

Basic concepts

1.1. General notation

1.1.1. Throughout this book, by $\mathbb{N} = \{0, 1, \dots\}$ we shall denote the set of all natural numbers. Moreover, for every positive integer n we set $[n] := \{1, \dots, n\}$.

For every set X by $|X|$ we shall denote its cardinality. If $k \in \mathbb{N}$ with $k \leq |X|$, then by $\binom{X}{k}$ we shall denote the set of all subsets of X of cardinality k , that is,

$$(1.1) \quad \binom{X}{k} = \{Y \subseteq X : |Y| = k\}.$$

On the other hand, if X is infinite, then $[X]^\infty$ stands for the set of all infinite subsets of X . The powerset of X will be denoted by $\mathcal{P}(X)$.

1.1.2. If X and Y are nonempty sets, then a map $c: X \rightarrow Y$ will be called a *Y-coloring of X*, or simply a *coloring* if X and Y are understood. A *finite coloring of X* is a coloring $c: X \rightarrow Y$ where Y is finite, and if $|Y| = r$ for some positive integer r , then c will be called an *r-coloring*. The nature of the set Y is irrelevant from a Ramsey theoretic perspective, and so we will view every r -coloring of X as a map $c: X \rightarrow [r]$.

Given a coloring $c: X \rightarrow Y$, a subset Z of X is said to be *monochromatic (with respect to the coloring c)* provided that $c(z_1) = c(z_2)$ for every $z_1, z_2 \in Z$, or equivalently, that $Z \subseteq c^{-1}(\{y\})$ for some $y \in Y$.

1.1.3. Let X be a (possibly infinite) nonempty set and let Y be a nonempty finite subset of X . For every $A \subseteq X$ the *density of A relative to Y* is defined by

$$(1.2) \quad \text{dens}_Y(A) = \frac{|A \cap Y|}{|Y|}.$$

If it is clear from the context which set Y we are referring to (for instance, if Y coincides with X), then we shall drop the subscript Y and we shall denote the above quantity simply by $\text{dens}(A)$. More generally, for every $f: X \rightarrow \mathbb{R}$ we set

$$(1.3) \quad \mathbb{E}_{y \in Y} f(y) = \frac{1}{|Y|} \sum_{y \in Y} f(y).$$

Notice that for every $A \subseteq X$ we have $\text{dens}_Y(A) = \mathbb{E}_{y \in Y} \mathbf{1}_A(y)$, where $\mathbf{1}_A$ stands for the characteristic function of A , that is,

$$(1.4) \quad \mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

The quantities $\text{dens}_Y(A)$ and $\mathbb{E}_{y \in Y} f(y)$ have a natural probabilistic interpretation which is very important in the context of density Ramsey theory. Specifically, denoting by μ_Y the uniform probability measure on X concentrated on Y , we see

that $\text{dens}_Y(A) = \mu_Y(A)$ and $\mathbb{E}_{y \in Y} f(y) = \int f d\mu_Y$. A review of those tools from probability theory which are needed in this book can be found in Appendix E.

1.1.4. Recall that a *hypergraph* is a pair $\mathcal{H} = (V, E)$, where V is a nonempty set and $E \subseteq \mathcal{P}(V)$. The elements of V are called the *vertices* of \mathcal{H} while the elements of E are called its *edges*. If E is a nonempty subset of $\binom{V}{r}$ for some $r \in \mathbb{N}$, then the hypergraph \mathcal{H} will be called *r-uniform*. Thus, a 2-uniform hypergraph is just a graph with at least one edge.

1.1.5. For every function $f: \mathbb{N} \rightarrow \mathbb{N}$ and every $\ell \in \mathbb{N}$ by $f^{(\ell)}: \mathbb{N} \rightarrow \mathbb{N}$, we shall denote the ℓ -th iteration of f defined recursively by the rule

$$(1.5) \quad \begin{cases} f^{(0)}(n) = n, \\ f^{(\ell+1)}(n) = f(f^{(\ell)}(n)). \end{cases}$$

Note that this is a basic example of primitive recursion (see Appendix A).

1.2. Words over an alphabet

Let A be a *nonempty alphabet*, that is, a nonempty set. For every $n \in \mathbb{N}$ by A^n , we shall denote the set of all sequences of length n having values in A . Precisely, A^0 contains just the empty sequence while if $n \geq 1$, then

$$(1.6) \quad A^n = \{(a_0, \dots, a_{n-1}) : a_i \in A \text{ for every } i \in \{0, \dots, n-1\}\}.$$

Also let

$$(1.7) \quad A^{<n+1} = \bigcup_{i=0}^n A^i \quad \text{and} \quad A^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} A^n.$$

The elements of $A^{<\mathbb{N}}$ are called *words over A*, or simply *words* if A is understood. The *length* of a word w over A , denoted by $|w|$, is defined to be the unique natural number n such that $w \in A^n$. For every $i \in \mathbb{N}$ with $i \leq |w|$ by $w \upharpoonright i$, we shall denote the word of length i which is an initial segment of w . (In particular, we have that $w \upharpoonright 0$ is the empty word.) More generally, if X is a nonempty subset of $A^{<\mathbb{N}}$ such that for every $w \in X$ we have $i \leq |w|$, then we set

$$(1.8) \quad X \upharpoonright i = \{w \upharpoonright i : w \in X\}.$$

If w and u are two words over A , then the concatenation of w and u will be denoted by $w \hat{\ } u$. Moreover, for every pair X, Y of nonempty subsets of $A^{<\mathbb{N}}$ we set

$$(1.9) \quad X \hat{\ } Y = \{w \hat{\ } u : w \in X \text{ and } u \in Y\}.$$

The *infimum* of w and u , denoted by $w \wedge u$, is defined to be the greatest common initial segment of w and u . Note that the infimum operation can be extended to nonempty sets of words. Specifically, for every nonempty subset X of $A^{<\mathbb{N}}$ the *infimum* of X , denoted by $\wedge X$, is the word over A of greatest length which is an initial segment of every $w \in X$. Observe that $w \wedge u = \wedge \{w, u\}$ for every $w, u \in A^{<\mathbb{N}}$.

If $<_A$ is a linear order on A , then for every distinct $w, u \in A^{<\mathbb{N}}$ we write $w <_{\text{lex}} u$ provided that (i) $|w| = |u| \geq 1$, and (ii) if $w = (w_0, \dots, w_{n-1})$, $u = (u_0, \dots, u_{n-1})$ and $i_0 = |w \wedge u|$, then $w_{i_0} <_A u_{i_0}$. Notice that for every positive integer n the partial order $<_{\text{lex}}$ restricted on A^n is the usual lexicographical order.

