Preface

This book gives a systematic presentation of the theory of Fokker–Planck–Kolmogorov equations, which are second order elliptic and parabolic equations for measures. This direction goes back to Kolmogorov's works [**527**], [**528**], [**529**] and a number of earlier works in the physics literature by Fokker [**377**], Smoluchowski [**863**], Planck [**781**], and Chapman [**235**]. One of our principal objects is the elliptic operator of the form

$$L_{A,b}f = \text{trace}(AD^2 f) + \langle b, \nabla f \rangle, \quad f \in C_0^\infty(\Omega),$$

where $A = (a^{ij})$ is a mapping on a domain $\Omega \subset \mathbb{R}^d$ with values in the space of nonnegative symmetric linear operators on $\mathbb{R}^d$ and $b = (b^i)$ is a vector field on Ω. In coordinate form, $L_{A,b}$ is given by the expression

$$L_{A,b}f = a^{ij}\partial_{x_i}\partial_{x_j}f + b^i\partial_{x_i}f,$$

where we always assume that the summation is taken over all repeated indices.

With this operator $L_{A,b}$, we associate the weak elliptic equation

$$(1) \qquad\qquad\qquad\qquad L_{A,b}^*\mu = 0$$

for Borel measures on Ω, which is understood in the following sense:

$$(2) \qquad\qquad\qquad \int_\Omega L_{A,b}f \, d\mu = 0 \quad \forall\, f \in C_0^\infty(\Omega),$$

where we assume that b^i, $a^{ij} \in L^1_{\text{loc}}(\mu)$. If μ has a density ϱ with respect to Lebesgue measure, then ϱ is sometimes called "an adjoint solution" and the equation is called "an equation in double divergence form". We use the above term "weak elliptic equation for measures". The corresponding equation for the density ϱ is

$$\partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho) = 0.$$

If $A = \mathrm{I}$, we obtain the equation $\Delta\varrho - \text{div}\,(\varrho b) = 0$.

Similarly, one can consider parabolic operators and parabolic Fokker–Planck–Kolmogorov equations for measures on $\Omega \times (0, T)$ of the type

$$\partial_t \mu = L_{A,b}^*\mu.$$

The corresponding equations for densities are

$$(3) \qquad\quad \partial_t \varrho(x, t) = \partial_{x_i}\partial_{x_j}\big(a^{ij}(x, t)\varrho(x, t)\big) - \partial_{x_i}\big(b^i(x, t)\varrho(x, t)\big),$$

and if we also have an initial distribution μ_0 in a suitable sense, then we arrive at the Cauchy problem for the Fokker–Planck–Kolmogorov equation. However, it is crucial that a priori Fokker–Planck–Kolmogorov equations are equations for measures, not for functions; this becomes relevant when the coefficients are singular

or degenerate and, in particular, in the infinite-dimensional case, where no Lebesgue measure exists. It is also important that equation (1) is meaningful under very broad assumptions about A and b: only their local integrability with respect to the regarded solution μ is needed. These coefficients may be quite singular with respect to Lebesgue measure even if the solution admits a smooth density. For example, for an arbitrary infinitely differentiable probability density ϱ on $\mathbb{R}^d$, the measure $\mu = \varrho\, dx$ satisfies the above equation with $A = \mathrm{I}$ and $b = \nabla\varrho/\varrho$, where we set $\nabla\varrho(x)/\varrho(x) = 0$ whenever $\varrho(x) = 0$. This is obvious from the integration by parts formula

$$\int_{\mathbb{R}^d}[\Delta f + \langle \nabla\varrho/\varrho, \nabla f\rangle]\varrho\, dx = \int_{\mathbb{R}^d}\varrho\Delta f + \int_{\mathbb{R}^d}\langle \nabla\varrho, \nabla f\rangle\, dx = 0.$$

Since ϱ may vanish on an arbitrary proper closed subset of $\mathbb{R}^d$, the vector field b can fail to be locally integrable with respect to Lebesgue measure, but it is locally integrable with respect to μ. Also note that in general our solutions need not be more regular than the coefficients (unlike in the case of usual elliptic equations). For example, if $d = 1$ and $b = 0$, then for an arbitrary positive probability density ϱ, the measure $\mu = \varrho\, dx$ satisfies the equation $L_{A,0}^*\mu = 0$ with $A = \varrho^{-1}$.

