'}{c} \langle \nabla r, \nabla R \rangle + \left(\frac{\eta'}{c} \Delta_f r + \frac{\eta''}{c^2} \right) R \\ &= \eta(-2|\text{Rc}|^2 - \varepsilon R) + \frac{2\eta'}{c\eta} \langle \nabla r, \nabla \Phi_c \rangle + \frac{\eta'}{c} (\Delta_f r) R + \frac{1}{c^2} \left(\eta'' - 2 \frac{(\eta')^2}{\eta} \right) R \end{aligned}$$

at all points where $\eta \neq 0$.

STEP 2. *Applying the maximum principle*¹ *to* Φ_c . Now suppose that there exists $x_c \in \mathcal{M}$ such that

$$(27.14) \quad \Phi_c(x_c) = \min_{\mathcal{M}} \Phi_c < 0.$$

Otherwise, we have $R \geq 0$ in all of $B_{\tilde{O}}(c)$.

Applying the first and second derivative tests to (27.13), using $|\text{Rc}|^2 \geq \frac{1}{n} R^2$, and dividing by $R(x_c) < 0$, we have that at x_c ,

$$(27.15) \quad 0 \geq \eta \left(-\frac{2}{n} R - \varepsilon \right) + \frac{\eta'}{c} \Delta_f r + \frac{1}{c^2} \left(\eta'' - 2 \frac{(\eta')^2}{\eta} \right).$$

We consider two cases, depending on the location of x_c .

Case (i): $x_c \in B_{\tilde{O}}(c)$. Then $\eta \circ \frac{r}{c} \equiv 1$ in a neighborhood of x_c , so that (27.15) and (27.14) imply

$$(27.16) \quad 0 \geq -\frac{2}{n} R(x_c) - \varepsilon = -\frac{2}{n} \Phi_c(x_c) - \varepsilon \geq -\frac{2}{n} \eta \left(\frac{r(x)}{c} \right) R(x) - \varepsilon$$

for all $x \in \mathcal{M}$. This yields the estimate

$$(27.17) \quad R(x) \geq -\frac{n\varepsilon}{2}$$

for all $x \in B_{\tilde{O}}(c)$ since $\eta \circ \frac{r}{c} = 1$ in $B_{\tilde{O}}(c)$.

Case (ii): $x_c \notin B_{\tilde{O}}(c)$. Regarding (27.15), since $\eta' \leq 0$, we wish to estimate the term $\Delta_f r$ from above. Recall from Lemma 18.6 in Part III (or the original Lemma 8.3 in Perelman [312]) that

$$(27.18) \quad \Delta r(x_c) \leq \int_0^{r(x_c)} \left((n-1)(\zeta')^2(s) - \zeta^2 \text{Rc}(\gamma'(s), \gamma'(s)) \right) ds$$

¹The distance function $r(x)$ is in general only Lipschitz continuous. When applying the maximum principle, to address the possible nonsmoothness of $r(x)$, one may use **Calabi's trick**; see p. 395 of [77] or pp. 453–456 in Part I for example.

for any unit speed minimal geodesic $\gamma : [0, r(x_c)] \rightarrow \mathcal{M}$ joining \tilde{O} to x_c and any continuous piecewise C^∞ function $\zeta : [0, r(x_c)] \rightarrow [0, 1]$ satisfying $\zeta(0) = 0$ and $\zeta(r(x_c)) = 1$, where $\gamma'(s) \doteq \frac{d\gamma}{ds}(s)$.

We also have

$$(27.19) \quad \begin{aligned} -\langle \nabla f, \nabla r \rangle(x_c) &= -\langle \nabla f(\tilde{O}), \gamma'(0) \rangle - \int_0^{r(x_c)} \frac{d}{ds} \langle \nabla f, \nabla r \rangle ds \\ &= -\langle \nabla f(\tilde{O}), \gamma'(0) \rangle - \int_0^{r(x_c)} \nabla^2 f(\gamma', \gamma') ds \end{aligned}$$

since $\nabla r = \gamma'(s)$ and $\nabla_{\gamma'(s)} \nabla r = 0$.

Therefore applying (27.1) to (27.19) and combining with (27.18), we have

$$\begin{aligned} \Delta_f r(x_c) &\leq (n-1) \int_0^{r(x_c)} (\zeta')^2 ds - \langle \nabla f(\tilde{O}), \gamma'(0) \rangle + \frac{\varepsilon}{2} r(x_c) \\ &\quad + \int_0^{r(x_c)} (1 - \zeta^2) \text{Rc}(\gamma', \gamma') ds. \end{aligned}$$

Let $\zeta(s) = s$ for $0 \leq s \leq 1$ and let $\zeta(s) = 1$ for $1 < s \leq r(x_c)$. Then

$$(27.20) \quad \Delta_f r(x_c) \leq n-1 - \langle \nabla f(\tilde{O}), \gamma'(0) \rangle + \frac{\varepsilon}{2} r(x_c) + \frac{2}{3} \max_{\bar{B}_{\tilde{O}}(1)} \text{Rc}_+,$$

where $\max_{\bar{B}_{\tilde{O}}(1)} \text{Rc}_+ \doteq \max_{V \in T_x \mathcal{M}, |V|=1, x \in \bar{B}_{\tilde{O}}(1)} \{\text{Rc}(V, V), 0\}$.

Applying (27.20) and (27.11) to (27.15) yields for all $x \in B_{\tilde{O}}(c)$

$$(27.21) \quad \begin{aligned} \frac{2}{n} R(x) &\geq \frac{2}{n} \Phi_c(x_c) \\ &\geq \frac{\eta'(\frac{r(x_c)}{c})}{c} \left(n-1 - \langle \nabla f(\tilde{O}), \gamma'(0) \rangle + \frac{2}{3} \max_{\bar{B}_{\tilde{O}}(1)} \text{Rc}_+ \right) \\ &\quad + \varepsilon \left(\frac{\eta'(\frac{r(x_c)}{c}) r(x_c)}{2c} - \eta\left(\frac{r(x_c)}{c}\right) \right) - \frac{\text{const}}{c^2}, \end{aligned}$$

where $\text{const} < \infty$ is independent of b and c .

Since $x_c \in B_{\tilde{O}}((1+b)c) - B_{\tilde{O}}(c)$ in Case (ii), we have

$$(27.22) \quad 1 \leq \frac{r(x_c)}{c} < 1+b.$$

We now consider the nonexpanding and expanding cases separately, where we primarily need to deal with Case (ii).

STEP 3. Proof of the theorem when $\varepsilon = 0$ or -1 . Here, Case (ii) must always hold since otherwise (27.16) contradicts assumption (27.14). We take $b = 2$. Then the inequality (27.21) and $-\text{const} \leq \eta' \leq 0$ imply that for all $x \in B_{\tilde{O}}(c)$

$$(27.23) \quad \frac{2}{n} R(x) \geq -\frac{\text{const}}{c} \left(n-1 + |\nabla f|(\tilde{O}) + \frac{2}{3} \max_{\bar{B}_{\tilde{O}}(1)} \text{Rc}_+ \right) - \frac{\text{const}}{c^2},$$

where const is independent of c . By taking $c \rightarrow \infty$, we conclude that $R(x) \geq 0$ for all $x \in \mathcal{M}$. This completes the proof of the theorem in the nonexpanding case.

STEP 4. *Proof of the theorem when $\varepsilon = 1$.* Define $\eta : [0, \infty) \rightarrow [0, 1]$ so as to satisfy the properties that η is C^∞ on $(0, 1 + c)$,

$$(27.24) \quad \eta(u) \doteq \begin{cases} 1 & \text{if } u \in [0, 1], \\ \frac{1+c-u}{c} & \text{if } u \in [2, 1+c), \\ 0 & \text{if } u \in [1+c, \infty), \end{cases}$$

and

$$(27.25) \quad -\frac{2}{c} \leq \eta' \leq 0 \quad \text{and} \quad |\eta''| \leq \text{const} \quad \text{on } (0, 1+c),$$

where $\text{const} < \infty$ is independent of c . Note that η satisfies the prior conditions in (27.11) with $b = c$. Moreover, for the purposes below, the nondifferentiability of η at $u = 1 + c$ shall not be an issue since $\eta(1 + c) = 0$.

If Case (i) holds, i.e., $x_c \in B_{\tilde{O}}(c)$, then we have the estimate (27.17) in $B_{\tilde{O}}(c)$.

Now assume that we are in Case (ii). Then $x_c \in B_{\tilde{O}}((1+c)c) - B_{\tilde{O}}(c)$.

(ii)(a) If $x_c \in B_{\tilde{O}}((1+c)c) - B_{\tilde{O}}(2c)$, then since $\eta(u) = \frac{1+c-u}{c}$ and $\eta'(u) = -\frac{1}{c}$ for $u \in [2, 1+c)$, from (27.21) we have that for all $x \in B_{\tilde{O}}(c)$

$$(27.26) \quad \begin{aligned} \frac{2}{n}R(x) &\geq -\frac{1}{c^2} \left(n-1 + |\nabla f|(\tilde{O}) + \frac{1}{2}r(x_c) + \frac{2}{3} \max_{B_{\tilde{O}}(1)} \text{Rc}_+ \right) \\ &\quad - \frac{1+c - \frac{r(x_c)}{c}}{c} - \frac{1}{c^2} \text{const} \\ &\geq -\frac{1+c}{c} - \frac{1}{c^2} \left(n-1 + |\nabla f|(\tilde{O}) + \frac{2}{3} \max_{B_{\tilde{O}}(1)} \text{Rc}_+ \right) - \frac{1}{c^2} \text{const}. \end{aligned}$$

(ii)(b) Otherwise, if $x_c \in B_{\tilde{O}}(2c) - B_{\tilde{O}}(c)$, then since $\eta \leq 1$, from (27.21) and from (27.25) we have that for all $x \in B_{\tilde{O}}(c)$

$$(27.27) \quad \frac{2}{n}R(x) \geq -\frac{2}{c^2} \left(n-1 + |\nabla f|(\tilde{O}) + c + \frac{2}{3} \max_{B_{\tilde{O}}(1)} \text{Rc}_+ \right) - 1 - \frac{1}{c^2} \text{const}.$$

From combining the estimates (27.26) and (27.27) for Case (ii) with the estimate (27.17) for Case (i), we have that for all $x \in B_{\tilde{O}}(c)$

$$(27.28) \quad \frac{2}{n}R(x) \geq \min \left\{ -\frac{1+c}{c} - \frac{1}{c^2} \text{const}, -\frac{2}{c^2} (\text{const} + c) - 1 - \frac{1}{c^2} \text{const}, -1 \right\}.$$

We then conclude when $\varepsilon = 1$ that, from taking $c \rightarrow \infty$ in (27.28), $\frac{2}{n}R(x) \geq -1$ for all $x \in \mathcal{M}$. This completes the proof of Theorem 27.2 in the expanding case. \square

1.3. Characterizing completeness of GRS structures.

Recall that the vector field ∇f is **complete** if for each $p \in \mathcal{M}$, the **integral curve** γ_p to ∇f with $\gamma_p(0) = p$ may be defined on all of \mathbb{R} . In this case, ∇f generates a 1-parameter group of diffeomorphisms $\{\varphi_t\}_{t \in \mathbb{R}}$ of \mathcal{M} which is given by $\varphi_t(p) = \gamma_p(t)$ for any $p \in \mathcal{M}$ and $t \in \mathbb{R}$. We are interested in finding conditions to ensure that the vector field ∇f is complete on a GRS.

First, by applying the lower bound for R in Theorem 27.2 to (27.6), we obtain the following result.

THEOREM 27.4 (Bounds for $|\nabla f|$). *Suppose that $(\mathcal{M}^n, g, f, \varepsilon)$ is a complete normalized GRS. Then, given any $\tilde{O} \in \mathcal{M}$, we have the following:*

(1) *For a steady,*

$$(27.29) \quad |\nabla f|(x) \leq 1 \quad \text{for all } x \in \mathcal{M}.$$

(2) *For a shrinker we have $f \geq 0$ and for all $x \in \mathcal{M}$,*

$$(27.30) \quad |\nabla f|(x) \leq f^{1/2}(x) \leq f^{1/2}(\tilde{O}) + \frac{1}{2}d(x, \tilde{O}).$$

(3) *For an expander we have $f \leq \frac{n}{2}$ and for all $x \in \mathcal{M}$,*

$$(27.31) \quad |\nabla f|(x) \leq \left(\frac{n}{2} - f(x)\right)^{1/2} \leq \frac{1}{2}d(x, \tilde{O}) + \left(\frac{n}{2} - f(\tilde{O})\right)^{1/2}.$$

PROOF. (1) $\varepsilon = 0$. Since $R \geq 0$, by (27.6b) we have

$$(27.32) \quad 1 \equiv R + |\nabla f|^2 \geq |\nabla f|^2.$$

(2) $\varepsilon = -1$. Again since $R \geq 0$, by (27.6a) we have

$$(27.33) \quad |\nabla f|^2 = -R + f \leq f.$$

For any $x \in \mathcal{M}$, let $\gamma : [0, d(x, \tilde{O})] \rightarrow \mathcal{M}$ be a unit speed minimal geodesic joining \tilde{O} to x . The function $F(s) \doteq f(\gamma(s))$ satisfies

$$\frac{dF}{ds}(s) = \nabla f(\gamma(s)) \cdot \gamma'(s) \leq |\nabla f|(\gamma(s)) \leq F^{1/2}(s).$$

Integrating this over $[0, d(x, \tilde{O})]$, we obtain²

$$F^{1/2}(d(x, \tilde{O})) \leq F^{1/2}(0) + \frac{1}{2}d(x, \tilde{O}).$$

Because $\gamma(d(x, \tilde{O})) = x$, this and (27.33) yield (27.30).