1.2.1. Located words. Let A be a nonempty alphabet. For every (possibly empty) finite subset J of \mathbb{N} by A^J we shall denote the set of all functions from J into A . An element of the set

$$(1.10) \quad \bigcup_{J \subseteq \mathbb{N} \text{ finite}} A^J$$

is called a *located word over A* . Clearly, every word over A is a located word over A . Indeed, notice that for every $n \in \mathbb{N}$ we have

$$(1.11) \quad A^{\{i \in \mathbb{N}: i < n\}} = A^n.$$

Conversely, we may identify located words over A with words over A as follows.

DEFINITION 1.1. *Let A be a nonempty alphabet and let J be a nonempty finite subset of \mathbb{N} . Set $j = |J|$ and let $n_0 < \dots < n_{j-1}$ be the increasing enumeration of J . The canonical isomorphism associated with J is the bijection $I_J: A^j \rightarrow A^J$ defined by the rule*

$$(1.12) \quad I_J(w)(n_i) = w_i$$

for every $i \in \{0, \dots, j-1\}$ and every $w = (w_0, \dots, w_{j-1}) \in A^j$.

Moreover, observing that $A^\emptyset = A^0 = \{\emptyset\}$, we define the canonical isomorphism I_\emptyset associated with the empty set to be the identity.

If J, K are two finite subsets of \mathbb{N} with $J \subseteq K$ and $w \in A^K$ is a located word over A , then by $w \upharpoonright J$ we shall denote the restriction of w on J . Notice that $w \upharpoonright J \in A^J$. Moreover, if I, J is a pair of finite subsets of \mathbb{N} with $I \cap J = \emptyset$, then for every $u \in A^I$ and every $v \in A^J$ by (u, v) we shall denote the unique element z of $A^{I \cup J}$ such that $z \upharpoonright I = u$ and $z \upharpoonright J = v$.

1.2.2. Variable words. Let A be a nonempty alphabet and let n be a positive integer. We fix a set $\{x_0, \dots, x_{n-1}\}$ of distinct letters which is disjoint from A . We view $\{x_0, \dots, x_{n-1}\}$ as a set of variables. An n -variable word over A is a word v over the alphabet $A \cup \{x_0, \dots, x_{n-1}\}$ such that (i) for every $i \in \{0, \dots, n-1\}$ the letter x_i appears in v at least once, and (ii) if $n \geq 2$, then for every $i, j \in \{0, \dots, n-1\}$ with $i < j$ all occurrences of x_i precede all occurrences of x_j .

If A is understood and $n \geq 2$, then n -variable words over A will be referred to as n -variable words. On the other hand, 1-variable words over A will simply be called *variable words* and their variable will be denoted by x . A *left variable word (over A)* is a variable word whose leftmost letter is x .

REMARK 1.1. The concept of a variable word is closely related to the notion of a parameter word introduced by Graham and Rothschild [GR]. Specifically, an n -parameter word over A is also a finite sequence having values in the alphabet $A \cup \{x_0, \dots, x_{n-1}\}$ which satisfies (i) above and such that (ii)' if $n \geq 2$, then for every $i, j \in \{0, \dots, n-1\}$ with $i < j$ the first occurrence of x_i precedes the first occurrence of x_j . In particular, every n -variable word is an n -parameter word. Of course, when $n = 1$, the two notions coincide.

Now let A, B be nonempty alphabets. If v is a variable word over A and $b \in B$, then by $v(b)$ we shall denote the unique word over $A \cup B$ obtained by substituting in v all appearances of the variable x with b . Notice that if $b \in A$, then $v(b)$ is a word over A , while $v(x) = v$. More generally, let v be an n -variable word over A and let $b_0, \dots, b_{n-1} \in B$. By $v(b_0, \dots, b_{n-1})$ we shall denote the unique word

over $A \cup B$ obtained by substituting in v all appearances of the letter x_i with b_i for every $i \in \{0, \dots, n-1\}$. Observe that if $B = A \cup \{x_0, \dots, x_{m-1}\}$ for some $m \in [n]$, then $v(b_0, \dots, b_{n-1})$ is a word over A if and only if (b_0, \dots, b_{n-1}) is a word over A ; on the other hand, $v(b_0, \dots, b_{n-1})$ is an m -variable word over A if and only if (b_0, \dots, b_{n-1}) is an m -variable word over A . Taking into account these remarks, for every $m \in [n]$ we set

(1.13)

$$\text{Subw}_m(v) = \{v(b_0, \dots, b_{n-1}) : (b_0, \dots, b_{n-1}) \text{ is an } m\text{-variable word over } A\},$$

and we call an element of $\text{Subw}_m(v)$ an m -variable subword of v . Note that for every $u \in \text{Subw}_m(v)$ and every $\ell \in [m]$, we have $\text{Subw}_\ell(u) \subseteq \text{Subw}_\ell(v)$.

1.3. Combinatorial spaces

Throughout this section let A be a finite alphabet with $|A| \geq 2$. A *combinatorial space* of $A^{<\mathbb{N}}$ is a set of the form

$$(1.14) \quad V = \{v(a_0, \dots, a_{n-1}) : a_0, \dots, a_{n-1} \in A\},$$

where n is a positive integer and v is a n -variable word over A . (Note that both n and v are unique since $|A| \geq 2$.) The positive integer n is called the *dimension* of V and is denoted by $\dim(V)$. The 1-dimensional combinatorial spaces will be called *combinatorial lines*.

Now let V be a combinatorial space of $A^{<\mathbb{N}}$ and set $n = \dim(V)$. Also let v be the (unique) n -variable word over A which generates V via formula (1.14) and notice that v induces a bijection between A^n and V . We will give this bijection a special name as follows.

DEFINITION 1.2. *Let V be a combinatorial space of $A^{<\mathbb{N}}$. Set $n = \dim(V)$ and let v be the n -variable word which generates V via formula (1.14). The canonical isomorphism associated with V is the bijection $I_V : A^n \rightarrow V$ defined by the rule*

$$(1.15) \quad I_V((a_0, \dots, a_{n-1})) = v(a_0, \dots, a_{n-1})$$

for every $(a_0, \dots, a_{n-1}) \in A^n$.

We will view an n -dimensional combinatorial space V as a ‘‘copy’’ of A^n and, using the canonical isomorphism, we will identify V with A^n for most practical purposes. This identification is very convenient and will be constantly used throughout this book.