In this general setting, a study of weak elliptic equations for measures on finite- and infinite-dimensional spaces was initiated in the 1990s in the papers of the first three authors. Actually, the infinite-dimensional case was even a starting point, which was motivated by investigations of infinite-dimensional diffusion processes and other applications in infinite-dimensional stochastic analysis (developed in particular in the works of Albeverio, Høegh-Krohn [21] as well as A.I. Kirillov [511]–[516]). It was realized in the course of these investigations that even infinite-dimensional equations with very nice coefficients often require results on finite-dimensional equations with quite general coefficients. For example, we shall see in Chapter 10 that the finite-dimensional projections μ_n of a measure μ satisfying an elliptic equation on an infinite-dimensional space satisfy elliptic equations whose coefficients are the conditional expectations of the original coefficients with respect to the σ-algebras generated by the corresponding projection operators. As a result, even for smooth infinite-dimensional coefficients, the only information about their conditional expectations is related to their integrability with respect to μ_n, not with respect to Lebesgue measure; in particular, no local boundedness is given.

The theory of elliptic and parabolic equations for measures is now a rapidly growing area with deep and interesting connections to many directions in real analysis, partial differential equations, and stochastic analysis. Let us briefly describe the probabilistic picture behind our analytic framework. Suppose that $\xi = (\xi_t^x)_{t\geqslant 0}$ is a diffusion process in $\mathbb{R}^d$ governed by the stochastic differential equation

$$d\xi_t^x = \sigma(\xi_t^x)dW_t + b(\xi_t^x)dt, \quad \xi_0 = x.$$

The basic concepts related to this equation are recalled in §1.3. The generator of the transition semigroup $\{T_t\}_{t\geqslant 0}$ has the form $L_{A,b}$, where $A = \sigma\sigma^*/2$. The matrix $A = (a^{ij})$ in the operator $L_{A,b}$ will be called the *diffusion matrix* or *diffusion coefficient*; this differs from the standard form of the diffusion generator by the absence of the factor $1/2$ in front of the second order derivatives, but is more convenient when one deals with equations. The vector field b is called the *drift coefficient* or just the *drift*. The transition probabilities of ξ satisfy the corresponding parabolic equation. Any invariant probability measure μ of ξ (if such exists) satisfies (1),

where μ is called invariant for $\{T_t\}_{t\geqslant 0}$ if the following identity holds:

$$(4) \qquad \int_{\mathbb{R}^d} T_t f\, d\mu = \int_{\mathbb{R}^d} f\, d\mu \quad \forall\, f \in C_b(\mathbb{R}^d).$$

Measures satisfying (1) are called *infinitesimally invariant*, because this equation has deep connections with invariance with respect to the corresponding operator semigroups. More precisely, if there is an invariant probability measure μ, then $\{T_t\}_{t\geqslant 0}$ extends to $L^1(\mu)$ and is strongly continuous. Let L be the corresponding generator with domain $D(L)$. Then (4) is equivalent to the equality

$$\int_{\mathbb{R}^d} Lf\, d\mu = 0 \quad \forall\, f \in D(L).$$

Under reasonable assumptions about A and b, the generator of the semigroup associated with the diffusion governed by the indicated stochastic equation coincides with $L_{A,b}$ on $C_0^\infty(\mathbb{R}^d)$. As we shall see, invariance of the measure in the sense of (4) is not the same as (2). The point is that the class $C_0^\infty(\mathbb{R}^d)$ may be much smaller than $D(L)$. What is important is that the equation is meaningful and can have solutions under assumptions that are much weaker than those needed for the existence of a diffusion, so that this equation can be investigated without any assumptions about the existence of semigroups. On the other hand, there exist very interesting and fruitful relations between equations (2) and (4). For example, if A and b are both Lipschitz and if A is nondegenerate, they are equivalent.