(3) $\varepsilon = 1$. Since $R \geq -\frac{n}{2}$, we have

$$(27.34) \quad -f = R + |\nabla f|^2 \geq -\frac{n}{2} + |\nabla f|^2.$$

Let the geodesic γ be as in part (2). The function $G(s) \doteq -f(\gamma(s)) + \frac{n}{2} \geq 0$ satisfies

$$\frac{dG}{ds}(s) = -\nabla f(\gamma(s)) \cdot \gamma'(s) \leq |\nabla f|(\gamma(s)) \leq G^{1/2}(s)$$

by (27.34). Therefore $G^{1/2}(d(x, \tilde{O})) \leq G^{1/2}(0) + \frac{1}{2}d(x, \tilde{O})$, which implies (27.31). \square

REMARK 27.5. The example of the Gaussian soliton, where we have $|\nabla f|(x) = \frac{|\varepsilon|}{2}|x|$ for $x \in \mathbb{R}^n$, shows that the above upper bounds for $|\nabla f|$ are qualitatively sharp.

The following is elementary.

LEMMA 27.6 (Criterion for completeness of a vector field). *Suppose $F : [0, \infty) \rightarrow [0, \infty)$ is a locally Lipschitz function with the property that any solution $u(t)$ to the ODE $\frac{du}{dt} = F(u)$, with $u(0) \in [0, \infty)$, exists for all $t \in [0, \infty)$. If a vector field X on a pointed complete Riemannian manifold $(\mathcal{M}^n, g, \tilde{O})$ satisfies $|X|(x) \leq F(d(x, \tilde{O}))$ for all $x \in \mathcal{M}$, then X is complete.*

²For each $\delta > 0$ we have $\frac{d}{ds}((F(s) + \delta)^{1/2}) \leq \frac{1}{2}$.

Combining Theorem 27.4 with Lemma 27.6 yields the following result of Z.-H. Zhang.

COROLLARY 27.7 (Completeness of g implies completeness of ∇f). *If $(\mathcal{M}^n, g, f, \varepsilon)$ is a complete GRS, then ∇f is a complete vector field.*

1.4. Equality case of the scalar curvature lower bound.

We now return to Theorem 27.2 and consider the equality case. We have the following result of Pigola, Rimoldi, and Setti.

PROPOSITION 27.8. *Let $(\mathcal{M}^n, g, f, \varepsilon)$ be a complete GRS.*

- (1) *If $\varepsilon = 1$ and if there exists $x_0 \in \mathcal{M}$ such that $R(x_0) = -\frac{n}{2}$, then $\text{Rc} \equiv -\frac{1}{2}g$.*
- (2) *If $\varepsilon = 0$ and if there exists $x_0 \in \mathcal{M}$ such that $R(x_0) = 0$, then $\text{Rc} \equiv 0$.*
- (3) *If $\varepsilon = -1$ and if there exists $x_0 \in \mathcal{M}$ such that $R(x_0) = 0$, then $(\mathcal{M}, g, f, -1)$ is a Gaussian soliton, so that (\mathcal{M}, g) is isometric to Euclidean space.*

PROOF. The idea is to apply the strong maximum principle. By Corollary 27.7 and by Theorem 4.1 in [77], we may extend the GRS structure $(\mathcal{M}, g, f, \varepsilon)$ canonically in time so that $g(t)$ is a complete solution of the Ricci flow with $g(0) = g$ and such that $g(t)$ and $f(t)$ satisfy

$$\text{Rc}_{g(t)} + \nabla_{g(t)}^2 f(t) + \frac{\varepsilon}{2(\varepsilon t + 1)} g(t) = 0.$$

Recall the standard equation

$$(27.35) \quad \frac{\partial R}{\partial t} = \Delta R + 2|\text{Rc}|^2.$$

We thus have that $\tilde{R} \doteq R + \frac{n\varepsilon}{2(\varepsilon t + 1)}$ satisfies

$$(27.36) \quad \frac{\partial \tilde{R}}{\partial t} = \Delta \tilde{R} + 2 \left| \text{Rc} - \frac{R}{n} g \right|^2 + \frac{2}{n} \left(\tilde{R} - \frac{n\varepsilon}{\varepsilon t + 1} \right) \tilde{R},$$

where we used

$$2|\text{Rc}|^2 - \frac{n\varepsilon^2}{2(\varepsilon t + 1)^2} = 2 \left| \text{Rc} - \frac{R}{n} g \right|^2 + \frac{2}{n} \left(\tilde{R} - \frac{n\varepsilon}{\varepsilon t + 1} \right) \tilde{R}.$$

Case i: $\varepsilon = 0$ or 1 . By Theorem 27.2 we have

$$R + \frac{n\varepsilon}{2(\varepsilon t + 1)} \geq 0.$$

Since there exists $x_0 \in \mathcal{M}$ such that $\tilde{R}(x_0, 0) = 0$, by applying the parabolic strong maximum principle (which is a local result) to (27.36), we have $\tilde{R}(x, t) \equiv 0$ for $x \in \mathcal{M}$ and $t \leq 0$ such that $\varepsilon t + 1 > 0$. Thus, by (27.36), we have $\text{Rc} \equiv \frac{R}{n} g \equiv -\frac{\varepsilon}{2} g$. This proves parts (1) and (2).

Case ii: $\varepsilon = -1$. By Theorem 27.2 we have $R \geq 0$. Applying the parabolic strong maximum principle to (27.35), we have $R \equiv 0$, which in turn implies $\text{Rc} \equiv 0$. Since g is a shrinker, we then obtain

$$(27.37) \quad \nabla^2 f = \frac{1}{2} g > 0.$$

Thus f is uniformly convex and proper. In particular, f attains its infimum at a unique point $O \in \mathcal{M}$ and \mathcal{M} is diffeomorphic to \mathbb{R}^n . Part (3) of Proposition 27.8 is now a consequence of the following lemma. \square

LEMMA 27.9 (A characterization of Euclidean space). *Let (\mathcal{M}^n, g) be a complete Riemannian manifold. If there exists a function f such that*

$$(27.38) \quad \nabla^2 f = \frac{1}{2}g,$$

then (\mathcal{M}, g) is isometric to Euclidean space. In particular, any complete Ricci flat shrinking GRS must be isometric to Euclidean space.

PROOF. By (27.38), we have

$$\nabla_i |\nabla f|^2 = 2\nabla_i \nabla_j f \nabla_j f = \nabla_i f,$$

so that adding a suitable constant to f yields

$$(27.39) \quad f = |\nabla f|^2 \geq 0,$$

which implies that $\inf_{\mathcal{M}} f = f(O) = 0$. Hence, defining $r \doteq 2\sqrt{f}$, we have on $\mathcal{M} - \{O\}$ that

$$(27.40) \quad \nabla^2 (r^2) = 2g, \quad |\nabla r|^2 = 1.$$

In particular, $\nabla_{\nabla r} \nabla r = 0$, so that the integral curves to ∇r are unit speed geodesics. Furthermore, by (27.40) we have that $\nabla (r^2)$ is a complete vector field which generates a 1-parameter group $\{\varphi_t\}_{t \in \mathbb{R}}$ of homotheties of g .

Since $r : \mathcal{M} \rightarrow [0, \infty)$, where r^2 is C^∞ , proper, and the only critical point of r is at O with $r(O) = 0$, and since (\mathcal{M}, g) is complete, by Morse theory we have that $\mathcal{S}_c \doteq r^{-1}(c)$ is diffeomorphic to \mathcal{S}^{n-1} for all $c \in (0, \infty)$.

Since $|\nabla r| = 1$, each homothety φ_t of g maps level sets of r to level sets of r . Hence g may be written as the warped product

$$g = dr^2 + r^2 \tilde{g},$$

where $\tilde{g} = g|_{\mathcal{S}_1}$. Since g is smooth at O , where $r = 0$, we have that $(\mathcal{S}_1, \tilde{g})$ must be isometric to the unit $(n-1)$ -sphere. Since $\bigcup_{c \in (0, \infty)} \mathcal{S}_c = \mathcal{M} - \{O\}$, we conclude

that (\mathcal{M}^n, g) is isometric to Euclidean space. \square

2. Estimates for potential functions of gradient solitons

A good qualitative understanding of the potential function is crucial in understanding the geometry of GRS. In this section we study the potential function f of a complete GRS structure $(\mathcal{M}^n, g, f, \varepsilon)$. In particular, we shall obtain bounds for $|\nabla f|$, upper and lower bounds for f , and, as a consequence, upper bounds for the scalar curvature R , all depending on the distance to a fixed point.

2.1. Bounds for the potential function f .

Immediate consequences of Theorem 27.4 are the following bounds for the potential function of a GRS.

COROLLARY 27.10 (Bounds on f for GRS). *Let $(\mathcal{M}^n, g, f, \varepsilon)$ be a complete normalized GRS and fix $\tilde{O} \in \mathcal{M}$.*

(1) *For a steady, for all $x \in \mathcal{M}$*

$$(27.41) \quad \left| f(x) - f(\tilde{O}) \right| \leq d(x, \tilde{O}).$$

(2) *For a shrinker, for all $x \in \mathcal{M}$*

$$(27.42) \quad f(x) \leq \frac{1}{4} \left(d(x, \tilde{O}) + 2\sqrt{f(\tilde{O})} \right)^2.$$

If O is a minimum point of f , then

$$(27.43) \quad f(O) \leq \frac{n}{2}$$

and

$$(27.44) \quad f(x) \leq \frac{1}{4} (d(x, O) + \sqrt{2n})^2.$$

(3) *For an expander, for all $x \in \mathcal{M}$*

$$(27.45) \quad -\frac{1}{4} \left(d(x, \tilde{O}) + \left(-4f(\tilde{O}) + 2n \right)^{1/2} \right)^2 + \frac{n}{2} \leq f(x) \leq \frac{n}{2}.$$

PROOF. (1) This follows from integrating inequality (27.29) along minimal geodesics.

(2) Since $f \geq 0$, this follows from squaring the right-hand inequality in (27.30).

Now suppose that O is a minimum point of f . Since $\Delta f(O) \geq 0$, by $R + \Delta f = \frac{n}{2}$ we have $R(O) \leq \frac{n}{2}$. Hence, by $R + |\nabla f|^2 = f$ and $|\nabla f|^2(O) = 0$, we conclude that $f(O) \leq \frac{n}{2}$. Applying this to (27.42) yields (27.44).

(3) Since $f \leq \frac{n}{2}$, this follows from squaring the right-hand inequality in (27.31). \square

For shrinkers, the potential function is in fact uniformly equivalent to the distance squared. The elegant proof of this relies on the second variation of arc length formula and integration by parts. Let $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. We have the following originally due to H.-D. Cao and D. Zhou and later refined by Haslhofer and Müller (with the sharper constants as presented here).

THEOREM 27.11 (Lower bound for f on shrinkers). *Let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized shrinking GRS. Given $\tilde{O} \in \mathcal{M}$, we have*

$$(27.46) \quad f(x) \geq \frac{1}{4} \left(\left(d(x, \tilde{O}) - 2\sqrt{f(\tilde{O})} - 4n + \frac{4}{3} \right)_+ \right)^2.$$

If O is a minimum point of f , then

$$(27.47) \quad f(x) \geq \frac{1}{4} \left(\left(d(x, O) - \frac{35}{8}n \right)_+ \right)^2.$$

PROOF. Let $x \in \mathcal{M} - B_{\tilde{O}}(2)$ be any point and let $\gamma : [0, r(x)] \rightarrow \mathcal{M}$ be a unit speed minimal geodesic joining \tilde{O} to x , where $r(x) \doteq d(x, \tilde{O})$. Define $\zeta : [0, r(x)] \rightarrow [0, 1]$ to be the piecewise linear function

$$(27.48) \quad \zeta(s) = \begin{cases} s & \text{if } s \in [0, 1], \\ 1 & \text{if } s \in (1, r(x) - 1], \\ r(x) - s & \text{if } s \in (r(x) - 1, r(x)]. \end{cases}$$

Let $\{E_1, \dots, E_{n-1}, \gamma'(0)\}$ be an orthonormal basis of $T_{\tilde{O}}\mathcal{M}$. Define $E_i(s) \in T_{\gamma(s)}\mathcal{M}$ to be the parallel translation of $E_i = E_i(0)$ along γ . Then the frame $\{E_1(s), \dots, E_{n-1}(s), \gamma'(s)\}$ forms an orthonormal basis of $T_{\gamma(s)}\mathcal{M}$ for $s \in [0, r(x)]$. Since γ is minimal, by the second variation of arc length formula, we have for each i

$$0 \leq \delta_{\zeta E_i}^2 L(\gamma) = \int_0^{r(x)} ((\zeta')^2 - \zeta^2 \langle \text{Rm}(\gamma', E_i) E_i, \gamma' \rangle) ds.$$

Summing over i , we obtain (compare with (27.18))

$$(27.49) \quad \int_0^{r(x)} \zeta^2 \text{Rc}(\gamma', \gamma') ds \leq (n-1) \int_0^{r(x)} (\zeta')^2 ds = 2(n-1).$$

By applying the shrinker equation to Rc and integrating by parts, we obtain

$$(27.50) \quad \begin{aligned} 2(n-1) &\geq \int_0^{r(x)} \zeta^2 \left(\frac{1}{2} - (f \circ \gamma)'' \right) ds \\ &= \frac{1}{2} r(x) - \frac{2}{3} + 2 \int_0^1 s (f \circ \gamma)' ds - 2 \int_{r(x)-1}^{r(x)} \zeta (f \circ \gamma)' ds \\ &\geq \frac{1}{2} r(x) - \frac{2}{3} - 2 \int_0^1 s \left(\sqrt{f}(\tilde{O}) + \frac{s}{2} \right) ds \\ &\quad - 2 \int_{r(x)-1}^{r(x)} \zeta(s) \left(\sqrt{f}(x) + \frac{\zeta(s)}{2} \right) ds, \end{aligned}$$

where for the last inequality we used by (27.30) that

$$|(f \circ \gamma)'(s)| \leq |\nabla f|(\gamma(s)) \leq \sqrt{f}(\tilde{O}) + \frac{s}{2}$$

for $s \in [0, 1]$ and $|(f \circ \gamma)'(s)| \leq \sqrt{f}(x) + \frac{\zeta(s)}{2}$ for $s \in [r(x) - 1, r(x)]$. Hence

$$2(n-1) \geq \frac{1}{2} r(x) - \frac{4}{3} - \sqrt{f}(\tilde{O}) - \sqrt{f}(x).$$

That is,

$$(27.51) \quad \sqrt{f}(x) + \sqrt{f}(\tilde{O}) \geq \frac{1}{2} d(x, \tilde{O}) - 2n + \frac{2}{3}.$$

If O is a minimum point of f , then $f(O) \leq \frac{n}{2}$ by (27.43) and hence we have $\sqrt{f}(x) \geq \frac{1}{2} r(x) - \frac{35}{16} n$. \square

REMARK 27.12. Compare (27.51) with Lemma 19.46 in Part III of Volume Two.