We proceed to discuss two alternative ways to define combinatorial spaces. First, for every nonempty finite sequence $(v_i)_{i=0}^{n-1}$ of variable words over A we set

$$(1.16) \quad V = \{v_0(a_0) \wedge \dots \wedge v_{n-1}(a_{n-1}) : a_0, \dots, a_{n-1} \in A\}$$

and we call V the *combinatorial space of $A^{<\mathbb{N}}$ generated by $(v_i)_{i=0}^{n-1}$* . Observe that two different finite sequences of variable words over A might generate the same combinatorial space of $A^{<\mathbb{N}}$.

Next, let V be a combinatorial space of $A^{<\mathbb{N}}$, set $n = \dim(V)$ and let v be the n -variable word over A which generates V via formula (1.14). Recall that v is a nonempty word over the alphabet $A \cup \{x_0, \dots, x_{n-1}\}$ and write $v = (v_0, \dots, v_{m-1})$ where $m = |v|$. For every $j \in \{0, \dots, n-1\}$, we set

$$(1.17) \quad X_j = \{i \in \{0, \dots, m-1\} : v_i = x_j\}.$$

Clearly, X_0, \dots, X_{n-1} are nonempty subsets of $\{0, \dots, m-1\}$, and if $n \geq 2$, then $\max(X_i) < \min(X_{i+1})$ for every $i \in \{0, \dots, n-2\}$. The sets X_0, \dots, X_{n-1} are called the *wildcard sets* of V . We also set

$$(1.18) \quad S = \{0, \dots, m-1\} \setminus \left(\bigcup_{j=1}^{n-1} X_j \right)$$

and we call S the *set of fixed coordinates* of V . Finally, the *constant part* of V is the located word $v \upharpoonright S \in A^S$. Note that the wildcard sets, the set of fixed coordinates, and the constant part completely determine a combinatorial space.

1.3.1. Subspaces. If V and U are two combinatorial spaces of $A^{<\mathbb{N}}$, then we say that U is a *combinatorial subspace* of V if U is contained in V . For every combinatorial space V of $A^{<\mathbb{N}}$ and every $m \in [\dim(V)]$ by $\text{Subsp}_m(V)$, we shall denote the set of all m -dimensional combinatorial subspaces of V .

We will present two different representations of the set $\text{Subsp}_m(V)$ which are both straightforward consequences of the relevant definitions. The first representation relies on the canonical isomorphism I_V associated with V .

FACT 1.3. *Let V be a combinatorial space of $A^{<\mathbb{N}}$ and set $n = \dim(V)$. Then for every $m \in [n]$ the map*

$$(1.19) \quad \text{Subsp}_m(A^n) \ni R \mapsto I_V(R) \in \text{Subsp}_m(V)$$

is a bijection.

The second representation will enable us to identify combinatorial subspaces with subwords. Specifically, we have the following fact.

FACT 1.4. *Let V be a combinatorial space of $A^{<\mathbb{N}}$. Set $n = \dim(V)$ and let v be the n -variable word over A which generates V via formula (1.14). Then for every $m \in [n]$, the map*

$$(1.20) \quad \text{Subw}_m(v) \ni u \mapsto \{u(a_0, \dots, a_{m-1}) : a_0, \dots, a_{m-1} \in A\} \in \text{Subsp}_m(V)$$

is a bijection.

1.3.2. Restriction on smaller alphabets. Let V be a combinatorial space of $A^{<\mathbb{N}}$ and let I_V be the canonical isomorphism associated with V . For every $B \subseteq A$ with $|B| \geq 2$, we define the *restriction of V on B* by the rule

$$(1.21) \quad V \upharpoonright B = \{I_V(u) : u \in B^{\dim(V)}\}.$$

Notice that the map $I_V : B^{\dim(V)} \rightarrow V \upharpoonright B$ is a bijection, and so we may identify the restriction of V on B with a combinatorial space of $B^{<\mathbb{N}}$. Having this identification in mind, for every $m \in [\dim(V)]$, we set

$$(1.22) \quad \text{Subsp}_m(V \upharpoonright B) = \{I_V(X) : X \in \text{Subsp}_m(B^{\dim(V)})\}.$$

By Definition 1.2 and (1.21), we have the following fact.

FACT 1.5. *Let V be a combinatorial space of $A^{<\mathbb{N}}$ and let $m \in [\dim(V)]$. Also let $B \subseteq A$ with $|B| \geq 2$. Then for every $R \in \text{Subsp}_m(V \upharpoonright B)$ there exists a unique $U \in \text{Subsp}_m(V)$ such that $R = U \upharpoonright B$.*

In particular, we have $\text{Subsp}_m(V \upharpoonright B) \subseteq \{U \upharpoonright B : U \in \text{Subsp}_m(V)\}$.

1.4. Reduced and extracted words

We are about to introduce two classes of combinatorial objects which are generated from sequences of variable words. In what follows, let A denote a finite alphabet with at least two letters.

1.4.1. Reduced words and variable words. Let $(w_i)_{i=0}^{n-1}$ be a nonempty finite sequence of variable words over A .

A *reduced word*¹ of $(w_i)_{i=0}^{n-1}$ is a word w over A of the form

$$(1.23) \quad w = w_0(a_0) \frown \dots \frown w_{n-1}(a_{n-1}),$$

where (a_0, \dots, a_{n-1}) is a word over A . The set of all reduced words of $(w_i)_{i=0}^{n-1}$ will be denoted by $[(w_i)_{i=0}^{n-1}]$. Observe that $[(w_i)_{i=0}^{n-1}]$ coincides with the combinatorial space of $A^{<\mathbb{N}}$ generated by $(w_i)_{i=0}^{n-1}$.

A *reduced variable word* of $(w_i)_{i=0}^{n-1}$ is a variable word v over A of the form

$$(1.24) \quad v = w_0(\alpha_0) \frown \dots \frown w_{n-1}(\alpha_{n-1}),$$

where $(\alpha_0, \dots, \alpha_{n-1})$ is a variable word over A . (Notice, in particular, that there exists $i \in \{0, \dots, n-1\}$ such that $\alpha_i = x$.) The set of all reduced variable words of $(w_i)_{i=0}^{n-1}$ will be denoted by $V[(w_i)_{i=0}^{n-1}]$.