Letting $P(x,t,\cdot)$ be the corresponding transition probabilities (the distributions of ξ_t^x), the semigroup property reads

$$(5) \qquad P(x,t+s,B) = \int_{\mathbb{R}^d} P(u,s,B)\, P(x,t,du),$$

or in the case where there exist densities $p(x,t,y)$,

$$p(x,t+s,y) = \int_{\mathbb{R}^d} p(u,s,y)\, p(x,t,u)\, du.$$

Identity (5) is called the Smoluchowski equation or the Chapman–Kolmogorov equation. In his seminal paper [**527**] Kolmogorov posed the following problems: find conditions for the existence and uniqueness of solutions to the Cauchy problem for (3) and investigate when (5) holds for these solutions. Now, 80 years later, these problems are still not completely solved. However, considerable progress has been achieved; results obtained and some related open problems are discussed in this book.

We shall consider the following problems.

1) Regularity of solutions of equation (2), for example, the existence of densities with respect to Lebesgue measure, the continuity and smoothness of these densities, and certain related estimates (such as L^2-estimates for logarithmic gradients of solutions). In particular, we shall see in Chapter 1 that the measure μ is always absolutely continuous with respect to Lebesgue measure on the set $\{\det A > 0\}$ and has a continuous density from the Sobolev class $W_{\mathrm{loc}}^{p,1}$ with $p > d$ provided that the diffusion coefficients a^{ij} are in this class, $|b| \in L_{\mathrm{loc}}^p(dx)$ or $|b| \in L_{\mathrm{loc}}^p(\mu)$, and the matrix A is positive definite. Global properties of solutions of equations with unbounded coefficients are studied in Chapter 3, where certain global upper and lower estimates for the densities are obtained. We shall also obtain analogous results for parabolic equations in Chapters 6–8.

2) Existence of solutions to elliptic equation (2) and existence of invariant measures in the sense of (4) as well as relations between these two concepts are the subjects of Chapter 2 and Chapter 5. In particular, we shall see in Chapter 5 that under rather general assumptions, for a given probability measure μ satisfying our elliptic equation (2), one can construct a strongly continuous Markov semigroup $\{T_t^\mu\}_{t\geqslant 0}$ on $L^1(\mu)$ such that μ is $\{T_t^\mu\}_{t\geqslant 0}$-invariant and the generator of $\{T_t^\mu\}_{t\geqslant 0}$ coincides with $L_{A,b}$ on $C_0^\infty(\mathbb{R}^d)$. For this, an easy to verify condition is the existence of a Lyapunov function for $L_{A,b}$. In the general case (without any additional assumptions), a bit less is true, namely, μ is only subinvariant for $\{T_t\}_{t\geqslant 0}$. We shall see examples where this really occurs, i.e., where μ is not invariant. Existence of solutions to parabolic equations is addressed in Chapter 6.

3) Various uniqueness problems are considered in Chapters 4 and 5; in particular, uniqueness of invariant measures in the sense of (4) and uniqueness of solutions to (2) in the class of all probability measures. Related interesting problems concern uniqueness of associated semigroups $\{T_t^\mu\}_{t\geqslant 0}$ and the essential self-adjointness of the operator $L_{A,b}$ on $C_0^\infty(\mathbb{R}^d)$ in the case when it is symmetric. Parabolic analogues are considered in Chapter 9.

First, we concentrate on the elliptic case, to which Chapters 1–5 are devoted. In Chapters 6–9 similar problems are studied for parabolic equations; however, parabolic equations appear already in Chapter 5 in relation to semigroups generated by elliptic operators. Chapter 10 is devoted to a brief discussion of infinite-dimensional analogues of the problems listed in 1)–3). The results obtained so far in the infinite-dimensional setting apply to various particular situations, although they cover many concrete examples arising in applications such as stochastic partial differential equations, infinite particle systems, Gibbs measures, and so on. The main purpose of Chapter 10 is to give applications of finite-dimensional results and to demonstrate the universality of certain ideas, methods, and techniques. Finally, in Chapters 2, 6, and 9 we discuss degenerate equations and nonlinear equations for measures; important examples of such equations are Vlasov-type equations. We made some effort to minimize dependencies between the chapters; the proofs of a number of fundamental results are rather difficult and can be omitted without any loss of understanding of the rest.

Every chapter opens with some synopsis mentioning the chief problems and results discussed. The last section of each chapter includes some complementary subsections (the numbers in brackets within these internal contents refer to the corresponding page numbers) and also brief historical and bibliographic comments and exercises. In the Bibliography each item is provided with indication of all pages where it is cited. The Subject Index also includes special notations used.