COROLLARY 27.13. *For a complete normalized shrinker, if f attains its minimum at O_1 and O_2 , then*

$$(27.52) \quad d(O_1, O_2) \leq \frac{35}{8} n + \sqrt{2n}.$$

PROOF. It follows from (27.47) that

$$d(x, O_1) \leq 2\sqrt{f}(x) + \frac{35}{8}n \quad \text{for all } x \in \mathcal{M}.$$

Taking $x = O_2$ and using $f(O_2) \leq \frac{n}{2}$ yields (27.52). \square

PROBLEM 27.14 (Potential functions of expanders). Regarding Corollary 27.10(3), what is the best qualitative estimate for f in the expanding case? Note that if an expander has $\text{Rc} > 0$, then the potential function is bounded from above by $-\frac{d^2(x, \tilde{O})}{4} + C$ (see Lemma 9.51 in [77] for example).

2.2. Upper bounds for R .

The estimates we proved in the previous subsections have various consequences, including bounds for the scalar curvature. For a shrinker, (27.6a) and (27.42) imply the following:

LEMMA 27.15 (Scalar curvature has at most quadratic growth). *Let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized noncompact shrinking GRS and let $\tilde{O} \in \mathcal{M}$. Then for all $x \in \mathcal{M}$,*

$$(27.53) \quad 0 \leq R(x) = f(x) - |\nabla f|^2(x) \leq \frac{1}{4} \left(d(x, \tilde{O}) + 2\sqrt{f}(\tilde{O}) \right)^2.$$

By the work of Gromoll and Meyer [124], there exist complete noncompact Riemannian manifolds with positive Ricci curvature and infinite topological type. On the other hand, Theorem 27.11 implies the following for shrinking GRS assuming a growth condition on R .

COROLLARY 27.16. *Let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized noncompact shrinking GRS. If its scalar curvature satisfies*

$$(27.54) \quad R(x) \leq \alpha(d(x, \tilde{O}) + C)^2$$

for some constants $\alpha < \frac{1}{4}$ and $C < \infty$, then \mathcal{M} has finite topological type. In particular, any complete noncompact shrinking GRS with bounded scalar curvature has finite topological type.

PROOF. By (27.46), there exists a positive constant C such that

$$(27.55) \quad f(x) \geq \frac{1}{4}(r(x) - C)^2$$

for $x \in \mathcal{M} - B_{\tilde{O}}(C)$. Hence f is a proper function; i.e., $f^{-1}((-\infty, c])$ is compact for any $c \in \mathbb{R}$.

On the other hand, by (27.6a), (27.55), and (27.54), we have

$$|\nabla f|^2 = f - R \geq \frac{1}{4}(r(x) - C)^2 - \alpha(r(x) + C)^2.$$

Thus we have $|\nabla f|^2(x) > 0$ provided $r(x)$ is sufficiently large.

Therefore, using the properness of f , we conclude that there exists $k \in \mathbb{R}$ such that the compact set $K \doteq f^{-1}((-\infty, k])$ contains all of the critical points of f . By the “deformation lemma” in Morse theory, we may deform f in a neighborhood of K , so that all of the critical points of f are nondegenerate (and still lie in a compact set). The fact that \mathcal{M} has finite topological type then follows from standard Morse theory (see Milnor [233]). \square

Since the assumption (27.54) is on the edge of holding true in the sense that it is true for $\alpha = \frac{1}{4}$ (see Lemma 27.15), we ask the following.

PROBLEM 27.17 (Conjecture 1.3 in [105]). Can one remove the condition (27.54) with $\alpha < \frac{1}{4}$ in Corollary 27.16?

A solution to Problem 27.17 would follow from obtaining a good lower estimate for $|\nabla f|^2$, i.e., one which would show that $|\nabla f|^2$ is positive outside a compact set. More ambitiously, one may pose the following:

OPTIMISTIC CONJECTURE 27.18. *For any complete noncompact shrinking GRS, the scalar/Ricci/sectional curvature is necessarily bounded from above.*

One of the best results in this direction is due to O. Munteanu and J. Wang; they proved the following.

THEOREM 27.19. *Let $(\mathcal{M}^4, g, f, -1)$ be a 4-dimensional complete noncompact shrinking GRS. If the scalar curvature R is bounded, then there exists a constant $C < \infty$ such that $|\text{Rm}| \leq CR$ on \mathcal{M} . In particular, $|\text{Rm}|$ is bounded.*

Some elementary evidence for Optimistic Conjecture 27.18 is given by inequality (27.79) below, which says that the average scalar curvatures on the sublevel sets of the potential function are bounded above by $\frac{n}{2}$.

Next we improve the scalar curvature upper bound for a net of points. We say that a countable collection of points $\{x_i\}$ in a Riemannian manifold (\mathcal{M}^n, g) is a δ -**net** if for every $y \in \mathcal{M}$ there exists i such that $d(y, x_i) \leq \delta$. The following result supports Problem 27.17.

LEMMA 27.20. *Let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized noncompact shrinking GRS and let $O \in \mathcal{M}$ be a minimum point of f . Then for any $\delta > 0$ there exists a constant $C(n, \delta) < \infty$ such that for any $x \in \mathcal{M} - B_O(C(n, \delta))$, there exists $y \in B_x(\delta)$ such that*

$$(27.56) \quad R(y) \leq C(n, \delta) (d(y, O) + 1).$$

In other words, the scalar curvature has at most linear growth on an δ -net, where the rate of growth depends on δ .

PROOF. Define

$$\zeta_\delta(s) = \begin{cases} \frac{s}{\delta} & \text{if } s \in [0, \delta], \\ 1 & \text{if } s \in (\delta, r(x) - \delta], \\ \frac{r(x) - s}{\delta} & \text{if } s \in (r(x) - \delta, r(x)]. \end{cases}$$

As in (27.50), we obtain from (27.49) that

$$\frac{2(n-1)}{\delta} \geq \frac{r(x)}{2} - \frac{2\delta}{3} + 2 \int_0^\delta \frac{s}{\delta^2} (f \circ \gamma)' ds - 2 \int_{r(x)-\delta}^{r(x)} \frac{r(x) - s}{\delta^2} (f \circ \gamma)' ds.$$

Using $|(f \circ \gamma)'(s)| \leq \sqrt{f}(O) + \frac{s}{2}$ for $s \in [0, \delta]$, the above formula implies that

$$2 \int_{r(x)-\delta}^{r(x)} \frac{r(x) - s}{\delta^2} (f \circ \gamma)' ds \geq \frac{r(x)}{2} - 2 \left(\frac{n-1}{\delta} + \frac{\delta}{2} \right) - \sqrt{f}(O).$$

Since $2 \int_{r(x)-\delta}^{r(x)} \frac{r(x)-s}{\delta^2} ds = 1$, there exists $\hat{s} \in [r(x) - \delta, r(x)]$ such that $y \doteq \gamma(\hat{s})$ satisfies

$$|\nabla f|(y) \geq \frac{r(y)}{2} - 2 \left(\frac{n-1}{\delta} + \frac{\delta}{2} \right) - \sqrt{\frac{n}{2}}$$

since $r(x) \geq r(y)$ and $\sqrt{f}(O) \leq \sqrt{\frac{n}{2}}$. Thus

$$|\nabla f|^2(y) \geq \frac{1}{4} ((d(y, O) - 2C(n, \delta))_+)^2,$$

where $C(n, \delta) \doteq 2 \left(\frac{n-1}{\delta} + \frac{\delta}{2} \right) + \sqrt{\frac{n}{2}}$.

On the other hand, by (27.44) we have

$$f(y) \leq \frac{1}{4} (d(y, O) + \sqrt{2n})^2.$$

Therefore, if $d(x, O) \geq 2C(n, \delta) + \delta$, then $d(y, O) \geq 2C(n, \delta)$ and hence

$$\begin{aligned} R(y) &= f(y) - |\nabla f|^2(y) \\ &\leq \frac{1}{4} (d(y, O) + \sqrt{2n})^2 - \frac{1}{4} (d(y, O) - 2C(n, \delta))^2 \\ &= d(y, O) \left(\sqrt{\frac{n}{2}} + C(n, \delta) \right) + \frac{n}{2} - C(n, \delta)^2. \end{aligned} \quad \square$$

This motivates us to consider

CONJECTURE 27.21 (Elliptic Harnack estimate for the scalar curvature). *Let $(\mathcal{M}^n, g, f, -1)$ be a complete noncompact shrinking GRS. There exists $\text{const} < \infty$ such that for any $x, y \in \mathcal{M}$ with $d(x, y) \leq 1$ we have*

$$(27.57) \quad R(x) \leq \text{const} R(y).$$

EXERCISE 27.22 (An elliptic Harnack estimate would imply finite topological type). Show that the truth of (27.57) would affirm Problem 27.17.

Returning to the lower bound for $|\nabla f|$ of a noncompact shrinker, note that the most optimistic conjecture would be that $|\nabla f|(x) \geq \frac{1}{2}d(x, \tilde{O}) - C$ on \mathcal{M} for some constant C . By (27.6a) and (27.46), we have

$$(27.58) \quad |\nabla f|^2 \geq \frac{1}{4} \left(\left(d(x, \tilde{O}) - C \right)_+ \right)^2 - R,$$

so that such an estimate would follow from a uniform upper bound for R .

Regarding the scalar curvature of a noncompact steady or expanding GRS, if we further assume a positive Ricci pinching condition, then R attains its maximum and has quadratic exponential decay (see D. Chen and L. Ma [215] or Theorem 9.56 in [77]). In particular, we have

PROPOSITION 27.23. *Let $\mathcal{G} = (\mathcal{M}^n, g, f, 1)$ be an expanding GRS with $\text{Rc} \geq \eta \text{R}g$, where $\eta > 0$ and $R > 0$. Then, for any $\bar{\eta} < \eta$, there exists $C < \infty$ such that*

$$(27.59) \quad R(x) \leq C e^{-\bar{\eta} d^2(x, \tilde{O})}.$$

We have the following results of Y. Deng and X. Zhu.

THEOREM 27.24. *Let $\mathcal{G} = (\mathcal{M}, g, f, 1)$ be an expanding Kähler GRS with $\dim_{\mathbb{C}} \mathcal{M} = n$ and let $\tilde{O} \in \mathcal{M}$. If $R \in o(d^{-2}(x, \tilde{O}))$, then (\mathcal{M}, g) is isometric to \mathbb{C}^n .*

COROLLARY 27.25. *There does not exist an expanding Kähler GRS with $\dim_{\mathbb{C}} \mathcal{M} \geq 2$ and $Rc \geq \eta Rg$, where $\eta > 0$ and $R > 0$.*

3. Lower bounds for the scalar curvature of nonflat nonexpanding gradient Ricci solitons

In this section, in the case of nonflat nonexpanding (i.e., shrinking and steady) GRS and with the aid of the estimates for the potential function of the previous section, we sharpen the lower bounds for the scalar curvature given in §1.

3.1. Lower estimate of R for shrinkers.

In this subsection we discuss a lower bound for the scalar curvatures of noncompact nonflat shrinkers. We have the following result due to the combined works of B. Wilking, B. Yang, and three of the authors.

THEOREM 27.26 (Scalar curvature of nonflat shrinkers decay at most quadratically). *Let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized noncompact nonflat shrinking GRS. Then for any point $\tilde{O} \in \mathcal{M}$ there exists a constant $C_0 > 0$ such that*

$$R(x) \geq C_0^{-1} d^{-2}(x, \tilde{O})$$

wherever $d(x, \tilde{O}) \geq C_0$. Consequently, the **asymptotic scalar curvature ratio** $\text{ASCR}(g) > 0$ (see (19.8) in Part III for its definition) and the asymptotic cone, if it exists, is not flat.