More generally, a finite sequence $(v_i)_{i=0}^{m-1}$ of variable words over A is said to be a *reduced subsequence* of $(w_i)_{i=0}^{n-1}$ if $m \in [n]$ and there exist a strictly increasing sequence $(n_i)_{i=0}^m$ in \mathbb{N} with $n_0 = 0$ and $n_m = n$, and a sequence $(\alpha_j)_{j=0}^{m-1}$ in $A \cup \{x\}$ such that for every $i \in \{0, \dots, m-1\}$ we have $x \in \{\alpha_j : n_i \leq j \leq n_{i+1} - 1\}$ and

$$(1.25) \quad v_i = w_{n_i}(\alpha_{n_i}) \frown \dots \frown w_{n_{i+1}-1}(\alpha_{n_{i+1}-1}).$$

For every $m \in [n]$ by $V_m[(w_i)_{i=0}^{n-1}]$, we shall denote the set of all reduced subsequences of $(w_i)_{i=0}^{n-1}$ of length m . Note that $V_1[(w_i)_{i=0}^{n-1}] = V[(w_i)_{i=0}^{n-1}]$.

The above notions can be extended to infinite sequences of variable words. Specifically, let $\mathbf{w} = (w_i)$ be a sequence of variable words over A . For every positive integer n let $\mathbf{w} \upharpoonright n = (w_i)_{i=0}^{n-1}$ and set

$$(1.26) \quad [\mathbf{w}] = \bigcup_{n=1}^{\infty} [\mathbf{w} \upharpoonright n] \quad \text{and} \quad V[\mathbf{w}] = \bigcup_{n=1}^{\infty} V[\mathbf{w} \upharpoonright n].$$

An element of $[\mathbf{w}]$ will be called a *reduced word* of \mathbf{w} while an element of $V[\mathbf{w}]$ will be called a *reduced variable word* of \mathbf{w} . Moreover, for every positive integer m we define the set of all *reduced subsequences* of \mathbf{w} of length m by the rule

$$(1.27) \quad V_m[\mathbf{w}] = \bigcup_{n=m}^{\infty} V_m[\mathbf{w} \upharpoonright n].$$

Finally, we say that an infinite sequence $\mathbf{v} = (v_i)$ of variable words over A is a *reduced subsequence* of \mathbf{w} if for every integer $m \geq 1$ we have $\mathbf{v} \upharpoonright m \in V_m[\mathbf{w}]$. The set of all reduced subsequences of \mathbf{w} of infinite length will be denoted by $V_{\infty}[\mathbf{w}]$.

We proceed to discuss some basic properties of reduced words and variable words. We first observe that if $(w_i)_{i=0}^{n-1}$ is a finite sequence of variable words over A , then every reduced subsequence of $(w_i)_{i=0}^{n-1}$ corresponds to a combinatorial subspace $[(w_i)_{i=0}^{n-1}]$. More precisely, we have the following fact.

¹This terminology is, of course, group theoretic. The reader should have in mind though that it has a somewhat different meaning in the present combinatorial context.

FACT 1.6. *Let $(w_i)_{i=0}^{n-1}$ be a nonempty finite sequence of variable words over A and set $W = [(w_i)_{i=0}^{n-1}]$. Then for every $m \in [n]$ the map*

$$(1.28) \quad V_m[(w_i)_{i=0}^{n-1}] \ni (v_i)_{i=0}^{m-1} \mapsto [(v_i)_{i=0}^{m-1}] \in \text{Subsp}_m(W)$$

is onto. Moreover, this map is a bijection between $V_1[(w_i)_{i=0}^{n-1}]$ and $\text{Subsp}_1(W)$.

We also have the following coherence properties.

FACT 1.7. *Let \mathbf{v}, \mathbf{w} be two nonempty sequences (of finite or infinite length) of variable words over A , and assume that \mathbf{v} is a reduced subsequence of \mathbf{w} . Then we have $[\mathbf{v}] \subseteq [\mathbf{w}]$, $V[\mathbf{v}] \subseteq V[\mathbf{w}]$, and $V_m[\mathbf{v}] \subseteq V_m[\mathbf{w}]$ for every positive integer m which is less than or equal to the length of \mathbf{v} . Moreover, if both \mathbf{v} and \mathbf{w} are infinite sequences, then we have $V_\infty[\mathbf{v}] \subseteq V_\infty[\mathbf{w}]$.*

1.4.2. Extracted words and variable words. As in the previous subsection, let $(w_i)_{i=0}^{n-1}$ be a nonempty finite sequence of variable words over A .

An *extracted word* of $(w_i)_{i=0}^{n-1}$ is a reduced word of a subsequence of $(w_i)_{i=0}^{n-1}$ while an *extracted variable word* of $(w_i)_{i=0}^{n-1}$ is a reduced variable word of a subsequence of $(w_i)_{i=0}^{n-1}$. (Thus, an extracted variable word of $(w_i)_{i=0}^{n-1}$ is of the form $w_{i_0}(\alpha_0) \wedge \dots \wedge w_{i_\ell}(\alpha_\ell)$, where $\ell \in \mathbb{N}$, $0 \leq i_0 < \dots < i_\ell \leq n-1$, and $(\alpha_0, \dots, \alpha_\ell)$ is a variable word over A .) An *extracted subsequence* of $(w_i)_{i=0}^{n-1}$ is a reduced subsequence of a subsequence of $(w_i)_{i=0}^{n-1}$. By $E[(w_i)_{i=0}^{n-1}]$ and $EV[(w_i)_{i=0}^{n-1}]$ we shall denote the sets of all extracted words and all extracted variable words of $(w_i)_{i=0}^{n-1}$ respectively. Moreover, for every $m \in [n]$ the set of all extracted subsequences of $(w_i)_{i=0}^{n-1}$ of length m will be denoted by $EV_m[(w_i)_{i=0}^{n-1}]$.

Next, let $\mathbf{w} = (w_i)$ be an infinite sequence of variable words over A . We set

$$(1.29) \quad E[\mathbf{w}] = \bigcup_{n=1}^{\infty} E[\mathbf{w} \upharpoonright n] \quad \text{and} \quad EV[\mathbf{w}] = \bigcup_{n=1}^{\infty} EV[\mathbf{w} \upharpoonright n]$$

and for every positive integer m let

$$(1.30) \quad EV_m[\mathbf{w}] = \bigcup_{n=m}^{\infty} EV_m[\mathbf{w} \upharpoonright n].$$

On the other hand, by $EV_\infty[\mathbf{w}]$ we shall denote the set of all infinite extracted subsequences of \mathbf{w} , that is, the set of all (infinite) sequences of variable words over A which are reduced subsequences of a subsequence of \mathbf{w} .

We close this section with the following analogue of Fact 1.7.

