

## CHAPTER 1

# Introduction

In this chapter we describe various instances of the Dynamical Mordell-Lang Conjecture which appear in seemingly different areas. We conclude our Introduction by giving a brief overview of the rest of the book.

### 1.1. Overview of the problem

We start by presenting several arithmetic questions which are all connected, though this may not be so obvious *a priori*. All these questions have in common the following theme: we have a dynamical system  $\Phi$  on a topological space  $X$ , and then for a point  $\alpha \in X$  and a closed subset  $V$  of  $X$ , we ask for what values of  $n \in \mathbb{N}_0$  we have  $\Phi^n(\alpha) \in V$ ? The underlying theme of this book is that all the questions we consider have, or are conjectured to have, the same answer to the above question: *finitely many arithmetic progressions*. We also recall our convention that an arithmetic progression of common difference equal to 0 is simply a *singleton*.

The cases we consider are the following ones:

- (1) Find all  $n \in \mathbb{N}_0$  such that  $a_n = 0$  where  $\{a_n\}_{n \in \mathbb{N}_0}$  is a linear recurrence sequence. Say that the recurrence relation verified by the sequence is given for all  $n \geq 0$  by

$$a_{n+m} = c_1 a_{n+m-1} + \cdots + c_m a_n,$$

for some given complex numbers  $c_1, \dots, c_m$ . Then the ambient space is the affine space  $\mathbb{A}^m$  with the Zariski topology, while the dynamical system is the one given by

$$\Phi((x_1, \dots, x_m)) = (x_2, \dots, x_m, c_1 x_m + \cdots + c_m x_1),$$

the starting point of the iteration is

$$x := (a_0, \dots, a_{m-1}),$$

and  $V \subset \mathbb{A}^m$  is the hyperplane given by the equation  $x_1 = 0$ . In Section 1.2 and Subsection 2.5.1, we explain this example in greater detail. In Section 2.5 we prove that the answer to this question is always a *finite union of arithmetic progressions*. A related, but more general problem involving (multi-dimensional) polynomial-exponential equations is discussed in Section 1.3.

- (2) Find all  $n \in \mathbb{N}_0$  such that given a matrix  $A \in M_n(\mathbb{C})$  acting on the complex affine space  $\mathbb{A}^n(\mathbb{C})$ , a point  $\alpha \in \mathbb{A}^n(\mathbb{C})$ , and a subvariety  $V \subset \mathbb{A}^n$ , then  $A^n \alpha \in V(\mathbb{C})$ . This case is discussed in Section 1.4 and it turns out to be equivalent with the problem (1) discussed above (see the equivalence proven in Proposition 2.5.1.4).

- (3) Find all  $n \in \mathbb{N}_0$  such that given an endomorphism  $\Phi$  of a quasiprojective variety  $X$  defined over  $\mathbb{C}$ , a point  $\alpha \in X(\mathbb{C})$ , and a subvariety  $V$  of  $X$ , then  $\Phi^n(x) \in V(\mathbb{C})$ . This problem, called the *Dynamical Mordell-Lang Conjecture* generalizes both of the above problems described above (see Section 1.5 for a first discussion of this conjecture). It is expected the answer to this question is again *finitely many arithmetic progressions*.
- (4) Given a power series

$$f(z) := \sum_{n=0}^{\infty} a_n z^n$$

which satisfies a linear differential equation with polynomial coefficients, describe the set

$$S_f := \{n \in \mathbb{N}_0 : a_n = 0\}.$$

Rubel [**Rub83**, Problem 16] conjectured that  $S_f$  is a finite union of arithmetic progressions. We discuss this problem in Subsection 3.2.1, and show that a positive answer for an extension of the above Dynamical Mordell-Lang Conjecture to rational maps would solve Rubel's question.

## 1.2. Linear recurrence sequences

Let  $\{F_n\}_{n \geq 0}$  be the Fibonacci sequence defined by

$$F_0 = 0, F_1 = 1 \text{ and } F_{n+2} = F_{n+1} + F_n \text{ for all } n \geq 0.$$

Also, let  $\{a_n\}_{n \geq 0}$  be the sequence defined recursively by

$$a_{n+2} = 5a_{n+1} - 6a_n,$$

where  $a_0 = \frac{7}{12}$  and  $a_1 = \frac{3}{2}$ .