PROOF. We modify the quantity R appearing in (27.10) by adding powers of f . For any $p \in \mathbb{R}$, using (27.9a) we compute that

$$(27.60) \quad \Delta_f \left(\frac{1}{p} f^{-p} \right) = f^{-p} - f^{-p-1} \left(\frac{n}{2} - (p+1) \frac{|\nabla f|^2}{f} \right).$$

In particular, we shall use the formulas ($p = 1, 2$)

$$(27.61) \quad \Delta_f (f^{-1}) = f^{-1} - f^{-2} \left(\frac{n}{2} - 2 \frac{|\nabla f|^2}{f} \right),$$

$$(27.62) \quad \Delta_f (f^{-2}) = 2f^{-2} - f^{-3} \left(n - 6 \frac{|\nabla f|^2}{f} \right).$$

Using (27.10) and (27.61), we compute for any $c > 0$ that

$$(27.63) \quad \Delta_f (R - cf^{-1}) \leq R - cf^{-1} + cf^{-2} \left(\frac{n}{2} - 2 \frac{|\nabla f|^2}{f} \right).$$

Keeping in mind that we wish to modify (27.63) so that more negative terms appear on the RHS, we define $\phi \doteq R - cf^{-1} - cnf^{-2}$. By (27.62), we obtain

$$(27.64) \quad \Delta_f \phi \leq \phi - cnf^{-3} \left(\frac{f}{2} - n \right) - cf^{-4} (2f + 6n) |\nabla f|^2.$$

By (27.42) and (27.46), we have good estimates on the potential function:

$$(27.65) \quad \frac{1}{4} \left[\left(d(x, \tilde{O}) - C_1 \right)_+ \right]^2 \leq f(x) \leq \frac{1}{4} \left(d(x, \tilde{O}) + 2\sqrt{f}(\tilde{O}) \right)^2$$

for some constant C_1 . Choosing $c > 0$ sufficiently small, we have $\phi > 0$ inside $B_{\tilde{O}}(C_1 + 3n)$. If $\inf_{\mathcal{M} - B_{\tilde{O}}(C_1 + 3n)} \phi \doteq -\delta < 0$, then by (27.65) there exists $\rho > C_1 + 3n$ such that $\phi > -\frac{\delta}{2}$ in $\mathcal{M} - B_{\tilde{O}}(\rho)$. Thus a negative minimum of ϕ is attained at some point x_0 outside of $B_{\tilde{O}}(C_1 + 3n)$. By the maximum principle, evaluating (27.64) at x_0 yields $\frac{f(x_0)}{2} - n \leq 0$. However, (27.65) implies that $f(x_0) \geq \frac{9n^2}{4}$, a contradiction. We conclude that

$$R \geq cf^{-1} + cnf^{-2} \quad \text{on } \mathcal{M}.$$

The theorem now follows from (27.65). \square

REMARK 27.27. M. Feldman, T. Ilmanen, and one of the authors [111] constructed complete noncompact Kähler shrinkers on the total spaces of k -th powers of tautological line bundles over the complex projective space $\mathbb{C}\mathbb{P}^{n-1}$ for $0 < k < n$. These examples, which have Euclidean volume growth and quadratic scalar curvature decay, show that Theorem 27.26 is sharp.

As a variation on the proof of Theorem 27.26, define on $\{f > \frac{n}{2}\}$ the function $\psi(f) = \frac{c}{f - \frac{n}{2}}$, where $c > 0$. In general, we compute that

$$\Delta_f(R - \psi(f)) = -2|\text{Rc}|^2 + R + \left(f - \frac{n}{2}\right) \psi'(f) - \psi''(f) |\nabla f|^2.$$

Since $(f - \frac{n}{2})\psi'(f) = -\psi(f)$ and $\psi'' \geq 0$, we obtain

$$(27.66) \quad \Delta_f(R - \psi(f)) \leq -2|\text{Rc}|^2 + R - \psi(f).$$

Choose c sufficiently small so that $R \geq c$ on $\{f = \frac{n}{2} + 1\}$, which implies that $R - \psi(f) \geq 0$ on $\{f = \frac{n}{2} + 1\}$. By applying the maximum principle to (27.66), since $\liminf_{x \rightarrow \infty} (R - \psi(f))(x) \geq 0$, we obtain a contradiction if $R - \psi(f) < 0$ somewhere in $\{f \geq \frac{n}{2} + 1\}$. Hence $R - \psi(f) \geq 0$ in $\{f \geq \frac{n}{2} + 1\}$.

3.2. Lower estimate of R for steadies.

By a similar argument to the previous subsection we may prove the following result, due to B. Yang and two of the authors, regarding steady GRS assuming a condition on the potential function.

THEOREM 27.28 (Scalar curvature of nonflat steadies decay at most exponentially). *Let $(\mathcal{M}^n, g, f, 0)$ be a complete normalized steady GRS. If $\lim_{x \rightarrow \infty} f(x) = -\infty$ and $f \leq 0$, then $R \geq \frac{1}{\sqrt{\frac{n}{2} + 2}} e^f$. Since $|\nabla f| \leq 1$, this implies that for any $\tilde{O} \in \mathcal{M}$ we have*

$$(27.67) \quad R(x) \geq ce^{-d(x, \tilde{O})} \quad \text{for all } x \in \mathcal{M},$$

where $c = (\sqrt{\frac{n}{2} + 2})^{-1} e^{f(\tilde{O})}$.

PROOF. The idea is to find a lower barrier function (an expression of f) for the scalar curvature R . Using (27.9b), we compute that

$$\Delta_f(e^f) = e^f \Delta_f f + e^f |\nabla f|^2 = -R e^f < 0.$$

By this and (27.10), we obtain for $c \in \mathbb{R}$,

$$\Delta_f (R - ce^f) \leq -\frac{2}{n}R^2 + cRe^f \leq \frac{nc^2}{8}e^{2f}.$$

Using $\Delta_f(e^{2f}) = 2e^{2f}(1 - 2R)$, we compute for any constant $b \in \mathbb{R}$ that

$$(27.68) \quad \Delta_f (R - ce^f - be^{2f}) \leq \left(\frac{nc^2}{8} - 2b + 4bR \right) e^{2f}.$$

Given $b, c > 0$ to be chosen below, suppose that $R - ce^f - be^{2f}$ is negative somewhere. Then, since $R \geq 0$ by Theorem 27.2(1) and since $\lim_{x \rightarrow \infty} e^{f(x)} = 0$ by hypothesis, a negative minimum of $R - ce^f - be^{2f}$ is attained at some point. By (27.68) and the maximum principle, at such a point we have

$$0 \leq \frac{nc^2}{8} - 2b + 4bR < \frac{nc^2}{8} - 2b + 4b(c + b)$$

since $f \leq 0$. Given $c \in (0, \frac{1}{2}]$, the minimizing choice $b = \frac{1-2c}{4}$ yields $\frac{(1-2c)^2}{4} < \frac{nc^2}{8}$. With this choice of b , we obtain a contradiction by then choosing $c = \frac{1}{\sqrt{\frac{n}{2}+2}}$. \square

REMARK 27.29. For the cigar soliton $(\mathbb{R}^2, \frac{dx^2+dy^2}{1+x^2+y^2})$ we have $R = e^f$, where $f(x, y) = -\ln(1 + x^2 + y^2)$.

4. Volume growth of shrinking gradient Ricci solitons

In this section we discuss the asymptotic volume ratio of noncompact shrinkers, including a Euclidean upper bound for their volume growth. We consider two approaches: sublevel sets of the potential function and the Riccati equation along geodesics.

4.1. A differential identity for the volume ratio of sublevel sets.

Given a complete noncompact Riemannian manifold (\mathcal{N}^n, h) and a basepoint $\tilde{O} \in \mathcal{N}$, the **asymptotic volume ratio** (AVR) is defined by

$$(27.69) \quad \text{AVR}(h) \doteq \lim_{r \rightarrow \infty} \frac{\text{Vol } B_{\tilde{O}}(r)}{\omega_n r^n},$$

provided the limit exists, where ω_n is the volume of the unit Euclidean n -ball. If $\text{Rc} \geq 0$, then $\text{AVR}(h) \in [0, 1]$ exists by the Bishop volume comparison theorem. In general, whenever the AVR exists, it is independent of \tilde{O} .

In the sublevel set approach, we shall need the co-area formula (see Schoen and Yau [355, p. 89] or Lemma 5.4 of [77] for example), which says the following.

PROPOSITION 27.30 (Co-area formula). *Let (\mathcal{M}^n, g) be a compact manifold with or without boundary. If f is a Lipschitz function and if h is an L^1 function or a nonnegative measurable function, then*

$$(27.70) \quad \int_{\mathcal{M}} h |\nabla f| d\mu = \int_{-\infty}^{\infty} dc \int_{\{f=c\}} h d\sigma,$$

where $d\mu$ is the Riemannian measure on \mathcal{M} and where $d\sigma$ is the induced measure on $\{f = c\}$.

Henceforth, let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized noncompact shrinking GRS structure. Extend this structure to a complete solution $g(t)$, $t \in (-\infty, 1)$, to the Ricci flow with $g(0) = g$. By Remark 13.32 in Part II and by the local nature of real analyticity, we may extend Bando's Theorem 13.21 in Part II to the noncompact case. Namely, using Shi's local derivative estimates, one can show that \mathcal{M} has a real analytic structure and that, given t , the metric components $g_{ij}(t)$ are real analytic functions in any normal coordinate system. In particular, g is real analytic. We shall use this fact in the sublevel set approach.

By (27.46) we have that f attains its minimum at some point $O \in \mathcal{M}$. Thus, from (27.2), we have

$$(27.71) \quad \inf_{x \in \mathcal{M}} R(x) \leq R(O) = \frac{n}{2} - \Delta f(O) \leq \frac{n}{2}.$$

We claim that, since \mathcal{M} is noncompact, we have $\inf_{x \in \mathcal{M}} R(x) < \frac{n}{2}$. If not, then on \mathcal{M} we have $\Delta f = \frac{n}{2} - R \leq 0$. Since f is superharmonic and attains its minimum, we conclude that f is a constant function by the strong maximum principle, which contradicts \mathcal{M} being a noncompact shrinker.

Define the functions

$$V : \mathbb{R} \rightarrow (0, \infty),$$

$$R : \mathbb{R} \rightarrow (0, \infty)$$

by³

$$(27.72a) \quad V(c) \doteq \int_{\{f < c\}} d\mu = \text{Vol}\{f < c\},$$

$$(27.72b) \quad R(c) \doteq \int_{\{f < c\}} R d\mu$$

for $c \in \mathbb{R}$, which are nondecreasing nonnegative functions of c since $R \geq 0$.

Since we have good control of f , to approach the AVR we shall use $\frac{V(c)}{c^{n/2}}$ instead of the volume ratios of geodesic balls. More precisely, by (27.42) and (27.46), given any $\tilde{O} \in \mathcal{M}$, there exists a constant $C_1 < \infty$ such that

$$\frac{1}{4} \left((d(x, \tilde{O}) - C_1)_+ \right)^2 \leq f(x) \leq \frac{1}{4} \left((d(x, \tilde{O}) + C_1) \right)^2.$$

Therefore, if $r \geq C_1$, then

$$\left\{ f < \frac{1}{4}(r - C_1)^2 \right\} \subset B_{\tilde{O}}(r) \subset \left\{ f < \frac{1}{4}(r + C_1)^2 \right\}.$$

Thus

$$\frac{V(\frac{1}{4}(r - C_1)^2)}{r^n} \leq \frac{\text{Vol} B_{\tilde{O}}(r)}{r^n} \leq \frac{V(\frac{1}{4}(r + C_1)^2)}{r^n}.$$

Thus $\lim_{c \rightarrow \infty} \frac{V(c)}{c^{n/2}}$ exists if and only if $\text{AVR}(g)$ exists. Thus, in this case,

$$(27.73) \quad 2^n \omega_n \text{AVR}(g) = \lim_{c \rightarrow \infty} \frac{V(c)}{c^{n/2}}.$$

Recall that by Sard's theorem, the set of singular values of f has Lebesgue measure zero. Note that in general $\partial\{f < c\} \subset \{f = c\}$, whereas for a regular value c of f we have that $\partial\{f < c\} = \{f = c\}$ is a smooth compact hypersurface

³Note the distinction between the function R and the scalar curvature R , whose notations look similar.

(here we recall that by (27.46), f is a proper function). Hence we verify by the co-area formula:

LEMMA 27.31. *If c is a regular value of f , then*

$$(27.74) \quad V'(c) = \int_{\{f=c\}} \frac{1}{|\nabla f|} d\sigma,$$

$$(27.75) \quad R'(c) = \int_{\{f=c\}} \frac{R}{|\nabla f|} d\sigma.$$

Both derivatives exist (note that $|\nabla f| > 0$ on $\{f=c\}$ since c is a regular value of f).