FACT 1.8. *Let \mathbf{v}, \mathbf{w} be two nonempty sequences (of finite or infinite length) of variable words over A and assume that \mathbf{v} is an extracted subsequence of \mathbf{w} . Then we have $E[\mathbf{v}] \subseteq E[\mathbf{w}]$, $EV[\mathbf{v}] \subseteq EV[\mathbf{w}]$, and $EV_m[\mathbf{v}] \subseteq EV_m[\mathbf{w}]$ for every positive integer m which is less than or equal to the length of \mathbf{v} . Moreover, if both \mathbf{v} and \mathbf{w} are infinite sequences, then we have $EV_\infty[\mathbf{v}] \subseteq EV_\infty[\mathbf{w}]$.*

1.5. Carlson–Simpson spaces

Let A be a finite alphabet with $|A| \geq 2$. This alphabet will be fixed throughout this section. A *finite-dimensional Carlson–Simpson system* over A is a pair $\langle t, (w_i)_{i=0}^{d-1} \rangle$ where t is a word over A and $(w_i)_{i=0}^{d-1}$ a nonempty finite sequence of left variable words over A . The length d of the finite sequence $(w_i)_{i=0}^{d-1}$ will be called the *dimension* of the system.

A *finite-dimensional Carlson–Simpson space* of $A^{<\mathbb{N}}$ is a set of the form

$$(1.31) \quad W = \{t\} \cup \{t \wedge w_0(a_0) \wedge \dots \wedge w_{m-1}(a_{m-1}) : m \in [d] \text{ and } a_0, \dots, a_{m-1} \in A\},$$

where $\langle t, (w_i)_{i=0}^{d-1} \rangle$ is a finite-dimensional Carlson–Simpson system over A . Note that the system $\langle t, (w_i)_{i=0}^{d-1} \rangle$ which generates W via formula (1.31) is unique; it will be called the *generating system* of W . The *dimension* of W , denoted by $\dim(W)$, is the dimension of its generating system (that is, the length of the finite sequence $(w_i)_{i=0}^{d-1}$). The 1-dimensional Carlson–Simpson spaces will be called *Carlson–Simpson lines*.

Let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$, set $d = \dim(W)$, and let $\langle t, (w_i)_{i=0}^{d-1} \rangle$ be its generating system. For every $m \in \{0, \dots, d\}$ we define the *m -level* $W(m)$ of W by setting $W(0) = \{t\}$ and

$$(1.32) \quad W(m) = \{t \wedge w_0(a_0) \wedge \dots \wedge w_{m-1}(a_{m-1}) : a_0, \dots, a_{m-1} \in A\}$$

if $m \in [d]$. Observe that $W = W(0) \cup \dots \cup W(d)$ and notice that for every $m \in [d]$ the m -level $W(m)$ of W is an m -dimensional combinatorial subspace of A^{n_m} where $n_m = |t| + \sum_{i=0}^{m-1} |w_i|$. The *level set* of W , denoted by $L(W)$, is defined by

$$(1.33) \quad L(W) = \{n \in \mathbb{N} : W(m) \subseteq A^n \text{ for some } m \in \{0, \dots, d\}\}.$$

Equivalently, we have $L(W) = \{|t|\} \cup \{|t| + \sum_{i=0}^{m-1} |w_i| : m \in [d]\}$.

1.5.1. Subsystems and subspaces. Let d, m be positive integers, and let $\mathbf{w} = \langle t, (w_i)_{i=0}^{d-1} \rangle$ and $\mathbf{u} = \langle s, (v_i)_{i=0}^{m-1} \rangle$ be two Carlson–Simpson systems over A of dimensions d and m , respectively. We say that \mathbf{u} is a *subsystem* of \mathbf{w} if $m \leq d$ and there exist a strictly increasing sequence $(n_i)_{i=0}^m$ in $\{0, \dots, d\}$ and a sequence $(a_j)_{j=0}^{n_m-1}$ in $A \cup \{x\}$ such that the following conditions are satisfied.

(C1) If $n_0 = 0$, then $s = t$. Otherwise, we have $a_0, \dots, a_{n_0-1} \in A$ and

$$s = t \wedge w_0(a_0) \wedge \dots \wedge w_{n_0-1}(a_{n_0-1}).$$

(C2) For every $i \in \{0, \dots, m-1\}$ we have $a_{n_i} = x$ and

$$v_i = w_{n_i}(a_{n_i}) \wedge \dots \wedge w_{n_{i+1}-1}(a_{n_{i+1}-1}).$$

The set of all m -dimensional subsystems of \mathbf{w} will be denoted by $\text{Subsys}_m(\mathbf{w})$.

On the other hand, if W and U are two finite-dimensional Carlson–Simpson spaces of $A^{<\mathbb{N}}$, then we say that U is a (*Carlson–Simpson*) *subspace* of W if U is contained in W . (This implies, in particular, that $\dim(U) \leq \dim(W)$.) For every $m \in [\dim(W)]$ by $\text{SubCS}_m(W)$ we shall denote the set of all m -dimensional Carlson–Simpson subspaces of W . Notice that, setting

$$(1.34) \quad W \upharpoonright m+1 = W(0) \cup \dots \cup W(m),$$

we have $W \upharpoonright m+1 \in \text{SubCS}_m(W)$.

There is a natural correspondence between subsystems and subspaces. Indeed, let W and U be two finite-dimensional Carlson–Simpson spaces of $A^{<\mathbb{N}}$ generated by the systems \mathbf{w} and \mathbf{u} , respectively, and observe that U is a subspace of W if and only if \mathbf{u} is a subsystem of \mathbf{w} . More precisely, we have the following fact.

FACT 1.9. *Let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$, and let \mathbf{w} be its generating system. For every Carlson–Simpson subspace U of W let \mathbf{w}_U be its generating system. Then for every $m \in [\dim(W)]$, the map*

$$(1.35) \quad \text{SubCS}_m(W) \ni U \mapsto \mathbf{w}_U \in \text{Subsys}_m(\mathbf{w})$$

is a bijection.

1.5.2. Canonical isomorphisms. Let d be a positive integer and note that the archetypical example of a d -dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$ is the set $A^{<d+1}$ of all finite sequences in A of length less than or equal to d . In fact, every d -dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$ can be viewed as a “copy” of $A^{<d+1}$. The philosophy is identical to that in Section 1.3.

DEFINITION 1.10. *Let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$, set $d = \dim(W)$, and let $\langle t, (w_i)_{i=0}^{d-1} \rangle$ be its generating system. The canonical isomorphism associated with W is the bijection $I_W : A^{<d+1} \rightarrow W$ defined by setting $I_W(\emptyset) = t$ and*

$$(1.36) \quad I_W((a_0, \dots, a_{m-1})) = t \wedge w_0(a_0) \wedge \dots \wedge w_{m-1}(a_{m-1})$$

for every $m \in [d]$ and every $(a_0, \dots, a_{m-1}) \in A^m$.