**QUESTION 1.2.0.1.** *What are the numbers which appear in both of the sequences  $\{F_m\}_{m \in \mathbb{N}_0}$  and  $\{a_n\}_{n \in \mathbb{N}_0}$ ?*

We can compute easily the first elements in both sequences:

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, \dots$$

and

$$a_0 = \frac{7}{12}, a_1 = \frac{3}{2}, a_2 = 4, a_3 = 11, a_4 = 31, a_5 = 89, a_6 = 259 \dots$$

One observes that  $F_{11} = 89 = a_5$ , and it is a reasonable question to ask whether this is the only answer to Question 1.2.0.1. This is a hard question since one would have to solve the equation  $F_m = a_n$  in nonnegative integers  $m$  and  $n$  (for more details, see [**Eve95**]). Moreover, since it is easy to find a formula for the general term of both of these sequences (see Proposition 2.5.1.4), Question 1.2.0.1 reduces to finding  $m, n \in \mathbb{N}$  such that

$$\frac{1}{\sqrt{5}} \cdot \left( \left( \frac{1 + \sqrt{5}}{2} \right)^m - \left( \frac{1 - \sqrt{5}}{2} \right)^m \right) = 2^{n-2} + 3^{n-1}.$$

On the other hand, if we were to ask the easier question of when the above equality holds when  $m = n$ , the answer would be *never* since (by a simple inductive argument) one can show that  $a_k > F_k$  for all  $k \in \mathbb{N}$ .

In general, given two linear recurrence sequences  $\{a_m\}_{m \in \mathbb{N}_0}$  and  $\{b_n\}_{n \in \mathbb{N}_0}$ , one would like to understand whether there exists an underlying *structure* for the solutions  $(m, n) \in \mathbb{N} \times \mathbb{N}$  for which  $a_m = b_n$ . Or, at least, for the easier case, one would like to understand the structure of the set of all  $n \in \mathbb{N}$  such that  $a_n = b_n$ . It is immediate to see that this last case reduces to understanding when a given linear recurrence sequence  $\{c_n\}_{n \in \mathbb{N}_0}$  (in this case,  $c_n = a_n - b_n$ ) takes the value 0. Then the answer is that if there exist infinitely many  $n \in \mathbb{N}$  such that  $c_n = 0$ , then there exists an *infinite* arithmetic progression  $\{\ell + nk\}_{n \in \mathbb{N}_0}$  such that  $c_{\ell+nk} = 0$ . This will be proven in Section 2.5. As described in Section 1.1, the proper dynamical setting for this example is as follows: given a linear recurrence sequence

$$\{a_m\}_{m \in \mathbb{N}_0} \subset \mathbb{C}$$

which satisfies the relation

$$a_{n+m} = c_1 a_{n+m-1} + \cdots + c_m a_n,$$

for some given complex numbers  $c_1, \dots, c_m$ , then the dynamical system is the one given by the map

$$\Phi((x_1, \dots, x_m)) = (x_2, \dots, x_m, c_1 x_m + \cdots + c_m x_1)$$

acting on the  $m$ -dimensional affine complex space  $\mathbb{A}^m$ . Then finding all  $n \in \mathbb{N}_0$  such that  $a_n = 0$  is equivalent with finding all  $n \in \mathbb{N}_0$  such that

$$\Phi^n((a_0, \dots, a_{m-1})) \in V(\mathbb{C}),$$

where  $V \subset \mathbb{A}^m$  is the hyperplane given by the equation  $x_1 = 0$ .