PROOF. Since g and $\nabla^2 f = -Rc + \frac{1}{2}g$ are both real analytic, we have that f is real analytic. Therefore $|\nabla f|^2$ is real analytic. In particular, either $|\nabla f|^2 \equiv 0$ on all of \mathcal{M} or $|\nabla f|^2 = 0$ only on a set of measure zero (see Krantz [171, p. 103]). Since f is not a constant function, we have that $|\nabla f|^{-1}$ is measurable. Hence, we may apply the co-area formula (27.70) with $h = |\nabla f|^{-1}$ and with $h = R|\nabla f|^{-1}$ to the compact manifold with boundary $\{f \leq \bar{c}\}$ to conclude that, for any regular value $\bar{c} \in [0, \infty)$ of $f \geq 0$, we have

$$(27.76) \quad V(\bar{c}) = \int_0^{\bar{c}} dc \int_{\{f=c\}} \frac{1}{|\nabla f|} d\sigma,$$

$$(27.77) \quad R(\bar{c}) = \int_0^{\bar{c}} dc \int_{\{f=c\}} \frac{R}{|\nabla f|} d\sigma,$$

respectively. The lemma follows. \square

Integrating the formula (27.2) over $\{f < c\}$ yields

$$(27.78) \quad \frac{n}{2} V(c) - R(c) = \int_{\{f < c\}} \Delta f d\mu = \int_{\{f=c\}} \frac{\partial f}{\partial \nu} d\sigma = \int_{\{f=c\}} |\nabla f| d\sigma$$

if c is a regular value of f . In particular,

$$(27.79) \quad R(c) \leq \frac{n}{2} V(c).$$

By (27.6a), we have wherever $|\nabla f| \neq 0$ that

$$(27.80) \quad |\nabla f| = \frac{|\nabla f|^2}{|\nabla f|} = \frac{f - R}{|\nabla f|}.$$

Applying this to (27.78) yields

LEMMA 27.32 (Differential identity relating V and R).

$$(27.81) \quad \frac{n}{2} V(c) - c V'(c) = R(c) - R'(c) \quad \text{for a.e. } c.$$

4.2. The AVR exists and is bounded.

For complete noncompact shrinkers, the AVR always exists. The result that the volume growth is at most Euclidean is due to H.-D. Cao and D. Zhou, with the assistance of Munteanu, who removed a technical condition that they assumed. That the constant stated below depends only on n is due to Haslhofer and Müller.

THEOREM 27.33 (The AVR exists and is bounded for shrinkers). *Let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized noncompact shrinking GRS. Then $\text{AVR}(g)$ exists and is bounded above by a constant depending only on n .*

PROOF. We modify the volume ratio $\frac{V(c)}{c^{\frac{n}{2}}}$ by defining the quantity

$$(27.82) \quad P(c) \doteq \frac{V(c)}{c^{\frac{n}{2}}} - \frac{R(c)}{c^{\frac{n}{2}+1}}.$$

For convenience, let $N(c) \doteq \frac{R(c)}{cV(c)}$. Note that $\frac{R(c)}{V(c)}$ is the average scalar curvature over the set $\{f < c\}$. Using the ODE (27.81), we compute that

$$(27.83) \quad \begin{aligned} P'(c) &= \frac{V'(c)}{c^{\frac{n}{2}}} - \frac{n}{2} \frac{V(c)}{c^{\frac{n}{2}+1}} - \frac{R'(c)}{c^{\frac{n}{2}+1}} + \frac{n+2}{2} \frac{R(c)}{c^{\frac{n}{2}+2}} \\ &= - \left(1 - \frac{n+2}{2c}\right) \frac{R(c)}{c^{\frac{n}{2}+1}} \\ &= - \frac{(1 - \frac{n+2}{2c}) N(c)}{1 - N(c)} P(c). \end{aligned}$$

By (27.79), we have monotonicity, i.e.,

$$(27.84) \quad P'(c) \leq 0$$

for $c \geq \frac{n+2}{2}$ and we have

$$(27.85) \quad \left(1 - \frac{n}{2c}\right) \frac{V(c)}{c^{\frac{n}{2}}} \leq P(c) \leq \frac{V(c)}{c^{\frac{n}{2}}}.$$

Hence, by (27.73) and (27.85), we have

$$(27.86) \quad 2^n \omega_n \text{AVR}(g) = \lim_{c \rightarrow \infty} \frac{V(c)}{c^{n/2}} = \lim_{c \rightarrow \infty} P(c),$$

where the limits exist by (27.84).

Finally, we show that $\text{AVR}(g)$ is bounded above by a constant depending only on n . We have for all $c \geq \frac{n+2}{2}$,

$$P(c) \leq P\left(\frac{n+2}{2}\right) \leq \frac{V(\frac{n+2}{2})}{(\frac{n+2}{2})^{\frac{n}{2}}} \leq \frac{\text{Vol } B_{\tilde{O}}(\sqrt{2(n+2)} + \frac{35}{8}n)}{(\frac{n+2}{2})^{\frac{n}{2}}};$$

the last inequality follows from (27.47). Since $\text{AVR}(g)$ is independent of the base-point, we may choose \tilde{O} to be a minimum point O of f . The desired result follows from

Claim. *There exists $C(n) < \infty$ such that*

$$\text{Vol } B_O\left(\sqrt{2(n+2)} + \frac{35}{8}n\right) \leq C(n).$$

Hence, by (27.86) we have $\text{AVR}(g) \leq \frac{C(n)}{\omega_n(2(n+2))^{n/2}}$.

Proof of the claim. Let $a_n = \sqrt{2(n+2)} + \frac{35}{8}n$. By (27.44), for $x \in B_O(a_n)$ we have

$$(27.87) \quad |\nabla f|(x) \leq \sqrt{f}(x) \leq \frac{1}{2}(\sqrt{2n} + d(x, O)) \leq \frac{1}{2}(\sqrt{2n} + a_n) \leq 3.4n.$$

Since the f -Ricci tensor is nonnegative, i.e., $\text{Rc}_f = \text{Rc} + \nabla^2 f = \frac{1}{2}g \geq 0$, the Bakry–Emery volume comparison theorem (see (27.104) below) implies that

$$(27.88) \quad \text{Vol}_f B_O(a_n) = \int_{B_O(a_n)} e^{-f} d\mu \leq \omega_n e^{3.4na_n} a_n^n.$$

Finally,

$$\text{Vol } B_O(a_n) \leq e^{\max_{B_O(a_n)} f} \text{Vol}_f B_O(a_n) \leq e^{\frac{1}{4}(a_n + \sqrt{2n})^2} \omega_n e^{3.4na_n} a_n^n \doteq C(n)$$

since $f \leq \frac{1}{4}(a_n + \sqrt{2n})^2$ in $B_O(a_n)$. \square

PROBLEM 27.34. For shrinkers, must we have $\text{AVR}(g) \leq 1$?

4.3. Characterizing when $\text{AVR} > 0$ for noncompact shrinkers.

Let $V(c)$ and $R(c)$ be defined as in (27.72). For shrinkers, we have the following characterization for $\text{AVR} > 0$. This is due to B. Yang and two of the authors, based on earlier works of others.

PROPOSITION 27.35 (Necessary and sufficient condition for $\text{AVR} > 0$). *Let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized noncompact shrinking GRS. Then*

$$(27.89) \quad \text{AVR}(g) > 0 \quad \text{if and only if} \quad \int_{n+2}^{\infty} \frac{R(c)}{cV(c)} dc < \infty,$$

or, equivalently,

$$(27.90) \quad \int_1^{\infty} \frac{\int_{B_{\bar{O}}(r)} R d\mu}{\text{Vol } B_{\bar{O}}(r)} \frac{dr}{r} < \infty.$$

That is, $\text{AVR}(g) = 0$ if and only if $\int_{n+2}^{\infty} \frac{R(c)}{cV(c)} dc = \infty$, or, equivalently,

$$\int_1^{\infty} \frac{\int_{B_{\bar{O}}(r)} R d\mu}{\text{Vol } B_{\bar{O}}(r)} \frac{dr}{r} = \infty.$$

PROOF. Integrating (27.83) yields

$$(27.91) \quad P(c) = P(n+2) e^{-\int_{n+2}^c \frac{(1-\frac{n+2}{2c})N(c)}{1-N(c)} dc}$$

for $c \geq n+2$. Regarding the integral on the RHS, from (27.79) it is easy to see that for any $c \in [n+2, \infty)$ we have

$$(27.92) \quad \frac{1}{2} \int_{n+2}^c N(c) dc \leq \int_{n+2}^c \left(1 - \frac{n+2}{2c}\right) \frac{N(c)}{1-N(c)} dc \leq 2 \int_{n+2}^c N(c) dc.$$

If $\int_{n+2}^{\infty} N(c) dc = \infty$, then by (27.91) we have $\text{AVR}(g) = \frac{1}{2^n \omega_n} \lim_{c \rightarrow \infty} P(c) = 0$.

On the other hand, if $\int_{n+2}^{\infty} N(c) dc < \infty$, then it follows from (27.91) and (27.92) that

$$P(c) \geq P(n+2) e^{-2 \int_{n+2}^{\infty} N(c) dc} > 0$$

for $c \geq n+2$; hence $\text{AVR}(g) > 0$. We have shown that $\text{AVR}(g) = 0$ if and only if $\int_{n+2}^{\infty} \frac{R(c)}{cV(c)} dc = \infty$.

Finally, we observe that by inequalities (27.42) and (27.46) we have that $\int_{n+2}^{\infty} \frac{R(c)}{cV(c)} dc = \infty$ if and only if $\int_1^{\infty} \frac{\int_{B_{\bar{O}}(r)} R d\mu}{\text{Vol } B_{\bar{O}}(r)} \frac{dr}{r} = \infty$. \square

Using Proposition 27.35, one may obtain the following qualitatively sharp result due to S.-J. Zhang.

COROLLARY 27.36 (Volume growth of shrinkers with $R \geq \delta \geq 0$). *If $\mathcal{G} = (\mathcal{M}^n, g, f, -1)$ is a complete noncompact normalized shrinking GRS with $R \geq \delta \geq 0$ and $\tilde{O} \in \mathcal{M}$, then*

$$(27.93) \quad \text{Vol } B_{\tilde{O}}(r) \leq \text{const}(\mathcal{G}, \tilde{O}) (1+r)^{n-2\delta}.$$

PROOF. By hypothesis, $N(c) = \frac{R(c)}{c\sqrt{V(c)}} \geq \frac{\delta}{c}$. Hence, combining this inequality with (27.91), for $c \geq n+2$ we have

$$P(c) \leq P(n+2) e^{-\int_{n+2}^c (1-\frac{n+2}{2c})^{\frac{\delta}{c}} dc} \leq (n+2)^\delta e^{\frac{\delta}{2}} P(n+2) c^{-\delta}.$$

From (27.85), we have $P(c) \geq \frac{V(c)}{2c^{n/2}}$ for $c \geq n+2$ and we conclude that

$$(27.94) \quad V(c) \leq 2(n+2)^\delta e^{\frac{\delta}{2}} P(n+2) c^{\frac{n}{2}-\delta}.$$

By (27.42) we obtain $B_{\tilde{O}}(2\sqrt{c}-C) \subset \{f < c\}$. Hence, using (27.94), we derive that

$$\text{Vol } B_{\tilde{O}}(2\sqrt{c}-C) \leq 2(n+2)^\delta e^{\frac{\delta}{2}} P(n+2) c^{\frac{n}{2}-\delta}$$

holds for $c \geq n+2$ almost everywhere, so that

$$\text{Vol } B_{\tilde{O}}(r) \leq 2(n+2)^\delta e^{\frac{\delta}{2}} P(n+2) \left(\frac{r+C}{2}\right)^{n-2\delta}$$

for all $r \geq 2\sqrt{n+2}-C$. This completes the proof of Corollary 27.36. \square

REMARK 27.37. For $2 \leq k \leq n$, we have the cylinder shrinkers $(\mathcal{M}^n, g) = (\mathcal{N}^k, h) \times \mathbb{R}^{n-k}$, where $R_{C_h} \equiv \frac{1}{2}h$ and hence $R_g \equiv \frac{k}{2}$. Since \mathcal{N} must be compact, we have $\text{Vol } B_{\tilde{O}}(r) \approx \text{const} (1+r)^{n-k}$.

We may ask the following:

QUESTION 27.38. *Can one show that if a simply-connected noncompact shrinker $(\mathcal{M}^n, g, f, -1)$ satisfies $R \geq \delta > 0$, then it must have a compact factor (\mathcal{N}^k, h) with $k \geq \min\{m \in \mathbb{Z} : m \geq 2\delta\}$?*

QUESTION 27.39. *Can one show that if a complete noncompact shrinking GRS $(\mathcal{M}^n, g, f, -1)$ satisfies $\text{AVR}(g) > 0$, then $\text{ASCR}(g) < \infty$?*

QUESTION 27.40. *Can one show that if a simply-connected noncompact shrinker $(\mathcal{M}^n, g, f, -1)$ does not have an \mathbb{R} factor, then g has Euclidean volume growth?*

4.4. Bakry–Emery volume comparison.

The analytic essence of the Bishop volume comparison theorem is the Bochner formula

$$(27.95) \quad \frac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u \rangle + \text{Rc}(\nabla u, \nabla u)$$

applied to the distance function. We now consider the Bakry–Emery volume comparison theorem, which was used to derive (27.88) above, from the same point of view.

