Preface

This volume is a follow-up of the study initiated in the first volume of this monograph, where we gave an introduction to operads, we provided a survey on the definition of the notion of an E_n-operad, and we explained the definition of the (pro-unipotent) Grothendieck–Teichmüller group from the viewpoint of the theory of algebraic operads.

Recall briefly that the class of E_n-operads consists of the topological operads which are weakly-equivalent to a reference model, namely the operad of little n-discs D_n (an equivalent choice is given by the operad of little n-cubes). Recall also that the fundamental groupoid of the little 2-discs operad D_2 is equivalent to an operad in groupoids, the operad of parenthesized braids PaB, which governs operations acting on braided monoidal categories. In our approach, we precisely define the (pro-unipotent) Grothendieck–Teichmüller group $GT(\Bbbk)$, where $\Bbbk$ is any characteristic zero ground field, as a group of automorphisms of a Malcev completion $\mathsf{PaB}^{\wedge}$ of this operad in groupoids PaB. The Malcev completion construction considered in this definition refers to a rationalization process for (possibly non-abelian) groups which we extend to groupoids and to operads in groupoids. These topics form the matter of the first part of this monograph, "From Operads to Grothendieck–Teichmüller Groups".

In this second volume, we set up general methods for the study of the (rational) homotopy of operads in topological spaces and we give the proof of the following statement which is the ultimate goal of this book: The pro-unipotent Grothendieck–Teichmüller group is isomorphic to the group of homotopy automorphism classes of the rationalization of the little 2-disc operad. These topics form the matter of the second and third parts of this monograph, entitled "Homotopy Theory and its Applications to Operads" and "The Computation of Homotopy Automorphism Spaces of Operads", which are both contained in this volume.

Most of this volume is independent from the results of the first volume. The reader interested in applications of homotopy theory methods and who is familiar enough with the general definition of an operad can tackle the study of this volume straight away by skipping the algebraic study of the first volume. (We also give a short reminder of our conventions in the next chapter of this premilinary section.)

The rational homotopy theory is the study of spaces modulo torsion. The idea of working modulo a class of groups in homotopy was introduced by Serre in [142]. The computation of the homotopy groups of spheres modulo the class of finite groups (which capture all the torsion in this case) was also achieved by Serre in [140] by relying on spectral sequence constructions (see also the article [31] by Cartan–Serre for an account of this method). The theory was revisited by Quillen in [128], who proved that the rational homotopy of simply connected spaces is captured by a model in the category of differential graded Lie algebras (Lie dg-algebras for short) and by a dual model in the category of differential graded cocommutative coalgebras (cocommutative dg-coalgebras).

In Quillen's formulation, the rational homotopy category of (simply connected) spaces is defined by formally inverting the maps which induce an isomorphism on the rationalized homotopy groups (the tensor products $\pi_*(-) \otimes_{\mathbb{Z}} \mathbb{Q}$ equivalent to the quotients of the groups $\pi_*(-)$ by the Serre class of torsion subgroups) and the rational homotopy type of a (simply connected) space is defined as the isomorphism class of a space in this localized category. We use the name *rational weak-equivalence* and we adopt the distinguishing mark $\sim_{\mathbb{Q}}$ for this class of maps which define the isomorphisms of the rational homotopy category of spaces.

To define the morphism sets of the localization of a category properly, Quillen axiomatized the usual construction of the homotopy category of spaces, where we essentially have to take a quotient with respect to the homotopy relation on maps to define the morphism sets of the localization. He coined the name *model category* for this general notion of category where the localization with respect to a class of weak-equivalences is defined by an analogue of the classical homotopy category of topological spaces. The category of Lie dg-algebras and the category of cocommutative dg-coalgebras inherit a natural model structure (we neglect some mild connectedness conditions) and Quillen precisely proved that the homotopy categories of both model categories are equivalent to the localization of the category of spaces with respect to the class of rational weak-equivalences.

The theory was again revisited by Sullivan in [151], who used a differential cochain graded algebra of piecewise linear differential forms $\mathtt{\Omega}^*(X)$ (the Sullivan cochain dg-algebra for short), which is defined for any simplicial set X. This cochain dg-algebra is equivalent to the dual unitary commutative dg-algebra of the cocommutative dg-coalgebra defined by Quillen when X is a simply connected space such that the rational homology $\mathtt{H}_*(X) = \mathtt{H}_*(X, \mathbb{Q})$ forms a finitely generated $\mathbb{Q}$-module degreewise. We elaborate on this model, introduced by Sullivan, to build our rational homotopy theory of operads.

In fact, both Quillen and Sullivan deal with simplicial sets, regarded as combinatorial models of spaces, rather than with actual topological spaces. In this context, we also use the name *space* to refer to the objects of the category of simplicial sets. In Sullivan's approach, the rational homotopy type of a simplicial set X is captured by a simplicial set $X^{\wedge}$ associated to X and equipped with a map $\eta^* : X \to X^{\wedge}$ which induces the rationalization on homotopy groups. To be explicit, we get that the fundamental group of this simplicial set $X^{\wedge}$ is identified with the Malcev completion of the fundamental group of our original simplicial set $\pi_1(X^{\wedge}) = \pi_1(X)^{\wedge}$, while we have $\pi_n(X^{\wedge}) = \pi_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ for $n \geq 2$. The map $\eta_* : \pi_n(X) \to \pi_n(X^{\wedge})$ is identified with the universal morphism associated to this algebraic rationalization construction in each case. We precisely prove that this construction lifts to operads.