### 1.3. Polynomial-exponential Diophantine equations

Let  $m, k \in \mathbb{N}$ , let  $F \in \mathbb{Z}[x_1, \dots, x_m, y_1, \dots, y_k]$ , and let  $r_1, \dots, r_k \in \mathbb{Z}$ . A polynomial-exponential equation has the form

$$F(j_1, \dots, j_m; r_1^{n_1}, \dots, r_k^{n_k}) = 0,$$

where the variables  $j_1, \dots, j_m \in \mathbb{Z}$ , respectively  $r_1, \dots, r_k \in \mathbb{N}_0$ . In general, there might be *many* solutions to the above equation, especially if the degree of  $f$  in  $x_i$  is 1 for at least one variable  $x_i$ . But, even if  $\deg_{x_i} f = 1$ , there might be no solutions due to some local constraints such as in the following case:

$$(1.3.0.1) \quad 21x_1^3 x_3 - 7 \cdot 3^{n_1} x_2 + 14 \cdot 5^{n_2} x_3^2 - 49x_1 x_3 + 2 = 0,$$

when there are no solutions  $x_1, x_2, x_3 \in \mathbb{Z}$  and  $n_1, n_2 \in \mathbb{N}_0$  by considering the congruence modulo 7 for the equation (1.3.0.1). Now, even if one assumes  $j_1 = j_2 = \cdots = j_m = j$ , and that the polynomial  $f$  has the variables  $x_i$  and  $y_j$  separated, the problem is not easier. Even also assuming that  $r_1 = r_2 = \cdots = r_k$  does not simplify the problem much. For example, we discuss in Chapter 13 the following special case:

$$g(x) = \sum_{i=1}^k c_i p^{n_i},$$

where  $g \in \mathbb{Z}[x]$ ,  $c_1, \dots, c_k \in \mathbb{Z}$  and  $p$  is a prime number. Essentially, one *expects* that if  $g(x)$  has *few* nonzero  $p$ -adic digits, then  $x$  (or a linear function evaluated at  $x$ ) would also have *few*  $p$ -adic digits. However, this is far from being proven even in simple cases such as  $g(x) = x^2$  and  $k \geq 5$  (for more details, see [BBM13, CZ00, CZ13] and the references therein).

On the other hand, if one assumes that

$$j_1 = \cdots = j_m = n_1 = n_2 = \cdots = n_k,$$

then the problem reduces essentially to the one discussed in Section 1.2 (see also Section 2.5). Thus one obtains that if there exist infinitely many  $n \in \mathbb{Z}$  such that

$$(1.3.0.2) \quad H(n, r_1^n, r_2^n, \dots, r_k^n) = 0,$$

where  $H \in \mathbb{Z}[z_0, z_1, z_2, \dots, z_k]$ , then there exists an infinite arithmetic progression  $\{\ell + nk\}_{n \in \mathbb{N}_0}$  such that each element of it is a solution to (1.3.0.2).

#### 1.4. Linear algebra

Let  $A$  be an invertible matrix in  $\mathrm{GL}_r(\mathbb{C})$ , let  $V$  be a linear subspace of  $\mathbb{C}^r$ , and let  $z \in \mathbb{C}^r$ .

**QUESTION 1.4.0.1.** *Is there a simple description of the set of positive integers  $n$  such that  $A^n z \in V$ ?*

We note that the problem discussed in this section could easily be asked for an arbitrary subvariety defined over  $\mathbb{C}$  of the affine space  $\mathbb{A}^r$ ; however this more general question reduces to the case  $V$  is a linear subvariety.

If  $V$  is a line passing through the origin of  $\mathbb{C}^r$ , then once there exist two distinct nonnegative integers  $m < n$  such that

$$(1.4.0.2) \quad A^m z \in V \text{ and } A^n z \in V,$$

then we immediately conclude that  $V$  is fixed by  $A^{n-m}$  and therefore

$$A^{m+\ell(n-m)} z \in V \text{ for all } \ell \in \mathbb{N}_0.$$

In particular, if  $k_0$  is the smallest positive integer  $k$  such that  $A^k$  fixes  $V$ , and if  $m_0$  is the smallest nonnegative integer  $m$  such that  $A^m z \in V$ , then  $A^n z \in V$  if and only if  $n = m_0 + \ell k_0$  for some nonnegative integer  $\ell$ .