Let (\mathcal{M}^n, g) be a complete Riemannian manifold and let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function. Adding (27.95) and

$$-\frac{1}{2}\langle \nabla f, \nabla |\nabla u|^2 \rangle = -\langle \nabla u, \nabla \langle \nabla f, \nabla u \rangle \rangle + \nabla^2 f(\nabla u, \nabla u)$$

together, we obtain the f -**Bochner formula**

$$(27.96) \quad \frac{1}{2} \Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \text{Rc}_f(\nabla u, \nabla u).$$

Let $\tilde{O} \in \mathcal{M}$ and let $r(x) = d(x, \tilde{O})$. Let $S(\tilde{O}, r) \doteq \{x \in \mathcal{M} : d(x, \tilde{O}) = r\}$ be the distance sphere. Let H denote the mean curvature of $S(\tilde{O}, r)$, wherever it is a smooth hypersurface. The f -**mean curvature** is

$$(27.97) \quad H_f = H - \langle \nabla f, \nabla r \rangle = \Delta_f r$$

since $\Delta r = H$. Recall that since $|\nabla r|^2 \equiv 1$ and $\nabla^2 r = \text{II}$, (27.96) with $u = r$ implies the f -**Riccati equation**

$$(27.98) \quad \frac{\partial H_f}{\partial r} = -\text{Rc}_f \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) - |\text{II}|^2 \leq -\text{Rc}_f \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) - \frac{H^2}{n-1}.$$

Let J denote the Jacobian of the exponential map and let $J_f \doteq e^{-f} J$ denote the f -**Jacobian**. We have $\frac{\partial}{\partial r} \ln J = H$ and $\frac{\partial}{\partial r} \ln J_f = H_f$. Thus

$$(27.99) \quad \frac{J_f(r_2)}{J_f(r_1)} = e^{\int_{r_1}^{r_2} H_f(r) dr}.$$

If $\text{Rc}_f \geq -\frac{\varepsilon}{2}g$ for some $\varepsilon \in \mathbb{R}$, then

$$\frac{\partial}{\partial r} (r^2 H) \leq 2rH - \frac{r^2 H^2}{n-1} - r^2 \text{Rc} \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) \leq n-1 + r^2 \left(\frac{\partial^2 f}{\partial r^2} + \frac{\varepsilon}{2} \right).$$

Hence

$$(27.100) \quad \frac{\partial}{\partial r} (r^2 H_f) \leq n-1 - 2r \frac{\partial f}{\partial r} + \frac{\varepsilon}{2} r^2.$$

Integrating this while using $\lim_{r \searrow 0} r^2 H_f(r) = 0$, we obtain

$$(27.101) \quad H(r) - \frac{\partial f}{\partial r}(r) = H_f(r) \leq \frac{n-1}{r} + \frac{\varepsilon}{6} r - \frac{2}{r^2} \int_0^r \bar{r} \frac{\partial f}{\partial r^2}(\bar{r}) d\bar{r}.$$

Now consider the case where $\varepsilon = 0$. Wherever we have the bound $|\nabla f| \leq A$,

$$(27.102) \quad H_f(r) \leq \frac{n-1}{r} + A.$$

By substituting this into (27.99), we have

$$\frac{J_f(r_2)}{J_f(r_1)} \leq e^{\int_{r_1}^{r_2} \left(\frac{n-1}{r} + A \right) dr} = e^{A(r_2 - r_1)} \left(\frac{r_2}{r_1} \right)^{n-1}.$$

Taking $r_1 \rightarrow 0$ and calling $r_2 = \bar{r}$, we obtain

$$(27.103) \quad J_f(\bar{r}) \leq e^{-f(\tilde{O}) + A\bar{r}} \bar{r}^{n-1}$$

since $\lim_{r \rightarrow 0} r^{n-1} J_f(r) = e^{-f(\tilde{O})}$. Integrating this yields the following (one deals with the possible nonsmoothness of $S(\tilde{O}, r)$ in the same way as the Bishop volume comparison theorem):

THEOREM 27.41 (Bakry–Emery volume comparison). *Let (\mathcal{M}^n, g) be a complete Riemannian manifold and let $f : \mathcal{M} \rightarrow \mathbb{R}$ be a smooth function. If $\tilde{O} \in \mathcal{M}$ and $r > 0$ are such that $\text{Rc}_f \geq 0$ and $|\nabla f| \leq A$ in $B_{\tilde{O}}(r)$ for some $A > 0$, then the f -volume*

$$(27.104) \quad \text{Vol}_f B_{\tilde{O}}(r) = \int_{B_{\tilde{O}}(r)} e^{-f} d\mu \leq n\omega_n \int_0^r e^{-f(\tilde{O}) + A\bar{r}} \bar{r}^{n-1} d\bar{r} \leq \omega_n e^{-f(\tilde{O}) + Ar} r^n,$$

where ω_n is the volume of the unit Euclidean n -ball. In particular, if $f \leq C$ in $B_{\tilde{O}}(r)$, then

$$(27.105) \quad \text{Vol} B_{\tilde{O}}(r) \leq \omega_n e^{-f(\tilde{O}) + Ar + Cr^n}.$$

From the form of the estimates we see that Bakry–Emery volume comparison is more effective at bounded distances.

4.5. Euclidean volume growth via the Riccati equation.

There is the following essentially equivalent version of the volume growth bound in Theorem 27.33. The proof we give is due to Munteanu and J. Wang.

THEOREM 27.42 (Shrinkers have at most Euclidean volume growth). *Let $(\mathcal{M}^n, g, f, -1)$ be a complete normalized noncompact shrinking GRS. Then for any $\tilde{O} \in \mathcal{M}$ we have*

$$(27.106) \quad \text{Vol} B_{\tilde{O}}(r) \leq \omega_n e^{f(\tilde{O})} r^n \quad \text{for all } r > 0,$$

where ω_n is the volume of the unit Euclidean n -ball. In particular, if O is a minimum point of f , then by (27.43) we have that $\text{Vol} B_O(r) \leq \omega_n e^{\frac{n}{2} r^n}$ for $r > 0$.

PROOF. On a shrinker we have $\text{Rc}_f = \frac{1}{2}g$. By (27.101), we have

$$(27.107) \quad \begin{aligned} \frac{\partial}{\partial r} \ln \left(\frac{J(r)}{r^{n-1}} \right) &= H(r) - \frac{n-1}{r} \\ &\leq -\frac{r}{6} + \frac{\partial f}{\partial r}(r) - \frac{2}{r^2} \int_0^r \tilde{r} \frac{\partial f}{\partial r}(\tilde{r}) d\tilde{r}. \end{aligned}$$

Since $\lim_{r \rightarrow 0} \frac{J(r)}{r^{n-1}} = 1$, we have

$$(27.108) \quad \begin{aligned} \ln \left(\frac{J(\bar{r})}{\bar{r}^{n-1}} \right) &\leq \int_0^{\bar{r}} \left(-\frac{r}{6} + \frac{\partial f}{\partial r}(r) - \frac{2}{r^2} \int_0^r \tilde{r} \frac{\partial f}{\partial r}(\tilde{r}) d\tilde{r} \right) dr \\ &= -\frac{\bar{r}^2}{12} - f(\bar{r}) + f(0) + \frac{2}{\bar{r}} \int_0^{\bar{r}} r \frac{\partial f}{\partial r}(r) dr, \end{aligned}$$

where we integrated by parts and used $\lim_{r \rightarrow 0} \frac{1}{r} \int_0^r \tilde{r} \frac{\partial f}{\partial r}(\tilde{r}) d\tilde{r} = 0$ to obtain the last line. Note that $f(\tilde{O}) = f(0)$. Another version of this formula is

$$(27.109) \quad \ln \left(\frac{J(\bar{r})}{\bar{r}^{n-1}} \right) \leq -\frac{\bar{r}^2}{12} + f(0) + f(\bar{r}) - \frac{2}{\bar{r}} \int_0^{\bar{r}} f(r) dr.$$

By combining (27.107) and (27.108), we obtain

$$\begin{aligned}
 (27.110) \quad \frac{\partial}{\partial \bar{r}} \left(\bar{r} \ln \frac{J(\bar{r})}{\bar{r}^{n-1}} \right) &= \ln \frac{J(\bar{r})}{\bar{r}^{n-1}} + \bar{r} \frac{\partial}{\partial \bar{r}} \ln \frac{J(\bar{r})}{\bar{r}^{n-1}} \\
 &\leq -\frac{\bar{r}^2}{4} - f(\bar{r}) + f(0) + \bar{r} \frac{\partial f}{\partial \bar{r}}(\bar{r}) \\
 &\leq f(0) - \left(\frac{\partial f}{\partial \bar{r}}(\bar{r}) - \frac{\bar{r}}{2} \right)^2 - \left(f(\bar{r}) - \left(\frac{\partial f}{\partial \bar{r}} \right)^2(\bar{r}) \right).
 \end{aligned}$$

Let ∇f^T be the tangential component to $\partial B_{\tilde{O}}(r)$ of ∇f . Then $f = |\nabla f|^2 + R = \left(\frac{\partial f}{\partial \bar{r}}\right)^2 + |\nabla f^T|^2 + R$. By integrating (27.110) from $\bar{r} = 0$ to $\bar{r} = r$, we obtain

$$\begin{aligned}
 (27.111) \quad \ln \frac{J(r)}{r^{n-1}} &\leq f(0) - \frac{1}{r} \int_0^r \left(\left(\frac{\partial f}{\partial \bar{r}}(\bar{r}) - \frac{\bar{r}}{2} \right)^2 + |\nabla f^T|^2(\bar{r}) + R(\bar{r}) \right) d\bar{r} \\
 &\leq f(0).
 \end{aligned}$$

Therefore

$$(27.112) \quad J(r) \leq e^{f(0)} r^{n-1}.$$

Since J is the volume density of g in spherical coordinates, by integrating (27.112), we obtain (27.106). \square

EXERCISE 27.43. Show that

$$\bar{r} \mapsto \left(\frac{J(\bar{r})}{e^{f(0)} \bar{r}^{n-1}} \right)^{\bar{r}}$$

is nonincreasing and that the limit as $\bar{r} \rightarrow 0$ is equal to 1.

REMARK 27.44. Suppose that the minimum of R is attained at \tilde{O} and that $|\nabla f|(\tilde{O}) = 0$. Then we have $R(\tilde{O}) = f(\tilde{O}) = f(0)$. This implies that $\frac{1}{r} \int_0^r R(\bar{r}) d\bar{r} \geq R(\tilde{O}) = f(0)$. Therefore, by (27.111),

$$(27.113) \quad \ln \frac{J(r)}{r^{n-1}} \leq f(0) - \frac{1}{r} \int_0^r R(\bar{r}) d\bar{r} \leq 0.$$

This implies that $\text{Vol} B_{\tilde{O}}(r) \leq \omega_n r^n$. On the other hand, by Corollary 27.36, we know that $\text{Vol} B_{\tilde{O}}(r) \leq C(\mathcal{G}, \tilde{O}) r^{n-2R(\tilde{O})}$.

EXERCISE 27.45. Show that

$$(27.114) \quad \bar{r} \mapsto \frac{e^{\frac{\bar{r}^2}{12} - f(\bar{r}) + \frac{2}{\bar{r}} \int_0^{\bar{r}} f(r) dr} J(\bar{r})}{\bar{r}^{n-1}} = \frac{e^{\frac{1}{\bar{r}} \int_0^{\bar{r}} \left(\left(\frac{\bar{r}}{2} - f'(r) \right)^2 + f(r) - (f'(r))^2 \right) dr} J(\bar{r})}{\bar{r}^{n-1}}$$

is nonincreasing and that the limit as $\bar{r} \rightarrow 0$ is equal to $e^{f(0)}$.

Let $\tilde{O} \in \mathcal{M}$, $x \notin \text{Cut}(\tilde{O})$, and $r(x) = d(x, \tilde{O})$, where $\text{Cut}(\tilde{O})$ is the cut locus of \tilde{O} . Define

$$h(x) = \frac{r^2(x)}{12} - f(x) + \frac{2}{r(x)} \int_0^{r(x)} f(\gamma(s)) ds,$$

where $\gamma : [0, r(x)] \rightarrow \mathcal{M}$ is the unique minimal unit speed geodesic joining \tilde{O} to x . The measure corresponding to the numerator in (27.114) is

$$dm(x) = e^{h(x)} d\mu(x) \quad \text{on } \mathcal{M} - \text{Cut}(\tilde{O}).$$

In general, for shrinkers, although $\lim_{r \rightarrow \infty} \frac{\text{Vol } B_{\tilde{O}}(r)}{r^n}$ exists for each $\tilde{O} \in \mathcal{M}$ and is bounded above only depending on n , we do not know if $\frac{\text{Vol } B_{\tilde{O}}(r)}{r^n}$ is bounded above independent of $\tilde{O} \in \mathcal{M}$ and $r \geq 1$. In particular, does there exist $C < \infty$ such that $\text{Vol } B_{\tilde{O}}(1) \leq C$ for all $\tilde{O} \in \mathcal{M}$?

5. Logarithmic Sobolev inequality

In this section we discuss the logarithmic Sobolev inequality on shrinkers due to Carrillo and one of the authors, which is based on the works of Bakry and Emery, Villani, and others. Let $\mathcal{G} = (\mathcal{M}^n, g, f, -1)$ be a complete noncompact shrinking GRS structure.

Taking $\tau = 1$ in Perelman's entropy functional, we define the (scale 1) **entropy** as

$$(27.115) \quad \mathcal{W}(g, \varphi) \doteq \mathcal{W}(g, \varphi, 1) = \int_{\mathcal{M}} (R + |\nabla \varphi|^2 + \varphi - n)(4\pi)^{-n/2} e^{-\varphi} d\mu.$$

Define the μ -invariant (or **logarithmic Sobolev constant**) of g by

$$\mu(g) \doteq \inf_{\varphi} \mathcal{W}(g, \varphi),$$

where the infimum is taken over all $\varphi : \mathcal{M} \rightarrow \mathbb{R} \cup \{\infty\}$ such that $w \doteq e^{-\varphi/2} \in C_c^\infty(\mathcal{M})$ satisfies the constraint $\int_{\mathcal{M}} w^2 d\mu = (4\pi)^{n/2}$.

