Prologue

> ... *Cela suffit pour faire comprendre que dans les cinq mémoires des Acta mathematica que j'ai consacrés à l'étude des transcendantes fuchsiennes et kleinéennes, je n'ai fait qu'effleurer un sujet très vaste, qui fournira sans doute aux géomètres l'occasion de nombreuses et importantes découvertes.*[1]
>
> – H. Poincaré, Acta Mathematica, **5**, 1884, p. 278.

The theory of discrete subgroups of real hyperbolic space has a long history. It was inaugurated by Poincaré, who developed the two-dimensional (Fuchsian) and three-dimensional (Kleinian) cases of this theory in a series of articles published between 1881 and 1884 that included numerous notes submitted to the C. R. Acad. Sci. Paris, a paper at Klein's request in Math. Annalen, and five memoirs commissioned by Mittag-Leffler for his then freshly-minted Acta Mathematica. One must also mention the complementary work of the German school that came before Poincaré and continued well after he had moved on to other areas, viz. that of Klein, Schottky, Schwarz, and Fricke. See [**80**, Chapter 3] for a brief exposition of this fascinating history, and [**79, 63**] for more in-depth presentations of the mathematics involved.

We note that in finite dimensions, the theory of *higher-dimensional Kleinian groups*, i.e., discrete isometry groups of the hyperbolic d-space $\mathbb{H}^d$ for $d \geq 4$, is markedly different from that in $\mathbb{H}^3$ and $\mathbb{H}^2$. For example, the Teichmüller theory used by the Ahlfors–Bers school (viz. Marden, Maskit, Jørgensen, Sullivan, Thurston, etc.) to study three-dimensional Kleinian groups has no generalization to higher dimensions. Moreover, the recent resolution of the Ahlfors measure conjecture [**3, 43**] has more to do with three-dimensional topology than with analysis and dynamics. Indeed, the conjecture remains open in higher dimensions [**106**, p. 526, last paragraph]. Throughout the twentieth century, there are several instances of theorems proven for three-dimensional Kleinian groups whose proofs extended easily to n dimensions (e.g. [**21, 133**]), but it seems that the theory of higher-dimensional Kleinian groups was not really considered a subject in its own right until around the 1990s. For more information on the theory of higher-dimensional Kleinian groups, see the survey article [**106**], which describes the state of the art up to the last decade, emphasizing connections with homological algebra.

[1] This is enough to make it apparent that in these five memoirs in Acta Mathematica which I have dedicated to the study of Fuschian and Kleinian transcendants, I have only skimmed the surface of a very broad subject, which will no doubt provide geometers with the opportunity for many important discoveries.

But why stop at finite n? Dennis Sullivan, in his IHÉS *Seminar on Conformal and Hyperbolic Geometry* [**164**] that ran during the late 1970s and early '80s, indicated a possibility of developing the theory of discrete groups acting by hyperbolic isometries on the open unit ball of a separable infinite-dimensional Hilbert space.[2] Later in the early '90s, Misha Gromov observed the paucity of results regarding such actions in his seminal lectures *Asymptotic Invariants of Infinite Groups* [**86**] where he encouraged their investigation in memorable terms: "The spaces like this [infinite-dimensional symmetric spaces] ... look as cute and sexy to me as their finite dimensional siblings but they have been for years shamefully neglected by geometers and algebraists alike".

Gromov's lament had not fallen to deaf ears, and the geometry and representation theory of infinite-dimensional hyperbolic space $\mathbb{H}^\infty$ and its isometry group have been studied in the last decade by a handful of mathematicians, see e.g. [**40, 65, 132**]. However, infinite-dimensional hyperbolic geometry has come into prominence most spectacularly through the recent resolution of a long-standing conjecture in algebraic geometry due to Enriques from the late nineteenth century. Cantat and Lamy [**47**] proved that the Cremona group (i.e. the group of birational transformations of the complex projective plane) has uncountably many non-isomorphic normal subgroups, thus disproving Enriques' conjecture. Key to their enterprise is the fact, due to Manin [**125**], that the Cremona group admits a faithful isometric action on a non-separable infinite-dimensional hyperbolic space, now known as the Picard–Manin space.