Things are not so simple in general. For example, when  $V$  is a line that does not pass through the origin, it is easy to see that you can have distinct  $m$  and  $n$  such that (1.4.0.2) holds without getting an entire arithmetic progression of such integers, just by choosing a line  $V$  which passes through two arbitrary points  $A^m z$  and  $A^n z$ . But in the case of lines not passing through the origin, once you have a large finite number of integers  $n$  such that

$$(1.4.0.3) \quad A^n z \in V,$$

you must have an infinite arithmetic progression of such  $n$ . There is even an explicit bound on that number due to Beukers-Schlickewei [BS96], which is likely nowhere near sharp. In fact, under the assumption that each eigenvalue of  $A$  is either equal to 1 or is not a root of unity, and furthermore for each two distinct eigenvalues  $\lambda_i$  and  $\lambda_j$  of  $A$  we have that  $\lambda_i/\lambda_j$  is not a root of unity, Beukers-Schlickewei [BS96] show that there are at most 61 integers  $n \in \mathbb{N}_0$  such that (1.4.0.3) holds. The general case of an arbitrary matrix  $A$  follows easily from this special case.

### 1.5. Arithmetic geometry

The subject of this book is a geometric generalization (see Conjecture 1.5.0.1) of all of the above problems, and it is also connected to the classical Mordell-Lang Conjecture (see Chapter 3). In each of the three problems discussed in Sections 1.2 to 1.4, we deal with a geometric object: the line  $V$  in Section 1.4, or the hypersurface  $F = 0$  in Section 1.3, or the hyperplane  $x_1 = 0$  in the affine space  $\mathbb{A}^m$  as in Section 1.2. And we want to understand when an arithmetic dynamical system intersects the geometric object. The arithmetic dynamical system is the iteration of the matrix  $A$  in Section 1.4, or the input of an integer number into the equation  $F = 0$  (which is a discrete dynamical system simply because all integers are obtained from 0 by repeated operations of either  $z \mapsto z + 1$  or  $z \mapsto z - 1$ ), or a linear recurrence sequence as in Section 1.2. And in each case one obtains that once there exist infinitely many instances of the intersection between the geometric object and the arithmetic dynamical object, then there is a *structure* for the intersection which is given by *finitely many arithmetic progressions*. This principle is formally stated in the Dynamical Mordell-Lang Conjecture (for more details, see Chapter 3).

**CONJECTURE 1.5.0.1 (Dynamical Mordell-Lang Conjecture).** *Let  $X$  be a quasi-projective variety defined over  $\mathbb{C}$ , let  $\Phi$  be any endomorphism of  $X$ , let  $\alpha \in X(\mathbb{C})$ , and let  $V \subseteq X$  be any subvariety. Then the set of all  $n \in \mathbb{N}_0$  such that  $\Phi^n(\alpha) \in V(\mathbb{C})$  is a union of finitely many arithmetic progressions.*

We note that the Dynamical Mordell-Lang Conjecture can be formulated over any field  $K$  of characteristic 0 (see Conjecture 3.1.1.1); however such a formulation reduces to proving the case when  $K = \mathbb{C}$  (see Proposition 3.1.2.1).

A special case of Conjecture 1.5.0.1 that is known is when  $X$  is an abelian variety, and  $\Phi$  is the translation-by- $P$  endomorphism of  $X$  for some point  $P \in X(\mathbb{C})$ . In this latter case we encounter the cyclic case of the classical Mordell-Lang Conjecture (for more details, see Section 3.4).

We present below a few cases of Conjecture 1.5.0.1; all our examples are set in the ambient space  $X = \mathbb{A}^3$  in which case there is at this time no general proof of the Dynamical Mordell-Lang Conjecture.

**EXAMPLE 1.5.0.2.** Consider the endomorphism

$$\Phi : \mathbb{A}^3 \longrightarrow \mathbb{A}^3$$

given by

$$\Phi(x, y, z) = (x^2 + x, y^2 + y, z^2 + z).$$

Let  $V \subset \mathbb{A}^3$  be the plane given by the equation

$$x + y + z = 1.$$

Then for *most* points  $\alpha \in \mathbb{A}^3(\overline{\mathbb{Q}})$ , the set

$$S := \{n \in \mathbb{N}_0 : \Phi^n(\alpha) \in V(\overline{\mathbb{Q}})\}$$

is finite. For example, this can be seen immediately if all three coordinates of  $\alpha$  are integers (in which case, at the very most,  $S$  has 1 element). However, if  $\alpha$  is an arbitrary point in  $\mathbb{A}^3(\overline{\mathbb{Q}})$ , then it is much harder to prove that  $S$  is always a finite union of arithmetic progressions (possibly with common difference equal to 0). However, we will see later (see Corollary 7.0.0.1) that for *any* subvariety