Now assume that f satisfies the normalization $\int_{\mathcal{M}} e^{-f} d\mu = (4\pi)^{n/2}$. Then for some constant $C_1(\mathcal{G})$ we have

$$(27.116) \quad R + |\nabla f|^2 - f \equiv C_1(\mathcal{G}).$$

Define the **entropy of \mathcal{G}** to be $\mu(\mathcal{G}) \doteq \mathcal{W}(g, f)$. In view of the exponentially decaying e^{-f} factor and the volume bound (27.106), one can prove using the estimates for f , $|\nabla f|^2$, and Δf in §2 of this chapter that the integral defining $\mu(\mathcal{G})$ converges and that we have the equality

$$\int_{\mathcal{M}} |\nabla f|^2 e^{-f} d\mu = \int_{\mathcal{M}} \Delta f e^{-f} d\mu.$$

Hence

$$(27.117) \quad \mu(\mathcal{G}) = \int_{\mathcal{M}} (R + 2\Delta f - |\nabla f|^2 + f - n)(4\pi)^{-n/2} e^{-f} d\mu.$$

By (27.2) and (27.116), we have

$$(27.118) \quad R + 2\Delta f - |\nabla f|^2 + f - n = f - R - |\nabla f|^2 = -C_1(\mathcal{G}).$$

Therefore, by (27.117) we have that $\mu(\mathcal{G}) = -C_1(\mathcal{G})$.

Carrillo and one of the authors proved that $\mu(g) = \mu(\mathcal{G})$, that is,

THEOREM 27.46 (Sharp logarithmic Sobolev inequality for shrinkers). *If $\mathcal{G} = (\mathcal{M}^n, g, f, -1)$ is a complete noncompact shrinking GRS, then*

$$\inf_{\varphi} \mathcal{W}(g, \varphi) = \mu(\mathcal{G}) = -C_1(\mathcal{G}),$$

where the infimum is taken over all $\varphi : \mathcal{M} \rightarrow \mathbb{R} \cup \{\infty\}$ such that $w \doteq e^{-\varphi/2} \in C_c^\infty(\mathcal{M})$ satisfies the constraint $\int_{\mathcal{M}} w^2 d\mu = (4\pi)^{n/2}$.

REMARK 27.47. In comparison, Corollary 6.38 in Part I is equivalent to the following statement. Given any closed Riemannian manifold (\mathcal{M}^n, g) and constant $b > 0$, there exists a constant $C(b, g)$ such that if a function φ satisfies $\int_{\mathcal{M}} e^{-\varphi} d\mu = (4\pi)^{n/2}$, then

$$\int_{\mathcal{M}} (b|\nabla\varphi|^2 + \varphi - n)e^{-\varphi} d\mu \geq -C(b, g).$$

Here we give a heuristic proof of the theorem following the Bakry–Emery method, which can be made rigorous in our noncompact setting; detailed proofs are given in [53], [16], and [425]. Let $\square_f \doteq \frac{\partial}{\partial t} - \Delta_f$ denote the **f -heat operator**. Let $\xi(x, t)$ be a solution to

$$(27.119) \quad \square_f \xi = |\nabla \xi|^2, \quad \text{equivalently,} \quad \square_f e^\xi = 0.$$

Henceforth we shall assume suitable growth conditions on ξ so that we can differentiate under the integral sign and so that we can integrate by parts to justify some of the equalities below. We first observe that

$$(27.120) \quad \frac{d}{dt} \int_{\mathcal{M}} e^{\xi-f} d\mu = \int_{\mathcal{M}} \left(\frac{\partial}{\partial t} e^\xi \right) e^{-f} d\mu = \int_{\mathcal{M}} (\Delta_f e^\xi) e^{-f} d\mu = 0.$$

Now assume that ξ satisfies the normalization $(4\pi)^{n/2} = \int_{\mathcal{M}} e^{\xi-f} d\mu$.

Define the **relative Fisher information** functional

$$(27.121) \quad \mathbf{I}(\xi) \doteq \int_{\mathcal{M}} |\nabla \xi|^2 e^{\xi-f} d\mu.$$

By (27.96), for any function $u(x, t)$ we have the parabolic Bochner formula

$$(27.122) \quad \frac{1}{2} \square_f |\nabla u|^2 = -|\nabla^2 u|^2 + \langle \nabla u, \nabla \square_f u \rangle - \text{Rc}_f(\nabla u, \nabla u).$$

Hence, for ξ satisfying (27.119) on a shrinking GRS, we have

$$(27.123) \quad \square_f |\nabla \xi|^2 = -2|\nabla^2 \xi|^2 + 2\langle \nabla \xi, \nabla |\nabla \xi|^2 \rangle - |\nabla \xi|^2.$$

From this we compute that

$$(27.124) \quad \square_f (|\nabla \xi|^2 e^\xi) = -e^\xi (2|\nabla^2 \xi|^2 + |\nabla \xi|^2).$$

Therefore

$$(27.125) \quad \begin{aligned} \frac{d}{dt} \mathbf{I}(\xi(t)) &= \int_{\mathcal{M}} \frac{\partial}{\partial t} (|\nabla \xi|^2 e^\xi) e^{-f} d\mu \\ &= - \int_{\mathcal{M}} e^\xi (2|\nabla^2 \xi|^2 + |\nabla \xi|^2) e^{-f} d\mu \\ &\leq -\mathbf{I}(\xi(t)). \end{aligned}$$

Now define the **Boltzmann relative entropy** (or **Nash entropy**) functional

$$(27.126) \quad \mathbf{H}(\xi) \doteq \int_{\mathcal{M}} \xi e^{\xi-f} d\mu.$$

We compute using (27.119) that⁴

$$(27.127) \quad \frac{d}{dt} \mathbf{H}(\xi(t)) = \int_{\mathcal{M}} \left(\frac{\partial}{\partial t} e^\xi \right) (\xi + 1) e^{-f} d\mu = \int_{\mathcal{M}} \Delta_f (e^\xi) (\xi + 1) e^{-f} d\mu = -\mathbf{I}(\xi(t)).$$

⁴If $f = 0$, then $\xi \doteq \ln u$ gives $\mathbf{H}(\xi) = \int_{\mathcal{M}} u \ln u d\mu$ and $\mathbf{I}(\xi) = \int_{\mathcal{M}} \frac{|\nabla u|^2}{u} d\mu$. As a special case of (27.127), we have that $\frac{\partial}{\partial t} \mathbf{H}(\ln u) = -\mathbf{I}(\ln u)$ provided $u > 0$ satisfies the heat equation.

Therefore, by (27.125) we have

$$\frac{d}{dt} H(\xi(t)) \geq \frac{d}{dt} I(\xi(t)).$$

Under suitable growth conditions on the function ξ , it can be shown that $\lim_{t \rightarrow \infty} \xi(t) = 0$ and thus $\lim_{t \rightarrow \infty} H(\xi(t)) = 0$. We conclude that if $\xi(t)$ satisfies (27.119), then

(27.128)

$$H(\xi(t)) \leq I(\xi(t)); \quad \text{that is,} \quad \int_{\mathcal{M}} |\nabla \xi(t)|^2 e^{\xi(t)-f} d\mu \geq \int_{\mathcal{M}} \xi(t) e^{\xi(t)-f} d\mu.$$

Given φ with $\int_{\mathcal{M}} e^{-\varphi} d\mu = (4\pi)^{n/2}$, take $\xi(0) = f - \varphi$. Then (27.128) at $t = 0$ implies that

$$\int_{\mathcal{M}} (f - \varphi) e^{-\varphi} d\mu \leq \int_{\mathcal{M}} (|\nabla f|^2 + |\nabla \varphi|^2 - 2 \langle \nabla f, \nabla \varphi \rangle) e^{-\varphi} d\mu$$

and

$$\int_{\mathcal{M}} (|\nabla \varphi|^2 + \varphi) e^{-\varphi} d\mu \geq \int_{\mathcal{M}} (2\Delta f - |\nabla f|^2 + f) e^{-\varphi} d\mu.$$

We conclude from this and (27.118) that

(27.129)

$$\begin{aligned} \int_{\mathcal{M}} (R + |\nabla \varphi|^2 + \varphi - n) e^{-\varphi} d\mu &\geq \int_{\mathcal{M}} (R + 2\Delta f - |\nabla f|^2 + f - n) e^{-\varphi} d\mu \\ &= -(4\pi)^{n/2} C_1(\mathcal{G}). \end{aligned}$$

This finishes our sketch of the proof of Theorem 27.46.

In terms of $w = e^{-\varphi/2}$, the logarithmic Sobolev inequality says that if $\int_{\mathcal{M}} w^2 d\mu = (4\pi)^{n/2}$, then

$$\int_{\mathcal{M}} (Rw^2 + 4|\nabla w|^2 - w^2 \ln(w^2)) d\mu \geq (n - C_1(\mathcal{G})) \int_{\mathcal{M}} w^2 d\mu.$$

If we do not impose any constraint on w , then this is equivalent to

$$(27.130) \quad \int_{\mathcal{M}} w^2 \ln(w^2) d\mu - \ln \left(\int_{\mathcal{M}} w^2 d\mu \right) \int_{\mathcal{M}} w^2 d\mu \leq \int_{\mathcal{M}} (4|\nabla w|^2 + Rw^2) d\mu + C_2(\mathcal{G}) \int_{\mathcal{M}} w^2 d\mu,$$

where $C_2(\mathcal{G}) = C_1(\mathcal{G}) - n - \ln((4\pi)^{n/2})$. Since $R = \frac{n}{2} - \Delta f$, we obtain

(27.131)

$$\begin{aligned} \int_{\mathcal{M}} w^2 \ln(w^2) d\mu - \ln \left(\int_{\mathcal{M}} w^2 d\mu \right) \int_{\mathcal{M}} w^2 d\mu &\leq \int_{\mathcal{M}} (4|\nabla w|^2 + \langle \nabla f, \nabla(w^2) \rangle) d\mu \\ &\quad + C_3(\mathcal{G}) \int_{\mathcal{M}} w^2 d\mu, \end{aligned}$$

where $C_3(\mathcal{G}) = C_1(\mathcal{G}) - \frac{n}{2} - \ln((4\pi)^{n/2})$.

REMARK 27.48. Compare the derivation of inequality (27.125) with Bakry and Emery's proof of their logarithmic Sobolev inequality [16] (for an exposition, see Proposition 5.40 in Part I).

By Perelman's proof of his no local collapsing theorem, one can demonstrate the following.

COROLLARY 27.49. *If $\mathcal{G} = (\mathcal{M}^n, g, f, -1)$ is a complete noncompact shrinking GRS, then there exists a constant $\kappa > 0$ depending only on $C_1(\mathcal{G})$ satisfying the following property. If $x_0 \in \mathcal{M}$ is a point and $r_0 > 0$ are such that $R \leq r_0^{-2}$ in $B_{x_0}(r_0)$, then $\text{Vol } B_{x_0}(r_0) \geq \kappa r_0^n$.*

The logarithmic Sobolev inequality plays a crucial role in the following result of Munteanu and J. Wang.

THEOREM 27.50 (Shrinkers must have at least linear volume growth). *If $\mathcal{G} = (\mathcal{M}^n, g, f, -1)$ is a complete noncompact shrinking GRS and $p \in \mathcal{M}$, then there exists a constant $c = c(n, f(p), \int e^{-f} d\mu) > 0$ such that*

$$\text{Area } S(p, r) \geq 2c \quad \text{for } r \geq \frac{1}{2}.$$

Hence

$$\text{Vol } B_p(r) \geq cr \quad \text{for } r \geq 1.$$

The constant c is nondecreasing in $f(p)$ and nonincreasing in $\int e^{-f} d\mu$. Hence, if \mathcal{G} is also a singularity model that is κ -noncollapsed below all scales, then by taking p to be a minimum point of f , we have that c depends only on n and κ .

Finally, we state a pair of important results on the geometry at infinity of shrinkers. First, we have the following uniqueness result of Kotschwar and L. Wang.

THEOREM 27.51. *Any two complete noncompact shrinking GRS which have a pair of ends which are asymptotic to the same Euclidean cone over a smooth $(n-1)$ -dimensional closed Riemannian manifold must have isometric universal covers.*

Second, we have the following result of Munteanu and J. Wang.

THEOREM 27.52. *Let $(\mathcal{M}^n, g, f, -1)$ be a complete noncompact shrinking GRS. If $|\text{Rc}|(x) \rightarrow 0$ as $x \rightarrow \infty$, then (\mathcal{M}, g) is asymptotic to a Euclidean cone over a smooth $(n-1)$ -dimensional closed Riemannian manifold.*

6. Gradient shrinkers with nonnegative Ricci curvature

If we consider shrinkers which are non-Ricci flat with nonnegative Ricci curvature, then we have the following improvement for the lower bound of the scalar curvature due to one of the authors.

