Preface

Generalized numbers are a multiplicative structure introduced by A. Beurling [**Be37**] in 1937 to investigate the degree to which prime number theory is independent of the additive properties of the natural numbers.

Beyond their own interest, the results and techniques of this theory apply to several other systems having the character of prime numbers and integers. Indeed, such ideas occurred already in a 1903 paper of E. Landau [**La03**] proving the prime number theorem for ideals of algebraic number fields. We shall introduce and use continuous (!) analogues of generalized (briefly: g-) numbers. As another application, these distributions provide an attractive path to the theories of Dickman and Buchstab for integers whose prime factors lie only in restricted ranges.

A central question that we shall examine is the following: if a sequence of g-integers is generated by a sequence of g-primes, and if one of the collections is "reasonably near" its classical counterpart, does the other collection also have this property? This monograph does not examine all facets of g-number theory; some interesting topics that are largely ignored include probabilistic theory, oscillatory counting functions, and collections of primes and integers that are unusually dense or sparse. We hope that the accompanying list of references will help interested readers to explore these topics further.

Our intended audience is readers having some familiarity with mathematical analysis and analytic number theory, particularly an analytic proof of the prime number theorem. Background material that we assume can be found in such books as those of Apostol [**Ap76**], Bateman-Diamond [**BD04**], Chandrasekharan [**Ch68**], [**Ch70**], Davenport [**Da00**], Ingham [**In32**], Montgomery-Vaughan [**MV07**], or Tenenbaum [**Te95**]. Specialized results will be developed as needed.

Many examples are provided to illustrate how various hypotheses affect the behavior of g-number systems. They are important! But readers put off by details are encouraged to at least note the point of each example.

This work contains published and new work of the authors. Also, we have benefited from the contributions of many others, and it is our pleasant duty to thank them here. These include Beurling, who originated the study and established its first important result; P. J. Cohen, who introduced the first author to this subject; P. T. Bateman and H. L. Montgomery, with whom we have worked in this area; and J.-P. Kahane, who established one of our main results. Also, we thank A. J. Hildebrand for his mathematical and TEXnical advice.

The authors request that readers advise us of errors or obscurities they find. Our email address is `beurlingbook@illinois.edu`. We maintain corrections and comments at `www.math.illinois.edu/~hdiamond/hgdwbz/corrigenda.pdf`.

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