$V \subseteq \mathbb{A}^3$ , the set  $S$  is a finite union of arithmetic progressions. Furthermore, using the classification of periodic curves under the coordinatewise action of a polynomial done by Medvedev-Scanlon [MS14], one can show that in the case of the above plane  $V$ , the set  $S$  is finite assuming  $\alpha$  is not preperiodic. Now, if  $\alpha$  is preperiodic, the question of whether  $S$  contains an infinite arithmetic progression is equivalent with finding three preperiodic points  $a$ ,  $b$  and  $c$  for the action of the polynomial

$$f(z) := z^2 + z,$$

such that

$$a + b + c = 1.$$

This last question is a deep question related to the problem of unlikely intersections in dynamics which we discuss in Subsection 14.2.2.

EXAMPLE 1.5.0.3. Consider the endomorphism

$$\Phi : \mathbb{A}^3 \longrightarrow \mathbb{A}^3$$

given by

$$\Phi(x, y, z) = (x^5, y^3, z^5).$$

Let  $\alpha = (0, i, 0)$  and let  $V \subset \mathbb{A}^3$  be the surface given by the equation

$$x^3 + y + z^3 = i.$$

We easily see that  $\alpha$  is periodic under the action of  $\Phi$  and moreover,  $\Phi^n(\alpha) \in V$  if and only if  $n$  is an even nonnegative integer. Actually, using Theorem 9.3.0.1 one can show that for any  $\alpha \in \mathbb{A}^3(\mathbb{C})$  and for any complex subvariety  $V \subseteq \mathbb{A}^3$ , the set  $S$  of all  $n \in \mathbb{N}_0$  such that  $\Phi^n(\alpha) \in V(\mathbb{C})$  is a finite union of arithmetic progressions. Furthermore, according to the classical Mordell-Lang conjecture for an algebraic torus (proven by Laurent [Lau84]; see also Section 3.4), one obtains that the above set  $S$  is finite unless  $V$  contains a translate of a positive dimensional algebraic torus.

EXAMPLE 1.5.0.4. Consider the endomorphism

$$\Phi : \mathbb{A}^3 \longrightarrow \mathbb{A}^3$$

given by

$$\Phi(x, y, z) = (x^2 + y, y^2 + z, z^2 + x).$$

Let  $\alpha = (1, 1, 1)$  and  $S \subset \mathbb{A}^3$  be the surface given by the equation

$$x + y^2 + z^3 = x^2 + y^3 + z.$$

It is immediate to see that the entire orbit  $\mathcal{O}_\Phi(\alpha)$  is contained in the surface  $S$ , and the reason for this is that  $V$  contains the line  $L$  given by the equation

$$x = y = z,$$

which is fixed by the action of  $\Phi$ . However, if  $V$  is an arbitrary subvariety of  $\mathbb{A}^3$ , and also  $\alpha$  is an arbitrary point in  $\mathbb{A}^3(\mathbb{C})$ , then it is not known whether Conjecture 1.5.0.1 holds. In some sense, the endomorphism  $\Phi$  from this Example lies outside all the presently known cases of the Dynamical Mordell-Lang Conjecture (see Chapter 3 for more details).

Examples 1.5.0.2 to 1.5.0.4 give us a glimpse into the philosophy behind the Dynamical Mordell-Lang Conjecture. With the notation as in Conjecture 1.5.0.1, the set

$$S := S(\Phi, V, \alpha) := \{n \in \mathbb{N}_0 : \Phi^n(\alpha) \in V\}$$

is finite unless one of the following two conditions holds:

- (1)  $\alpha$  is preperiodic under the action of  $\Phi$  and  $V$  contains a point from the periodic cycle of  $\mathcal{O}_\Phi(\alpha)$ ; or
- (2)  $V$  contains a positive dimensional subvariety  $W$  which is periodic under the action of  $\Phi$ , and moreover,  $W$  intersects  $\mathcal{O}_\Phi(\alpha)$  (for the definition of periodic subvarieties, see Section 2.2).