THEOREM 27.53 ($\text{Rc} \geq 0$ and $\text{Rc} \not\equiv 0$ shrinker $\Rightarrow R \geq \delta > 0$). *Let $(\mathcal{M}^n, g, f, -1)$ be a complete noncompact non-Ricci flat shrinking GRS with nonnegative Ricci curvature. Then there exists $\delta > 0$ such that*

$$(27.132) \quad R \geq \delta \quad \text{on } \mathcal{M}.$$

PROOF OF THE THEOREM (MODULO A CLAIM). Let $\tilde{O} \in \mathcal{M}$ and let $x_0 \in \mathcal{M} - B_{\tilde{O}}(8r_1)$, where the constant $r_1 < \infty$ is chosen to satisfy (27.147) below. Let σ be an integral curve of ∇f with $\sigma(0) = x_0$. Since ∇f is a complete vector field (by Corollary 27.7), $\sigma(u)$ is defined for all $u \in \mathbb{R}$. Using $\nabla R = 2 \text{Rc}(\nabla f)$, we have

$$(27.133) \quad \frac{d}{du} R(\sigma(u)) = \langle \nabla R, \nabla f \rangle = 2 \text{Rc}(\nabla f, \nabla f) \geq 0$$

and hence

$$(27.134) \quad R(x_0) \geq R(\sigma(u)) \quad \text{for all } u \in (-\infty, 0].$$

Claim. For any $x_0 \in \mathcal{M} - B_{\tilde{O}}(8r_1)$, we must have either **Case (1)**:

$$(27.135) \quad R(\sigma(u_1)) \geq 1 \quad \text{for some } u_1 \in (-\infty, 0]$$

or **Case (2)**:

$$(27.136) \quad \sigma(u_2) \in \bar{B}_{\tilde{O}}(8r_1) \quad \text{for some } u_2 \in (-\infty, 0).$$

Now assume the claim. If Case (1) holds, then (27.134) implies

$$R(x_0) \geq R(\sigma(u_1)) \geq 1.$$

On the other hand, if Case (2) holds, then (27.134) implies

$$(27.137) \quad R(x_0) \geq R(\sigma(u_2)) \geq \min_{x \in \bar{B}_{\tilde{O}}(8r_1)} R(x).$$

Since g is not Ricci flat, by Proposition 27.8(3) the RHS is positive. Therefore, in either Case (1) or Case (2) we have

$$R(x_0) \geq \min \left\{ 1, \min_{x \in \bar{B}_{\tilde{O}}(8r_1)} R(x) \right\} \doteq \delta > 0.$$

Since $x_0 \in \mathcal{M} - B_{\tilde{O}}(8r_1)$ is arbitrary, the theorem is proved, modulo the claim. \square

It remains for us to present the following:

PROOF OF THE CLAIM. Suppose that the claim is false. Then there exists $x_0 \in \mathcal{M} - B_{\tilde{O}}(8r_1)$ such that

$$(27.138) \quad \begin{aligned} R(\sigma(u)) &< 1 && \text{for all } u \in (-\infty, 0], \\ \sigma(u) &\notin \bar{B}_{\tilde{O}}(8r_1) && \text{for all } u \in (-\infty, 0). \end{aligned}$$

Let $r(\cdot) = d(\cdot, \tilde{O})$. The whole aim of the proof is to show that

$$(27.139) \quad \frac{d_-}{du} r(\sigma(u)) \geq \frac{1}{8} r(\sigma(u))$$

for all $u \in (-\infty, 0)$, where $\frac{d_-}{du}$ denotes the lim inf of backward difference quotients. This will prove the claim since it contradicts $\sigma(u) \notin \bar{B}_{\tilde{O}}(8r_1)$ for all $u \in (-\infty, 0)$.

STEP 1. For $x \in \mathcal{M} - B_{\tilde{O}}(4(n-1))$ with $R(x) \leq 1$, we have

$$(27.140) \quad \langle \nabla f, \gamma'(r(x)) \rangle \geq \frac{1}{4} r(x) - C_1 - |\nabla f|(\tilde{O}),$$

where $\gamma : [0, r(x)] \rightarrow \mathcal{M}$ is a minimal unit speed geodesic joining \tilde{O} to x and where C_1 is given by (27.146) below.

Proof of (27.140). By integrating the shrinker equation $\text{Rc}(\gamma', \gamma') + (f \circ \gamma)'' = \frac{1}{2}$, we have

$$(27.141) \quad \langle \nabla f, \gamma'(r(x)) \rangle = \frac{1}{2} r(x) - \int_0^{r(x)} \text{Rc}(\gamma', \gamma') ds + \langle \nabla f, \gamma'(0) \rangle.$$

Regarding the second term on the RHS, the second variation formula implies (see (27.49))

$$\int_0^{r(x)} \zeta^2 \text{Rc}(\gamma', \gamma') ds \leq (n-1) \int_0^{r(x)} (\zeta')^2 ds$$

for any continuous piecewise C^∞ function $\zeta : [0, r(x)] \rightarrow \mathbb{R}$ satisfying $\zeta(0) = \zeta(r(x)) = 0$. Now define ζ to be

$$\zeta(s) = \begin{cases} s & \text{if } 0 \leq s \leq 1, \\ 1 & \text{if } 1 < s \leq r(x) - r_0, \\ \frac{r(x)-s}{r_0} & \text{if } r(x) - r_0 < s \leq r(x), \end{cases}$$

where r_0 is to be chosen below. We have

$$\int_0^{r(x)} \zeta^2 \operatorname{Rc}(\gamma', \gamma') ds \leq (n-1)(r_0^{-1} + 1).$$

From this we obtain

(27.142)

$$\begin{aligned} \int_0^{r(x)} \operatorname{Rc}(\gamma', \gamma') ds &\leq (n-1)(r_0^{-1} + 1) + \int_0^1 (1 - \zeta^2) \operatorname{Rc}(\gamma', \gamma') ds \\ &\quad + \int_{r(x)-r_0}^{r(x)} (1 - \zeta^2) \operatorname{Rc}(\gamma', \gamma') ds \\ &\leq (n-1)(r_0^{-1} + 1) + \frac{2}{3} \max_{V \in T_y \mathcal{M}, |V|=1, y \in \bar{B}_{\tilde{O}}(1)} \operatorname{Rc}(V, V) \\ &\quad + \int_{r(x)-r_0}^{r(x)} R(\gamma(s)) ds \end{aligned}$$

since $\operatorname{Rc} \geq 0$.

We now estimate $\int_{r(x)-r_0}^{r(x)} R(\gamma(s)) ds$. Since $\nabla R = 2 \operatorname{Rc}(\nabla f)$ and $\operatorname{Rc} \geq 0$, we have for all $y \in \mathcal{M}$

$$|\nabla R|(y) \leq 2 |\operatorname{Rc}|(y) |\nabla f|(y) \leq 2R(y) \left(\frac{1}{2} r(y) + \sqrt{f}(\tilde{O}) \right),$$

using (27.30). Thus, if $y \in \mathcal{M}$ satisfies $r(y) \geq 1$, then

$$(27.143) \quad |\nabla \ln R|(y) \leq (1 + 2\sqrt{f}(\tilde{O}))r(y).$$

Now choose $r_0 = \frac{4(n-1)}{r(x)} \leq 1$. Suppose that $\bar{s} \in [r(x) - r_0, r(x)]$. Using $r(\gamma(\bar{s})) \geq 1$, the assumption that $R(x) \leq 1$, and (27.143), we compute that

$$\begin{aligned} \ln R(\gamma(\bar{s})) &\leq \ln \frac{R(\gamma(\bar{s}))}{R(x)} \\ &\leq \int_{\bar{s}}^{r(x)} |\nabla \ln R|(\gamma(s)) ds \\ &\leq (1 + 2\sqrt{f}(\tilde{O})) \int_{r(x)-r_0}^{r(x)} r(\gamma(s)) ds \\ &\leq (1 + 2\sqrt{f}(\tilde{O}))r_0 r(x) \\ &\leq 4(n-1)(1 + 2\sqrt{f}(\tilde{O})). \end{aligned}$$

That is, for $\bar{s} \in [r(x) - r_0, r(x)]$,

$$R(\gamma(\bar{s})) \leq e^{4(n-1)(1+2\sqrt{f}(\tilde{O}))}.$$

Therefore

$$(27.144) \quad \int_{r(x)-r_0}^{r(x)} R(\gamma(s)) ds \leq e^{4(n-1)(1+2\sqrt{f}(\tilde{O}))} r_0.$$

Combining this with (27.142), we have

$$(27.145) \quad \int_0^{r(x)} \text{Rc}(\gamma', \gamma') ds \leq (n-1)r_0^{-1} + C_1,$$

where

$$(27.146) \quad C_1 = n-1 + \frac{2}{3} \max_{V \in T_y \mathcal{M}, |V|=1, y \in \bar{B}_{\tilde{O}}(1)} \text{Rc}(V, V) + e^{4(n-1)(1+2\sqrt{f}(\tilde{O}))}.$$

Applying (27.145) to (27.141) while using $r_0^{-1} = \frac{r(x)}{4(n-1)} \geq 1$, we obtain (27.140).

STEP 2. *Applying the inequality (27.140) to prove the claim.* Now define

$$(27.147) \quad r_1 = \max \left\{ \frac{n-1}{2}, C_1 + |\nabla f|(\tilde{O}) \right\}.$$

By (27.138) and (27.140) with $x = \sigma(u)$, we have that

$$\langle \nabla f, \gamma'(r(\sigma(u))) \rangle \geq \frac{1}{4} r(\sigma(u)) - C_1 - |\nabla f|(\tilde{O}) \geq \frac{1}{8} r(\sigma(u))$$

for any $u \in (-\infty, 0]$. Here, $\gamma : [0, r(\sigma(u))] \rightarrow \mathcal{M}$ is a minimal unit speed geodesic joining \tilde{O} to $\sigma(u)$. Since $\frac{d}{du} r(\sigma(u)) \geq \langle \nabla f, \gamma'(r(\sigma(u))) \rangle$, we obtain (27.139) for all $u \in (-\infty, 0)$. \square

As a consequence, we have the following result, originally due to Carrillo and one of the authors.

THEOREM 27.54 (Ric ≥ 0 and Ric $\not\equiv 0$ shrinker \Rightarrow AVR = 0). *If $(\mathcal{M}^n, g, f, -1)$ is a complete noncompact non-Ricci flat shrinking GRS with nonnegative Ricci curvature, then AVR(g) = 0.*

PROOF. This follows directly from a combination of Theorem 27.53 and Corollary 27.36. \square

REMARK 27.55 (Ricci flat manifolds with AVR > 0). There are many examples of 4-dimensional complete noncompact Ricci flat manifolds (hyper-Kähler and asymptotically locally Euclidean) with AVR(g) > 0 .

Note that Feldman, Ilmanen, and one of the authors [111] have described examples of complete noncompact Kähler shrinkers, which have AVR > 0 and have Ricci curvature which is negative somewhere. Related to Remark 27.37, we ask the following:

PROBLEM 27.56. Is there an example of a noncompact shrinker with Ric ≥ 0 which does not split as the product of a compact shrinker and a Euclidean space?

7. Notes and commentary

There are many works on GRS which we have not discussed in this chapter. We have not discussed some important advances in the construction of Kähler and non-Kähler Ricci solitons. Some works we would like to mention are Xu-Jia Wang and Xiaohua Zhu [431], Fabio Podestà and Andrea Spiro [329], and Andrew Dancer and McKenzie Wang [90]. The results in this chapter are due to various authors. We give some citations below.

§1. For Exercise 27.1, see §1.3 of [312].

The proof of Theorem 27.2(1), using the elliptic maximum principle, follows Theorem 1.3(ii) in [455]. The proof of Theorem 27.2(2) follows Theorem 0.5 in [451]. For an extension to ancient solutions (discussed in the next chapter), see [61].

Theorem 27.4 follows from the formula (27.6) for GRS due to Hamilton combined with Theorem 27.2. For the shrinking case, see (2.3) in [48] (see also [45]).

Corollary 27.7 is Theorem 1.3(i) in [455].

For Proposition 27.8, see [326] or [451]. For the claim stated in its proof, see also Proposition 2 in [319].

§2. Regarding shrinkers, (27.42) is (2.2) in [48].

For Theorem 27.11, see Theorem 1.1 in [48] and the refinement in [146]. If the Ricci curvature of (\mathcal{M}^n, g) is bounded, then this estimate is in [312].

For Corollary 27.16, see [105].

Also related is the work in [105], where the properness of f is proven and from which a nonsharp form of the quadratic growth of f may be derived.

Theorem 27.19 is in [268].

Theorem 27.24 and Corollary 27.25 are in [95].

§3. For Theorem 27.26, see [291] and [78].

See [441] and [264] for an estimate for the potential functions of steady GRS.

See [32], [46], [132], [138], and [264] for further works on the qualitative aspects of steady GRS.

§4. For (27.81), see (3.7) of [48].

The result that shrinkers have at most Euclidean volume growth, in Theorem 27.33 and Theorem 27.42, is primarily due to [48], with a technical hypothesis they first assumed removed in [263]. That the lim sup of the volume ratios, as the radius tends to infinity, is bounded above by a constant depending only on dimension is in [146].

The part of Theorem 27.33 that the AVR of a shrinker exists is in [79]. Their work built upon the earlier works in [53], [45], [451], [64], and [290]. Some work related to the above is in [105] and is by Hamilton (see Proposition 9.46 in [77]).

For Proposition 27.35, see [79].

Corollary 27.36 is in [451]. It extends the earlier results in [48] and [263]. The alternate proof we give is due to Bo Yang.

For Theorem 27.41, see Theorem 1.2(a) in [434] for example. See also [442]. For some earlier related work, see [202] and [16].

The proof of Theorem 27.42 that we present, using the Riccati equation, is in [267].

For further work on GRS, see [453].

§5. Theorem 27.46 is in [53]. Their proof is based on the earlier work in [16] and [425], among others.

Theorem 27.50 is in [265] (see Theorem 1.4(2) in [267] for a generalization).

Theorem 27.51 is in [170].

§6. Theorem 27.53 is Proposition 1.1 in [280]. Its proof is a modification of an idea of Perelman (see Proposition 9.81 in [77]). Theorem 27.54 was proved in [53].