The canonical isomorphism preserves all structural properties one is interested in while working in the category of Carlson–Simpson spaces. In particular, we have the following analogue of Fact 1.3.

FACT 1.11. *Let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$ and set $d = \dim(W)$. Then for every $m \in [d]$ the map*

$$(1.37) \quad \text{SubCS}_m(A^{<d+1}) \ni R \mapsto I_W(R) \in \text{SubCS}_m(W)$$

is a bijection.

Another basic property of canonical isomorphisms is that they preserve infima.

FACT 1.12. *Let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$. Then for every nonempty subset F of $A^{<\dim(W)+1}$ we have $I_W(\wedge F) = \wedge I_W(F)$.*

We continue by presenting a method to produce Carlson–Simpson spaces from combinatorial spaces. The method is based on canonical isomorphisms. Specifically, let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$ and let $m \in [\dim(W)]$. Recall that for every $U \in \text{SubCS}_m(W)$ the m -level $U(m)$ of U is an m -dimensional combinatorial space of $A^{<\mathbb{N}}$. We set

$$(1.38) \quad \text{SubCS}_m^{\max}(W) = \{U \in \text{SubCS}_m(W) : U(m) \subseteq W(d)\}.$$

That is, $\text{SubCS}_m^{\max}(W)$ is the set of all m -dimensional Carlson–Simpson subspaces of W whose last level is contained in the last level of W .

LEMMA 1.13. *Let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$, and set $d = \dim(W)$. Then for every $m \in [d]$, the map*

$$(1.39) \quad \text{SubCS}_m^{\max}(W) \ni U \mapsto U(m) \in \text{Subsp}_m(W(d))$$

is a bijection.

PROOF. Notice that every $V \in \text{Subsp}_m(A^d)$ is of the form $V = R(m)$ for some unique $R \in \text{SubCS}_m^{\max}(A^{<d+1})$. By Fact 1.11 and taking into account this remark, the result follows. \square

1.5.3. Restriction on smaller alphabets. Let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$ and let I_W be the canonical isomorphism associated

with W . As in Subsection 1.3.2, for every $B \subseteq A$ with $|B| \geq 2$, we define the *restriction of W on B* by the rule

$$(1.40) \quad W \upharpoonright B = \{I_W(u) : u \in B^{<\dim(W)+1}\},$$

and for every $m \in [\dim(W)]$ we set

$$(1.41) \quad \text{SubCS}_m(W \upharpoonright B) = \{I_W(X) : X \in \text{SubCS}_m(B^{<\dim(W)+1})\}.$$

Observe that the map $I_W : B^{<\dim(W)+1} \rightarrow W \upharpoonright B$ is a bijection. Moreover, we have the following fact.

FACT 1.14. *Let W be a finite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$. Also let $m \in [\dim(W)]$ and $B \subseteq A$ with $|B| \geq 2$. Then for every $R \in \text{SubCS}_m(W \upharpoonright B)$ there exists a unique $U \in \text{SubCS}_m(W)$ such that $R = U \upharpoonright B$.*

In particular, we have $\text{SubCS}_m(W \upharpoonright B) \subseteq \{U \upharpoonright B : U \in \text{SubCS}_m(W)\}$.

1.5.4. Infinite-dimensional Carlson–Simpson spaces. We proceed to discuss the infinite versions of the concepts introduced in this section so far.

An *infinite-dimensional Carlson–Simpson system* over A is a pair $\langle t, (w_i) \rangle$ where t is a word over A and (w_i) is a sequence of left variable words over A . On the other hand, an *infinite-dimensional Carlson–Simpson space* of $A^{<\mathbb{N}}$ is a set of the form

$$(1.42) \quad W = \{t\} \cup \{t \wedge w_0(a_0) \wedge \dots \wedge w_m(a_m) : m \in \mathbb{N} \text{ and } a_0, \dots, a_m \in A\},$$

where $\langle t, (w_i) \rangle$ is an infinite-dimensional Carlson–Simpson system over A . This system is clearly unique and will also be called the *generating system* of W . Respectively, for every $m \in \mathbb{N}$ the *m -level $W(m)$* of W is defined by setting $W(0) = t$ and $W(m) = \{t \wedge w_0(a_0) \wedge \dots \wedge w_{m-1}(a_{m-1}) : a_0, \dots, a_{m-1} \in A\}$ if $m \geq 1$, while the *level set* of W is defined by

$$(1.43) \quad L(W) = \{n \in \mathbb{N} : W(m) \subseteq A^n \text{ for some } m \in \mathbb{N}\}.$$

We have the following analogue of Definition 1.10.

DEFINITION 1.15. *Let W be an infinite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$ and let $\langle t, (w_i) \rangle$ be its generating system. The canonical isomorphism associated with W is the bijection $I_W : A^{<\mathbb{N}} \rightarrow W$ defined by setting $I_W(\emptyset) = t$ and*

$$(1.44) \quad I_W((a_0, \dots, a_m)) = t \wedge w_0(a_0) \wedge \dots \wedge w_m(a_m)$$

for every $m \in \mathbb{N}$ and every $a_0, \dots, a_m \in A$.

Now let W be an infinite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$. A *Carlson–Simpson subspace* of W is a finite- or infinite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$ which is contained in W . For every positive integer m by $\text{SubCS}_m(W)$ we shall denote the set of all m -dimensional Carlson–Simpson subspaces of W while $\text{SubCS}_\infty(W)$ stands for the set of all infinite-dimensional Carlson–Simpson subspaces of W . We close this section with the following analogue of Fact 1.11.

FACT 1.16. *Let W be an infinite-dimensional Carlson–Simpson space of $A^{<\mathbb{N}}$. Also let m be a positive integer. Then the maps*

$$(1.45) \quad \text{SubCS}_m(A^{<\mathbb{N}}) \ni R \mapsto I_W(R) \in \text{SubCS}_m(W)$$

and

$$(1.46) \quad \text{SubCS}_\infty(A^{<\mathbb{N}}) \ni R \mapsto I_W(R) \in \text{SubCS}_\infty(W)$$

are both bijections.