It is easy to see that either (1) or (2) above yield a corresponding infinite set  $S = S(\Phi, V, \alpha)$ . Moreover, it is also clear that if either case (1) holds, or  $V$  itself is a periodic variety, then the set  $S$  consists of finitely many arithmetic progressions. So, the content of the Dynamical Mordell-Lang Conjecture is to prove that when  $\alpha$  is not preperiodic, then the *only* possibility for the set  $S$  to be infinite is when condition (2) holds.

## 1.6. Plan of the book

We sketch here the contents of the remaining chapters of our book. Also, at the end of this section, we suggest several plans for studying from this book.

**1.6.1. Description of each chapter.** In Chapter 2 we present the necessary background material for the rest of the book, focusing on the notions from algebraic and arithmetic geometry, valuation theory,  $p$ -adic analysis and their applications to the problems studied in our book. Of special importance for our study is Theorem 2.5.4.1, which also constituted the starting point for our  $p$ -adic approach to the Dynamical Mordell-Lang Conjecture. Theorem 2.5.4.1 is the classical result of Skolem [Sko34] (later generalized by Mahler [Mah35] and Lech [Lec53]) that solves the problem discussed in Section 1.2: given a linear recurrence sequence  $\{a_n\} \subset \mathbb{C}$ , the set of nonnegative integers  $n$  such that  $a_n = 0$  is a finite union of arithmetic progressions.

In Chapter 3 we discuss Conjecture 1.5.0.1 and its connection to the classical Mordell-Lang conjecture (proven by Faltings [Fal83]) and to the Denis-Mordell-Lang Conjecture (see [Den92a]). We also discuss a multi-dimensional problem stemming from the Dynamical Mordell-Lang Conjecture, which turns out to be false in general, but sometimes, in outstanding cases, such as the classical Mordell-Lang conjecture itself, it has a positive answer. We explore in more depth this multi-dimensional analogue of the Dynamical Mordell-Lang Conjecture in Chapter 5. Also, in Chapter 5 we prove an interesting instance of Conjecture 1.5.0.1 when  $X = \mathbb{A}^2$ ,  $V$  is the diagonal line, and

$$\Phi(x, y) := (f(x), g(y))$$

for arbitrary polynomials  $f, g \in \mathbb{C}[z]$  (see [GTZ08, GTZ12]). The proof of the main result from Chapter 5 is one of the very few instances when a special case of the Dynamical Mordell-Lang Conjecture is proven without using a  $p$ -adic approach; other special cases of Conjecture 1.5.0.1 proven without the explicit use of  $p$ -adic analysis are the works of Ng and Wang [NW13] and Xie [Xie14, Xieb].

In Chapter 4 we expand on the discussion from Section 2.5 by giving a geometric twist of the classical result of Skolem-Mahler-Lech regarding the occurrence of zeros in an arithmetic progression; essentially, Theorem 2.5.4.1 can be easily reformulated in terms of automorphisms of  $\mathbb{P}^n$  (see Denis [Den94]). In particular, we show that given an étale endomorphism  $\Phi$  of a quasiprojective variety  $X$  defined over  $\mathbb{C}$ , and given a point  $\alpha \in X(\mathbb{C})$  there exists a prime number  $p$ , a suitable embedding into  $\mathbb{Q}_p$ , and a positive integer  $k$  such that the map  $n \mapsto \Phi(kn)$  is  $p$ -adic analytic (see [BGT10] whose main result builds on previous work of Bell [Bel06]). This method of finding a  $p$ -adic analytic parametrization of the orbit of a point is called the  *$p$ -adic arc lemma*.

In Chapter 6 we present the main results from  $p$ -adic dynamics which allow us to parametrize the orbit of a point under a non-étale endomorphism  $\Phi$  of a quasiprojective variety  $X$ . The main results are for rational maps  $\Phi$  acting on  $X = \mathbb{P}^1$ , and they are due to Rivera-Letelier [RL03]. As an application of these  $p$ -adic analytic parametrizations we obtain several interesting results in Chapters 7 and 11 (see [BGKT10, BGKT12, BGHKST13]). In Chapter 8 we present heuristics regarding the general case of the Dynamical Mordell-Lang Conjecture (for more details, see [BGHKST13]). In particular, these heuristics suggest that the  $p$ -adic approach *might* not work to prove the general case of Conjecture 1.5.0.1.

