Preface

This book presents a fairly comprehensive account of the modern theory of quasiconformal mappings in Euclidean n-space for $n \geq 2$, starting from the elementary theory of conformal mappings and building towards the more general aspects by carefully developing the necessary analytic and geometric tools. This book is primarily aimed at graduate students and researchers who seek to understand quasiconformal mappings, particularly in three or more dimensions, perhaps after having seen applications of the two-dimensional theory in Teichmüller spaces of Riemann surfaces, or in conformal dynamical systems and elsewhere. However, as we carefully develop most of the necessary analytic theory only a basic background course in multi-dimensional real analysis is assumed.

The theory of quasiconformal mappings seeks to generalise the remarkable geometric and analytic theory of conformal mappings in the plane to higher dimensions. This is since Liouville's rigidity theorem implies an extreme paucity—a finite dimensional family—of conformal mappings defined on domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$. Of course in two dimensions the conformal mappings of a domain form an infinite-dimensional family and one has the Riemann mapping theorem. The reasons for seeking this generalisation are manifold with wide application. For instance in the theory of partial differential equations, quasiconformal mappings preserve the ellipticity of second order equations of divergence type—those with the widest application in physics—so the solution to mapping problems enables the transfer of equations from one domain to another, potentially nicer, domain where a solution might be found.

In higher dimensions few manifolds admit a conformal structure, yet D. Sullivan has shown that every topological manifold admits a quasiconformal structure, that is, a covering with quasiconformal local coordinate charts. This presents the opportunity to compute analytic invariants on a topological manifold or to compute topological invariants analytically—for instance in the work of A. Connes, D. Sullivan, and N. Teleman. Unfortunately we will only touch on these deep applications in this work. Nevertheless the reader will find—for the first time in book form—a solid foundation to explore these remarkable results and applications.

We approach the theory of quasiconformal mappings from the geometric point of view, using conformal invariants such as the moduli of curve families and capacities. These ideas are of independent interest and again of wide utility in many areas of mathematics, and so we give a fairly thorough account of them.

We begin by developing the basics of the theory—including the study of conformal mappings in space, elementary aspects of higher-dimensional hyperbolic geometry and its isometries, along with the associated matrix groups. This leads quickly to the celebrated rigidity theorem of Liouville for smooth mappings, the proof for

which follows an argument of Nevanlinna. To get Liouville's theorem in complete generality, more theory—in particular the theory of conformal modulus—is developed. The geometric aspects of the theory of quasiconformal mappings rely to a great deal on understanding and estimating these conformal invariants. Indeed the very definition of a quasiconformal mapping here is via the distortion of moduli by a multiplicative factor.

We then consider deeper properties of conformal modulus such as symmetrisation, continuity, the structure of sets of capacity zero and the existence and uniqueness of extremal functions. These give us powerful tools to study quasiconformal mappings which enable us to not only establish analytic properties, but also to develop the compactness and normal family properties of sequences of quasiconformal mappings.

We then turn our attention to the mapping problem in its various forms, basically seeking a higher-dimensional version of the Riemann mapping theorem for the class of quasiconformal mappings. We present the classical geometric obstructions to existence and then turn to positive results. We give a proof for the Schoenflies theorem in the quasiconformal category and subsequently give a fairly complete proof of Väisälä's mapping theorem for cylindrical domains, perhaps the best result to date answering this question.

We then present the sophisticated and important work of Tukia-Väisälä developed using Sullivan's machinery. In particular we give a proof for their solution of the lifting problem. Many of the last chapters of this book—part of a central theme in the area to develop quasiconformal versions of classical theorems in geometric topology—have never previously appeared in book form. Indeed many aspects of the approach to the theory given here are novel among recent monographs on the subject, as these primarily focus on the analytic approach through the associated nonlinear partial differential equations and differential inequalities.

We close with a presentation of the Mostow rigidity theory, one of the most compelling and important applications of the higher-dimensional theory of quasiconformal mappings. We take a fairly roundabout approach here so as to be able to clearly exhibit the remarkable interaction between quasiconformal theory, hyperbolic geometry, and modern aspects of geometric group theory. In particular we give a fairly comprehensive discussion of quasi-isometries and isomorphisms of hyperbolic groups.

During the long gestation of this book the first-named author Fred Gehring passed away. He was of course a major figure in the area, and much of the important work presented in this book is due to him and his coauthors. He is sadly missed.

We would also like to thank the team at the American Mathematical Society (in particular Ina Mette who tirelessly pressed us to complete) who skillfully guided us through the production process and whose considerable efforts improved this book.

Gaven Martin and Bruce Palka

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