Preface

This book originated from the authors' desire to give an explanation of several recent applications of p-adic analysis to number theory and especially to arithmetic geometry. Central to this end has been the work done by several people (including the authors) to prove the *Dynamical Mordell-Lang conjecture*, which gives predictions about how the orbits of points in a variety under self-maps should intersect subvarieties. As the name suggests, this can be interpreted as a dynamical analogue of the classical Mordell-Lang Conjecture (proved by Faltings and Vojta) concerning intersections between finitely generated subgroups and subvarieties in a semiabelian variety.

Many results working towards this conjecture have used p-adic analysis, and we describe all known (to us) partial results up to this point in time—both those using p-adic analysis and those using alternative approaches—towards the Dynamical Mordell-Lang Conjecture. In some cases, we present entire proofs of results, while in other cases only a sketch is given, and in certain cases only a brief overview of the idea of the proof is provided. Our choice should not be interpreted as our opinion about the relative importance of the included results, but is instead an editorial choice regarding which material we thought best fits the overarching theme of this book.

We also give other applications of p-adic analysis to number theory and arithmetic geometry. In these cases, our list of applications is not meant to be exhaustive, but rather our goal is to show the wide reach of applications and potential applications of p-adic analysis to arithmetic geometry. While the uses of p-adic analytic methods we give do not always explicitly relate to the Dynamical Mordell-Lang Conjecture, we have generally favored applications of p-adic analysis to problems with some relation to the Dynamical Mordell-Lang Conjecture.

We thank all our colleagues with whom we wrote many of the papers whose results are detailed in this book; obviously, without the joint efforts we put towards solving the Dynamical Mordell-Lang Conjecture we would not have had a topic for this book. So, we thank Rob Benedetto, Ben Hutz, Par Kurlberg, Jeff Lagarias, Tom Scanlon, Yu Yasufuku, Umberto Zannier, and Mike Zieve. We are also grateful to the referees for their careful reading of a previous version of this book, and for suggesting many improvements for our work. Last, but definitely not least, we thank our families for their love and support while writing this book.

Notation

We let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of integer, rational, real, respectively, complex numbers. $\mathbb{N}_0$ is the set of all nonnegative integers, while $\mathbb{N}$ is the set of all positive integers.

An *arithmetic progression* is a set of the form $\{a+rn\}_{n\in\mathbb{N}_0}$, where the *common difference* r may be equal to 0 (in which case the set consists of a single element). If the common difference r is nonzero, then the arithmetic progression is infinite. Note that in the literature, sometimes one calls such a sequence a *one-sided* arithmetic progression in order to distinguish it from a *two-sided arithmetic progression*, which is a set of the form $\{a+rn\}_{n\in\mathbb{Z}}$. However, since in this book we mainly encounter one-sided arithmetic progressions and only occasionally encounter two-sided arithmetic progressions, our convention is to call arithmetic progression a sequence $\{a+rn\}_{n\in\mathbb{N}_0}$, while a sequence $\{a+rn\}_{n\in\mathbb{Z}}$ is called a two-sided arithmetic progression.

For a matrix A, we denote by A^t its transpose.

For a set U, we denote by id_U the identity function on U.

For any field K, we denote by $\mathrm{char}(K)$ its characteristic. By $\overline{K}$ we denote a fixed algebraic closure of K.

For any subfield $K \subseteq \overline{\mathbb{Q}}$, we denote by $\mathfrak{o}_K$ the ring of algebraic integers contained in K. If K is a number field, and $\mathfrak{p}$ is a prime ideal of K, then $k_\mathfrak{p}$ is the residue field corresponding to $\mathfrak{p}$, i.e., $k_\mathfrak{p} \xrightarrow{\sim} \mathfrak{o}_K/\mathfrak{p}$.

The usual affine space of dimension m is denoted by $\mathbb{A}^m$; for any field K, we have that $\mathbb{A}^m(K)$ consists of all m-tuples of points with coordinates in K. Similarly, we denote by $\mathbb{P}^m$ the projective space of dimension m; for any field K, we have that $\mathbb{P}^m(K)$ consists of all equivalence classes of $(m+1)$-tuples of points with coordinates in K not all equal to 0, under the equivalence relation

$$[x_0 : x_1 \cdots : x_m] \sim [y_0 : y_1 : \cdots : y_m]$$

if and only if there exists a nonzero scalar $c \in K$ such that

$$y_i = cx_i \text{ for all } i = 1, \ldots, m.$$

By *affine variety* we mean a subset of an affine space defined by a set of algebraic equations. Note that we do not ask *a priori* the variety be irreducible. Similarly, by *projective variety* we mean a subset of a projective space defined by a set of algebraic equations. We endow both the affine space and the projective space with the Zariski topology where the closed sets are precisely the (affine, respectively projective) varieties. We say that X is a quasiprojective variety if it is the open subset of a projective subvariety of some projective space. We say that a variety X is defined over a field K if it may be defined by a set of equations with coefficients in K. For a variety X defined over a field K, we denote by $X(K)$ the set of K-rational points of X.

We denote by $\mathbb{G}_a$ the affine line $\mathbb{A}^1$ endowed with the additive group law; we extend this law coordinatewise to $\mathbb{G}_a^n$. We denote by $\mathbb{G}_m$ the (Zariski open subset of the affine line) $\mathbb{A}^1 \setminus \{0\}$, i.e., the affine line without the origin, endowed with the multiplicative group law. Similarly to $\mathbb{G}_a^n$, we extend the multiplicative group law to $\mathbb{G}_m^n$.

An *abelian variety* is an irreducible projective variety which has the structure of an algebraic group.

For a set X, a map $\Phi : X \longrightarrow X$ is called a *self-map*. In general, for a self-map $\Phi : X \longrightarrow X$ and for any integer $n \geq 0$, we denote by Φ^n the n-th compositional iterate of Φ, i.e. $\Phi^n = \Phi \circ \cdots \circ \Phi$ (n times), with the convention that Φ^0 is the identity map. The *orbit* of a point $x \in X$ is denoted as $\mathcal{O}_\Phi(x)$ and it is the set of all $\Phi^n(x)$ for $n \in \mathbb{N}_0$.

A *dynamical system* consists of a topological space X endowed with a continuous self-map Φ.

For two real-valued functions f and g, we write $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$. Similarly, we write $f(x) = O(g(x))$ if the function $x \mapsto f(x)/g(x)$ is bounded as $x \to \infty$.

In a metric space $(X, d(\cdot,\cdot))$, for $x \in X$ and $r \in \mathbb{R}_{>0}$ we denote by $D(x,r)$ the open disk

$$D(x,r) = \{y \in X : d(x,y) < r\}.$$

We denote by $\overline{D}(x,r)$ the closure of $D(x,r)$.