There are fewer instances of  $p$ -adic analytic parametrizations of orbits under endomorphisms  $\Phi$  of higher dimensional varieties  $X$ ; the main result in this area is an older theorem of Herman and Yoccoz [HY83]. However, this last result is sufficient for us to prove certain special cases of the Dynamical Mordell-Lang Conjecture in Chapter 9.

In Chapter 10 we present two results of Scanlon [Sca, Sca11] towards the Dynamical Mordell-Lang Conjecture which both use analytic parametrizations of the orbit – one of the parametrizations using  $p$ -adic analysis, and the other one using real analytic functions. In Chapter 10, we also discuss briefly Xie’s proofs [Xie14, Xieb] of the special cases of the Dynamical Mordell-Lang Conjecture for endomorphisms of  $\mathbb{A}^2$ . We point out right from the beginning that Xie proved one of the most outstanding open case of the dynamical Mordell-Lang Conjecture – the case of endomorphisms of  $\mathbb{A}^2$  – however, due to our emphasis in this book for the  $p$ -adic analytic approach to the Dynamical Mordell-Lang Conjecture and also due to the fact that Xie’s results are very recent (actually, [Xieb] was not even released when we submitted our first draft of our book), we do not include a thorough description of Xie’s theorems. But we encourage the reader interested in the problem we study in this book to consult Xie’s almost 100 pages preprint [Xieb] (which uses a previously released almost equally long preprint [Xiea]); in a way, together [Xiea] and [Xieb] contain enough material for another book on the topic of the Dynamical Mordell-Lang Conjecture!

In Chapter 11 we prove a weaker version of the Dynamical Mordell-Lang Conjecture. In the highest possible generality (even surpassing the world of algebraic geometry and dealing with continuous self-maps  $\Phi$  on Noetherian spaces  $X$ ), we prove in Theorem 11.4.2.2 that for any closed subset  $Y \subseteq X$  and for any  $\alpha \in X$ , the set

$$S := \{n \in \mathbb{N}_0 : \Phi^n(\alpha) \in Y\}$$

is a union of finitely many arithmetic progressions along with a set of Banach density 0. In other words, we prove that if  $X$  contains no closed subset which is

both periodic under  $\Phi$  and also intersects the orbit of  $\alpha$ , then the set  $S$  has Banach density 0. The Dynamical Mordell-Lang Conjecture (when  $\Phi$  is an endomorphism of a quasiprojective variety  $X$ ) asks that  $S$  must be finite in this case. Nevertheless, Theorem 11.4.2.2 (see also [BGT15b] where this result was first published) presents very strong evidence towards Conjecture 1.5.0.1. Furthermore, also in Chapter 11 we present various strengthenings of Theorem 11.4.2.2 for various special cases of the Dynamical Mordell-Lang Conjecture (for more details, we refer the reader to [BGKT10] and [BGT15b]).

In Chapter 12 we discuss the Denis-Mordell-Lang Conjecture which may be viewed as a hybrid between the classical Mordell-Lang problem and Conjecture 1.5.0.1 over a field of positive characteristic. We continue the exploration of the Dynamical Mordell-Lang Conjecture in characteristic  $p$  in Chapter 13; very little is known for the characteristic  $p$  analogue of Conjecture 1.5.0.1 (see Conjecture 13.2.0.1) even for the case of endomorphisms of  $\mathbb{G}_m^n$ .

In Chapter 14 we discuss various other questions in arithmetic geometry whose solution was obtained (or it *might* be obtained) using a  $p$ -adic analytic approach. Among these questions we mention the Dynamical Manin-Mumford Conjecture and the unlikely intersection problem in dynamics (see also [Zan12] for more details on related problems in arithmetic geometry). In Chapter 15 we conclude by speculating on the future of the Dynamical Mordell-Lang Conjecture.