To be explicit, to any operad in simplicial sets R, we associate another operad $\mathsf{R}^{\wedge}$ whose components $\mathsf{R}^{\wedge}(r)$ are (under mild finiteness assumptions) weakly-equivalent to the Sullivan rationalization $\mathsf{R}(r)^{\wedge}$ of the individual spaces $\mathsf{R}(r)$. In good cases (when the classical Sullivan model works properly), this rationalized operad $\mathsf{R}^{\wedge}$ captures the rational homotopy type of the object R in the category of operads in spaces, where, to be precise, we define the rational homotopy type of an operad R as the isomorphism class of our object R in the localization of the category of operads with respect to the operad morphisms $\phi : \mathsf{P} \to \mathsf{Q}$ which define a rational weak-equivalence of spaces aritywise $\phi : \mathsf{P}(r) \xrightarrow{\sim_{\mathbb{Q}}} \mathsf{Q}(r)$.

We also establish that the Sullivan cochain dg-algebra admits an operadic enhancement which associates a cooperad in the category of unitary commutative cochain dg-algebras $\Omega^*_\sharp(\mathsf{R})$ to any operad in simplicial set R. We call cooperad the structure, dual to an operad in the categorical sense, which we naturally get in this context, because we rely on contravariant functors to build our model. We also use the short name *Hopf cochain dg-cooperad* for the objects of the category of cooperads in unitary commutative cochain dg-algebras. We precisely prove that (in good cases again) the Hopf cochain dg-cooperad $\Omega^*_\sharp(\mathsf{R})$ captures the rational homotopy type of the operad in simplicial sets R just like the Sullivan cochain dg-algebra $\Omega^*(X)$ captures the rational homotopy type of any (good) space X.

The latter result is the main goal of the second part of this book, "Homotopy Theory and its Applications to Operads". We comprehensively review the homotopical background of our constructions, the theory of model categories, and the rational homotopy theory of spaces, before tackling the applications to operads. We also define a new model structure for the study of the homotopy of unitary operads (the Reedy model category of Λ-operads) by relying on ideas introduced in the first volume of this work. We use this model structure to adapt the definition of our Hopf dg-cooperad model to the case of unitary operads.

To complete this study of the applications of homotopy theory to operads, we make explicit the definition of rational models of E_n-operads. In short, we explain a result of [66] which asserts that the rational homotopy of E_n-operads in simplicial sets (and in topological spaces) is determined by a model which we deduce from the cohomology of these operads (we say that E_n-operads are rationally formal as operads). This formality result follows from a counterpart, in the category of Hopf cochain dg-cooperads, of the formality of the chain operad of little n-discs, established by Tamarkin in the case $n = 2$ (see [152]) and by Kontsevich [97] for $n \geq 2$. In the case $n = 2$, we also check that a model for the rationalization $\mathsf{D}_2^{\widehat{}}$ of the little 2-discs operad D_2 (and a model of a rational E_2-operad) is given by the classifying space of the Malcev completion of the operad of parenthesized braids $\mathsf{PaB}^{\widehat{}}$ studied in the previous volume. We explicitly have $\mathsf{D}_2^{\widehat{}} = \mathtt{B}(\mathsf{PaB}^{\widehat{}})$. We use this observation and the functoriality of the classifying space construction to define our map from the Grothendieck–Teichmüller group $GT(\mathbb{Q})$ to the group of homotopy automorphism classes of the rationalization of the little 2-discs operad $\mathsf{D}_2^{\widehat{}}$.

We prove that this map defines an isomorphism in the third and concluding part of this book, "The Computation of Homotopy Automorphism Spaces of Operads". We briefly explained in the preface of the first volume that the group of homotopy classes of homotopy automorphisms of a (unitary) operad R is identified with the degree zero homotopy of a homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{O}p_*}(\mathsf{R})$, which in short consists of invertible connected components of a simplicial endomapping space $\mathtt{Map}_{\mathcal{O}p_*}(\mathsf{R}, \mathsf{R})$ associated to our object R.

We use homotopy spectral sequence methods to determine the homotopy of this homotopy automorphism space $\mathtt{Aut}^h_{\mathcal{O}p_*}(\mathsf{R})$ for our rationalization of the little 2-discs operad $\mathsf{R} = \mathsf{D}_2^{\widehat{}}$. We give a short survey of the general definition of such homotopy spectral sequences, which we borrow from Bousfield-Kan [25], before tackling the applications to operads. We use an operadic cotriple cohomology theory and the Koszul duality theory of operads [73] to compute this homotopy spectral sequence.

We give a detailed account of the applications of these methods to our problem. We also provide an account of the Koszul duality of operads in an appendix of this volume, "Cofree Cooperads and the Bar Duality of Operads".

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