1.6. Trees

By the term *tree* we mean a (possibly empty) partially ordered set $(T, <_T)$ such that the set $\{s \in T : s <_T t\}$ is finite and linearly ordered under $<_T$ for every $t \in T$. The cardinality of this set is defined to be the *length of t in T* and will be denoted by $|t|_T$. For every $n \in \mathbb{N}$ the *n -level of T* , denoted by $T(n)$, is defined to be the set

$$(1.47) \quad T(n) = \{t \in T : |t|_T = n\}.$$

The *height of T* , denoted by $h(T)$, is defined as follows. First, we set $h(T) = 0$ if T is empty. If T is nonempty and there exists $n \in \mathbb{N}$ such that $T(n) = \emptyset$, then we set

$$h(T) = \max\{n \in \mathbb{N} : T(n) \neq \emptyset\} + 1;$$

otherwise, we set $h(T) = \infty$. Notice that the height of a nonempty finite tree T is the cardinality of the set of all nonempty levels of T .

An element of a tree T is called a *node* of T . For every node t of T by $\text{Succ}_T(t)$ we shall denote the set of all *successors of t in T* , that is,

$$(1.48) \quad \text{Succ}_T(t) = \{s \in T : t = s \text{ or } t <_T s\}.$$

The set of *immediate successors of t in T* is the subset of $\text{Succ}_T(t)$ defined by

$$(1.49) \quad \text{ImmSucc}_T(t) = \{s \in T : t <_T s \text{ and } |s|_T = |t|_T + 1\}.$$

A node t of T is called *maximal* if $\text{ImmSucc}_T(t)$ is empty.

A nonempty tree T is said to be *finitely branching* (respectively, *pruned*) if for every $t \in T$ the set of immediate successors of t in T is finite (respectively, nonempty). It is said to be *rooted* if $T(0)$ is a singleton; in this case, the unique node of $T(0)$ is called the *root of T* .

A *chain* of a tree T is a subset of T which is linearly ordered under $<_T$. A maximal (with respect to inclusion) chain of T is called a *branch* of T . The tree T is said to be *balanced* if all branches of T have the same cardinality. Note that a tree of infinite height is balanced if and only if it is pruned.

Now let T be a tree and let D be a subset of T . The *level set of D in T* , denoted by $L_T(D)$, is defined to be the set

$$(1.50) \quad L_T(D) = \{n \in \mathbb{N} : D \cap T(n) \neq \emptyset\}.$$

Moreover, for every $n \in \mathbb{N}$ let

$$(1.51) \quad D \upharpoonright n = \bigcup_{\{m \in \mathbb{N} : m < n\}} D \cap T(m).$$

(In particular, $D \upharpoonright 0$ is the empty tree.) Note that if $D = T$ and $n \geq 1$, then

$$(1.52) \quad T \upharpoonright n = T(0) \cup \dots \cup T(n-1).$$

More generally, for every $M \subseteq \mathbb{N}$ we set

$$(1.53) \quad D \upharpoonright M = \bigcup_{m \in M} D \cap T(m).$$

Finally, if D is finite, then we define the *depth of D in T* , denoted by $\text{depth}_T(D)$, to be the least $n \in \mathbb{N}$ such that $D \subseteq T \upharpoonright n$. Observe that for every nonempty finite subset D of T we have $\text{depth}_T(D) = \max(L_T(D)) + 1$.

1.6.1. Strong subtrees. A *subtree* of a tree $(T, <_T)$ is a subset of T viewed as a tree equipped with the induced partial ordering. An *initial subtree* of T is a subtree of T of the form $T \upharpoonright n$ for some $n \in \mathbb{N}$ with $n < h(T)$. The following class of subtrees is of particular importance in the context of Ramsey theory.

DEFINITION 1.17. *A subtree S of a tree T is said to be strong if either S is empty, or S is nonempty and satisfies the following conditions.*

- (a) *The tree S is rooted and balanced.*
- (b) *Every level of S is a subset of some level of T , that is, for every $n \in \mathbb{N}$ with $n < h(S)$ there exists $m \in \mathbb{N}$ such that $S(n) \subseteq T(m)$.*
- (c) *For every nonmaximal node $s \in S$ and every $t \in \text{ImmSucc}_T(s)$, the set $\text{ImmSucc}_S(s) \cap \text{Succ}_T(t)$ is a singleton.*

For every $k \in \mathbb{N}$ with $k < h(T)$ by $\text{Str}_k(T)$ we shall denote the set of all strong subtrees of T of height k . Notice that $\text{Str}_0(T)$ contains only the empty tree; on the other hand, we have $\text{Str}_1(T) = \{\{t\} : t \in T\}$ and so we may identify the set $\text{Str}_1(T)$ with T . If T has infinite height, then by $\text{Str}_{<\infty}(T)$ and $\text{Str}_\infty(T)$ we shall denote the set of all strong subtrees of T of finite and infinite height, respectively.

We isolate below some elementary (though basic) properties of strong subtrees.

FACT 1.18. *Let T be a tree and let S be a strong subtree of T .*

- (a) *Every strong subtree of S is also a strong subtree of T .*
- (b) *If T is pruned and $S \in \text{Str}_k(T)$ for some $k \in \mathbb{N}$, then there is $R \in \text{Str}_\infty(T)$ (not necessarily unique) such that $S = R \upharpoonright k$.*

1.6.2. Homogeneous trees. Let $b \in \mathbb{N}$ with $b \geq 2$ and recall that $[b]^{<\mathbb{N}}$ stands for the set of all finite sequences having values in $[b]$. We view $[b]^{<\mathbb{N}}$ as a tree equipped with the (strict) partial order \sqsubset of end-extension. In particular, for every positive integer n the set $[b]^{<n}$ is the initial subtree of $[b]^{<\mathbb{N}}$ of height $n + 1$.

A *homogeneous tree* is a nonempty strong subtree T of $[b]^{<\mathbb{N}}$ for some $b \in \mathbb{N}$ with $b \geq 2$. The (unique) integer b is called the *branching number* of T and is denoted by b_T . Note that a homogeneous tree T of height k is just a ‘‘copy’’ of $[b_T]^{<k}$ inside $[b_T]^{<\mathbb{N}}$. More precisely, we have the following definition.

DEFINITION 1.19. *Let T be a homogeneous tree of finite height. The canonical isomorphism associated with T is the unique bijection $I_T: [b_T]^{<h(T)} \rightarrow T$ such that for every $t, t' \in [b_T]^{<h(T)}$ we have*

- (P1) $|t| = |t'|$ if and only if $|I_T(t)|_T = |I_T(t')|_T$,
- (P2) $t \sqsubset t'$ if and only if $I_T(t) \sqsubset I_T(t')$, and
- (P3) $t <_{\text{lex}} t'$ if and only if $I_T(t) <_{\text{lex}} I_T(t')$.

Respectively, the canonical isomorphism associated with a homogeneous tree T of infinite height is the unique bijection $I_T: [b_T]^{<\mathbb{N}} \rightarrow T$ satisfying (P1), (P2), and (P3) for every $t, t' \in [b_T]^{<\mathbb{N}}$.

