Preface

I. Ramsey theory is the general area of combinatorics devoted to the study of the pigeonhole principles that appear in mathematical practice. It originates from the works of Ramsey [**Ra**] and van der Waerden [**vdW**], and at the early stages of its development the focus was on structural properties of graphs and hypergraphs. However, the last 40 years or so, Ramsey theory has expanded significantly, both in scope and in depth, and is now constantly interacting with analysis, ergodic theory, logic, number theory, probability theory, theoretical computer science, and topological dynamics.

This book (which inherits, to some extent, the diversity of the field) is a detailed exposition of a number of Ramsey-type results concerning *product spaces* or, more accurately, finite Cartesian products $F_1 \times \cdots \times F_n$ where the factors $F_1, \dots, F_n$ may be equipped with an additional structure depending upon the context. Product spaces are ubiquitous in mathematics and are admittedly elementary objects, yet they exhibit a variety of Ramsey properties which depend on the *dimension* n and the *size* of each factor. Quantifying properly this dependence is one of the main goals of Ramsey theory, a goal which can sometimes be quite challenging.

I.1. The first example of a product space of interest to us in this book is the *discrete hypercube*

$$A^n := \underbrace{A \times \cdots \times A}_{n-\text{times}},$$

where n is a positive integer and A is a nonempty finite set. In fact, we will be mostly interested in the *high-dimensional* case (that is, when the dimension n is large compared with the cardinality of A), but apart from this assumption no further constraints will be imposed on the set A.

A classical result concerning the structure of high-dimensional hypercubes was discovered in 1963 by Hales and Jewett [**HJ**]. It asserts that for every partition of A^n into, say, two pieces, one can always find a "sub-cube" of A^n which is entirely contained in one of the pieces of the partition. The Hales–Jewett theorem paved the way for a thorough study of the Ramsey properties of discrete hypercubes and related structures, and it triggered the development of several infinite-dimensional extensions. This material is the content of Chapters 2, 4, and 5.

Around 30 years after the work of Hales and Jewett, another fundamental result of Ramsey theory was proved by Furstenberg and Katznelson [**FK4**]. It is a natural, yet quite deep, refinement of the Hales–Jewett theorem and asserts that every *dense* subset of A^n (that is, every subset of A^n whose cardinality is proportional to that of A^n) must contain a "sub-cube" of A^n. Much more recently, the work of Furstenberg and Katznelson was revisited by several authors, and a number of different proofs of this important result have been found. This line of research eventually led to a better understanding of the structure of dense subsets of hypercubes both in the finite- and the infinite-dimensional setting. We present these developments in Chapters 8 and 9.

I.2. A second example relevant to the theme of this book is the product space

$$T_1 \times \cdots \times T_d,$$

where d is a positive integer and $T_1, \dots, T_d$ are nonempty trees. Partitions of products spaces of this form appear in the context of Ramsey theory for trees. However, in this case we are interested in the somewhat different regime where the dimension d is regarded as being fixed while the trees $T_1, \dots, T_d$ are assumed to be sufficiently large and even possibly infinite. Chapter 3 is devoted to this topic.

I.3. The last main example of a product space which we are considering in this book is of the form

$$\Omega_1 \times \cdots \times \Omega_n,$$

where n is a positive integer and for each $i \in \{1, \dots, n\}$ the set Ω_i is the sample space of a probability space $(\Omega_i, \Sigma_i, \mu_i)$. We view, in this case, the set $\Omega_1 \times \cdots \times \Omega_n$ also as a probability space equipped with the product measure $\mu_1 \times \cdots \times \mu_n$.

A powerful result concerning products of probability spaces, with several consequences in Ramsey theory, was proved around 10 years ago. It asserts that for every finite family $\mathcal{F}$ of measurable events of $\Omega_1 \times \cdots \times \Omega_n$ whose joint probability is negligible, one can approximate the members of $\mathcal{F}$ by lower-complexity events (that is, by events which depend on fewer coordinates) whose intersection is empty. This result is known as the *removal lemma*, and in this generality it is due to Tao [**Tao1**], though closely related discrete analogues were obtained earlier by Gowers [**Go5**] and, independently, by Nagle, Rödl, Schacht, and Skokan [**NRS, RSk**]. We present these results in Chapter 7.

Finally, in Chapter 6 we discuss certain aspects of the *regularity method*. It originates from the work of Szemerédi [**Sz1, Sz2**] and is used to show that dense subsets of discrete structures are inherently pseudorandom. We follow a probabilistic approach in the presentation of the method, emphasizing its relevance not only in the context of graphs and hypergraphs, but also in the analysis of high-dimensional product spaces.

II. This book is addressed to researchers in combinatorics, but also working mathematicians and advanced graduate students who are interested in this part of Ramsey theory. The prerequisites for reading this book are rather minimal; it only requires familiarity, at the graduate level, with probability theory and real analysis. Some familiarity with the basics of Ramsey theory (as exposed, for instance, in the book of Graham, Rothschild and Spencer [**GRS**]) would also be beneficial, though it is not necessary.

To assist the reader, we have included six appendices, thus making this book essentially self-contained. In Appendix A we briefly discuss some properties of primitive recursive functions, while in Appendix B we present a classical estimate for the Ramsey numbers due to Erdős and Rado [**ER**]. In Appendix C we recall some results related to the Baire property which are needed in Section 3.2. Appendix D contains an exposition of a part of the theory of ultrafilters and idempotents in compact semigroups; we note that this material is used only in Section 4.1. Finally, in Appendix E we present the necessary background from probability theory, and in Appendix F we discuss open problems.

It is needless to say that this book is based on the work of many researchers who made Ramsey theory a rich and multifaceted area. Several new results are also included. Bibliographical information on the content of each chapter is contained in its final section named “Notes and remarks”.

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