**1.6.2. Suggested plans for studying.** Of course, we hope the interested reader will find time to read our entire book, but in case one wants to read only a subset of our book in order to gain understanding to some of the more important results and methods discussed in the book, we present in each of the following Subsections a possible reading plan. We leave out from our suggested reading plans most of the background Chapter 2 and also the Chapters 14 and 15 which talk about related questions to the Dynamical Mordell-Lang Conjecture and also speculate about its future. Obviously, we encourage all readers to read Chapter 2 in its entirety to familiarize with the notions from algebraic geometry and number theory that we use in our book. Also, we hope Chapters 14 and 15 will motivate the reader to study in the future various cases of the Dynamical Mordell-Lang Conjecture or related questions from arithmetic dynamics. On the other hand, we encourage all readers to include Chapter 3 in their study of our book since it provides a comprehensive introduction into the Dynamical Mordell-Lang Conjecture and its connection to other important questions from arithmetic geometry.

The plans listed in the subsequent Subsections are given a name that informally describes the goal of each such reading plan.

**1.6.3. The  $p$ -adic arc lemma.** The central method used in this book for attacking the Dynamical Mordell-Lang Conjecture is the  $p$ -adic arc lemma, which is formally introduced in Chapter 4. So, if a reader wants to understand this important tool, we recommend one to read first Section 2.5 (where a very special case of the  $p$ -adic arc lemma, known as the Skolem's method, is introduced) and then to proceed to Chapter 4. Then, the reader could study Chapters 6 and 7, and also Sections 11.5 and 11.11 for more examples of use of the  $p$ -adic arc lemma and of similar  $p$ -adic uniformization techniques for an orbit of a point. Finally, the reader should read Chapter 8 to understand the limitations one encounters in the use of the  $p$ -adic arc lemma for the Dynamical Mordell-Lang Conjecture.

**1.6.4. The Dynamical Mordell-Lang for algebraic groups.** The motivation for Conjecture 1.5.0.1 comes from the classical Mordell-Lang Conjecture; for more details, see Chapter 3. So, a reader could focus on the connection between these two conjectures, in particular on the special case of Conjecture 1.5.0.1 when the ambient variety is a semiabelian variety. Hence, one can read in Chapter 9 the proof of the Dynamical Mordell-Lang Conjecture for endomorphisms of semiabelian varieties. This proof from chapter 9, which avoids the use of the  $p$ -adic arc lemma (after all, an alternative proof of the Dynamical Mordell-Lang Conjecture for all semiabelian varieties is found in Chapter 4; see Corollary 4.4.1.2) has the advantage of extending to proving some special cases of Question 3.6.0.1, which is a question generalizing both the classical Mordell-Lang conjecture and the Dynamical Mordell-Lang Conjecture.

Then one can read about a hybrid version of the classical Mordell-Lang conjecture and the Dynamical Mordell-Lang Conjecture, which is the Denis-Mordell-Lang conjecture in the context of Drinfeld modules. Actually, this conjecture of Denis [Den92a] was the starting point for formulating Conjecture 1.5.0.1 in [GT09]. Finally, one can read in Chapter 13 about a characteristic  $p$  version of the Dynamical Mordell-Lang Conjecture, and also read about partial results on this conjecture in the context of semiabelian varieties using the same approach as in Chapter 9 but also using an alternative approach coming from the theory of automata.

**1.6.5. The intersection of orbits.** While trying to prove a very special case of Conjecture 1.5.0.1 for lines in the plane under the coordinatewise action of two one-variable polynomials, the authors of [GTZ08] discovered a more general result regarding the intersection of two orbits under the action of two polynomials. Briefly, the results of [GTZ08] (and their extension from [GTZ12]) say that if two polynomials  $f, g \in \mathbb{C}[z]$  of degrees larger than 1 have the property that there exist  $\alpha, \beta \in \mathbb{C}$  such that

$$\mathcal{O}_f(\alpha) \cap \mathcal{O}_g(\beta) \text{ is infinite,}$$

then there exist linear polynomials  $\mu, \nu \in \mathbb{C}[z]$ , some polynomial  $h \in \mathbb{C}[z]$  of degree larger than 1, and positive integers  $m$  and  $n$  such that

$$f = \mu \circ h^m \text{ and } g = \nu \circ h^n.$$

These results can be viewed as a possible bridge towards Question 3.6.0.1, which is presented in Chapter 3. The interested reader will find all about these questions and results in Chapter 5, which is mainly self-contained.