1.6.3. Vector trees. A *vector tree* is a finite sequence $\mathbf{T} = (T_1, \dots, T_d)$ of trees having common height. This common height is defined to be the *height* of \mathbf{T} and will be denoted by $h(\mathbf{T})$. A vector tree $\mathbf{T} = (T_1, \dots, T_d)$ is said to be *finitely branching* (respectively, *pruned*, *rooted*, *balanced*) if for every $i \in [d]$ the tree T_i is finitely branching (respectively, pruned, rooted, balanced).

If $\mathbf{T} = (T_1, \dots, T_d)$ is a vector tree, then a *vector subset* of \mathbf{T} is a finite sequence $\mathbf{D} = (D_1, \dots, D_d)$ where $D_i \subseteq T_i$ for every $i \in [d]$. As in (1.51), for every $n \in \mathbb{N}$ let

$$(1.54) \quad \mathbf{D} \upharpoonright n = (D_1 \upharpoonright n, \dots, D_d \upharpoonright n)$$

and, more generally, for every $M \subseteq \mathbb{N}$ let

$$(1.55) \quad \mathbf{D} \upharpoonright M = (D_1 \upharpoonright M, \dots, D_d \upharpoonright M).$$

In particular, if $\mathbf{D} = \mathbf{T}$ and $n \geq 1$, then we have $\mathbf{T} \upharpoonright n = (T_1 \upharpoonright n, \dots, T_d \upharpoonright n)$.

Now let $\mathbf{D} = (D_1, \dots, D_d)$ be a vector subset of $\mathbf{T} = (T_1, \dots, T_d)$. If D_i is finite for every $i \in [d]$, then the *depth of \mathbf{D} in \mathbf{T}* , denoted by $\text{depth}_{\mathbf{T}}(\mathbf{D})$, is defined by

$$(1.56) \quad \text{depth}_{\mathbf{T}}(\mathbf{D}) = \min\{n \in \mathbb{N} : \mathbf{D} \text{ is a vector subset of } \mathbf{T} \upharpoonright n\}.$$

On the other hand, we say that \mathbf{D} is *level compatible* if there exists $L \subseteq \mathbb{N}$ such that $L_{T_i}(D_i) = L$ for every $i \in [d]$. The (unique) set L will be denoted by $L_{\mathbf{T}}(\mathbf{D})$ and will be called the *level set of \mathbf{D} in \mathbf{T}* . Moreover, for every $n \in L_{\mathbf{T}}(\mathbf{D})$, we set

$$(1.57) \quad \otimes \mathbf{D}(n) = (D_1 \cap T_1(n)) \times \cdots \times (D_d \cap T_d(n))$$

and we define the *level product* of \mathbf{D} by the rule

$$(1.58) \quad \otimes \mathbf{D} = \bigcup_{n \in L_{\mathbf{T}}(\mathbf{D})} \otimes \mathbf{D}(n).$$

In particular, for every $n \in \mathbb{N}$ with $n < h(\mathbf{T})$ we have

$$\otimes \mathbf{T}(n) = T_1(n) \times \cdots \times T_d(n)$$

and

$$\otimes \mathbf{T} = \bigcup_{n < h(\mathbf{T})} \otimes \mathbf{T}(n).$$

Finally, for every $\mathbf{t} = (t_1, \dots, t_d) \in \otimes \mathbf{T}$ by $|\mathbf{t}|_{\mathbf{T}}$ we shall denote the unique natural number n such that $\mathbf{t} \in \otimes \mathbf{T}(n)$.

1.6.4. Vector strong subtrees. The concept of a strong subtree is naturally extended to vector trees. Specifically, we have the following definition.

DEFINITION 1.20. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a vector tree. A vector strong subtree of \mathbf{T} is a vector subset $\mathbf{S} = (S_1, \dots, S_d)$ of \mathbf{T} which is level compatible (that is, there exists $L \subseteq \mathbb{N}$ with $L_{T_i}(S_i) = L$ for every $i \in [d]$) and such that S_i is a strong subtree of T_i for every $i \in [d]$.*

Notice that every vector strong subtree \mathbf{S} of a vector tree \mathbf{T} is a vector tree on its own, and observe that its height $h(\mathbf{S})$ coincides with the common height of S_1, \dots, S_d . For every $k \in \mathbb{N}$ with $k < h(\mathbf{T})$ by $\text{Str}_k(\mathbf{T})$ we shall denote the set of all vector strong subtrees of \mathbf{T} of height k . If, in addition, \mathbf{T} is of infinite height, then by $\text{Str}_{<\infty}(\mathbf{T})$ and $\text{Str}_{\infty}(\mathbf{T})$ we shall denote the set of all vector strong subtrees of \mathbf{T} of finite and infinite height, respectively.

We close this subsection with the following analogue of Fact 1.18.

FACT 1.21. *Let \mathbf{T} be a vector tree and let \mathbf{S} be a vector strong subtree of \mathbf{T} .*

- (a) *Every vector strong subtree of \mathbf{S} is also a vector strong subtree of \mathbf{T} .*
- (b) *If \mathbf{T} is pruned and $\mathbf{S} \in \text{Str}_k(\mathbf{T})$ for some $k \in \mathbb{N}$, then there is $\mathbf{R} \in \text{Str}_{\infty}(\mathbf{T})$ (not necessarily unique) such that $\mathbf{S} = \mathbf{R} \upharpoonright k$.*

1.6.5. Vector homogeneous trees. A *vector homogeneous tree* is a vector tree $\mathbf{T} = (T_1, \dots, T_d)$ such that T_i is homogeneous for every $i \in [d]$. Observe that a vector homogeneous tree $\mathbf{T} = (T_1, \dots, T_d)$ is a vector strong subtree of $([b_{T_1}]^{<\mathbb{N}}, \dots, [b_{T_d}]^{<\mathbb{N}})$ with $h(\mathbf{T}) \geq 1$.

1.7. Notes and remarks

1.7.1. The notion of a combinatorial line originates from the classical paper of Hales and Jewett [HJ]. On the other hand, the concepts of a reduced and an extracted word appeared first in the work of Carlson [C]. Carlson–Simpson spaces were introduced in [CS]; however, our exposition follows later presentations (see, e.g., [DKT3, McC1]).

1.7.2. There are several (essentially) equivalent ways to define trees. We followed the set theoretic approach mainly for historical reasons (see, in particular, the discussion in Section 3.4). We also note that the notion of a strong subtree was introduced by Laver in the late 1960s (see also [M2, M3]).