Preface

This is a collection of three introductory tutorials coming out of three courses held at the CIMPA Research School "Galois Theory of Difference Equations" in Santa Marta, Columbia, July 23–August 1, 2012. The aim of these tutorials is to introduce the reader to three Galois theories of linear difference equations and their interrelations. Each of the three articles addresses a different galoisian aspect of linear difference equations and introduces the reader to the basic ideas and techniques, giving him or her an entryway to current research.

The article by Michael F. Singer presents the Galois theory of linear difference equations as introduced by van der Put and Singer in their book *Galois Theory of Linear Difference Equations*, Lecture Notes in Mathematics, vol. 1666, Springer-Verlag, Heidelberg, 1997. This Galois theory is especially effective in describing algebraic properties of sequences satisfying linear recurrences. This theory has led to applications in combinatorics as well as algorithms that have been implemented in computer algebra software. The article assumes the reader is only familiar with basic notions from algebra (groups, rings, fields, ideals, ...). Section 1 gives the reader examples of questions that this theory addresses. Sections 2 and 3 give a quick introduction to affine algebraic geometry and the theory of linear algebraic groups, the groups that appear as Galois groups in this Galois theory. Using the tools developed in these two sections, the Galois theory is presented in Sections 4 and 5. Section 6 introduces the reader to computational issues.

The article by Charlotte Hardouin presents a Galois theory of linear difference equations aimed at understanding the differential behavior of solutions. This Galois theory was introduced by Hardouin and Singer in *Differential Galois theory of linear difference equations*, Mathematische Annalen, 342(2):333-377, 2008. This theory has been used to give a new conceptual proof of Hölder's result that the Gamma function satisfies no polynomial differential equation as well as give group-theoretic criteria for the integrability (isomonodromy) of certain dynamical systems. Hardouin's article also assumes only a very basic background as above. After a brief overview in Section 1, Hardouin presents an introduction to differential algebra in Section 2. In Section 3, she develops the more geometric side of differential algebra ending with an introduction to differential algebraic groups. These are the groups that appear as Galois groups in the Galois theory that Hardouin exposes but they are also of interest in themselves and are a topic of current studies by model theorists and number theorists. Section 4 is devoted to developing the parameterized Picard-Vessiot theory, the Galois theory that studies the differential algebraic properties of solutions of linear differential equations.

The article by Jacques Sauloy introduces the reader to an analytic approach to q-difference equations and an associated Galois theory. Although the study of q-difference equations is classical, the recent work of André, Di Vizio, van der Put, Ramis, Reversat, Sauloy, Singer, and Zhang has given new insights and results and has made this topic one of current research interest. This article is an entryway to this area and prepares the reader to study the recent papers. In addition to a knowledge of algebra at the level of a basic graduate level algebra course, Sauloy assumes a knowledge of the basic theory of functions of a complex variable and, later in the article, a slight acquaintance with homological algebra. Sauloy begins, in Section 1, with examples and motivations showing how q-series and q-difference equations arise in mathematics. Section 2 is an introduction to the difference algebra needed in what follows. Section 3 begins the study of q-difference equations with fuchsian singularities. Section 4 begins by reviewing the notions of local monodromy and local Galois groups for linear differential equations and then introduces the appropriate analogies for q-difference equations as well as their global counterparts. Section 4 also contains a breezy introduction to Tannaka duality. Section 5 goes into the finer structure of linear q-difference operators, focusing on a useful factorization that is one of the properties that distinguishes q-difference operators from differential operators. This factorization allows us to give normal forms for q-difference equations and make statements about asymptotics of solutions. Appendices discuss the role of q-difference equations as q-deformations of differential equations.

All three articles aim at bringing the reader to a point where he or she can delve further into the current literature and begin research in these areas. These articles motivate and give elementary examples of the basic concepts. In addition each article contains an extensive bibliography including very recent papers. In the texts of the articles the authors have provided pointers to these articles allowing the interested reader to explore further.

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