Our project was motivated by a desire to answer Gromov's plea by exposing a coherent general theory of groups acting isometrically on the infinite-dimensional hyperbolic space $\mathbb{H}^\infty$. In the process we came to realize that a more natural domain for our inquiries was the much larger setting of semigroups acting on Gromov hyperbolic metric spaces – that way we could simultaneously answer our own questions about $\mathbb{H}^\infty$ and construct a theoretical framework for those who are interested in more exotic spaces such as the curve graph, arc graph, and arc complex [**95, 126, 96**] and the free splitting and free factor complexes [**89, 27, 104, 96**]. These examples are particularly interesting as they extend the well-known dictionary [**26**, p.375] between mapping class groups and the groups $\mathrm{Out}(\mathbb{F}_N)$. In another direction, a dictionary is emerging between mapping class groups and Cremona groups, see [**30, 66**]. We speculate that developing the Patterson–Sullivan theory in these three areas would be fruitful and may lead to new connections and analogies that have not surfaced till now.

In a similar spirit, we believe there is a longer story for which this monograph lays the foundations. In general, infinite-dimensional space is a wellspring of outlandish examples and the wide range of new phenomena we have started to uncover has no analogue in finite dimensions. The geometry and analysis of such groups should pique the interests of specialists in probability, geometric group theory, and metric geometry. More speculatively, our work should interact with the ongoing and still nascent study of geometry, topology, and dynamics in a variety of infinite-dimensional spaces and groups, especially in scenarios with sufficient

[2] This was the earliest instance of such a proposal that we could find in the literature, although (as pointed out to us by P. de la Harpe) infinite-dimensional hyperbolic spaces without groups acting on them had been discussed earlier [**130**, §27], [**131, 60**]. It would be of interest to know whether such an idea may have been discussed prior to that.

negative curvature. Here are three concrete settings that would be interesting to consider: the universal Teichmüller space, the group of volume-preserving diffeomorphisms of $\mathbb{R}^3$ or a 3-torus, and the space of Kähler metrics/potentials on a closed complex manifold in a fixed cohomology class equipped with the Mabuchi–Semmes–Donaldson metric. We have been developing a few such themes. The study of thermodynamics (equilibrium states and Gibbs measures) on the boundaries of Gromov hyperbolic spaces will be investigated in future work [**57**]. We speculate that the study of stochastic processes (random walks and Brownian motion) in such settings would be fruitful. Furthermore, it would be of interest to develop the theory of discrete isometric actions and limit sets in infinite-dimensional spaces of higher rank.

Acknowledgements. This monograph is dedicated to our colleague Bernd O. Stratmann, who passed away on the 8th of August, 2015. Various discussions with Bernd provided inspiration for this project and we remain grateful for his friendship. The authors thank D. P. Sullivan, D. Mumford, B. Farb, P. Pansu, F. Ledrappier, A. Wilkinson, K. Biswas, E. Breuillard, A. Karlsson, I. Assani, M. Lapidus, R. Guo, Z. Huang, I. Gekhtman, G. Tiozzo, P. Py, M. Davis, M. Roychowdury, M. Hochman, J. Tao, P. de la Harpe, T. Barthelme, J. P. Conze, and Y. Guivarc'h for their interest and encouragement, as well as for invitations to speak about our work at various venues. We are grateful to S. J. Patterson, J. Elstrodt, and É. Ghys for enlightening historical discussions on various themes relating to the history of Fuchsian and Kleinian groups and their study through the twentieth century, and to D. P. Sullivan and D. Mumford for suggesting work on diffeomorphism groups and the universal Teichmüller space. We thank X. Xie for pointing us to a solution to one of the problems in our Appendix A. The research of the first-named author was supported in part by 2014-2015 and 2016-2017 Faculty Research Grants from the University of Wisconsin–La Crosse. The research of the second-named author was supported in part by the EPSRC Programme Grant EP/J018260/1. The research of the third-named author was supported in part by the NSF grant DMS-1361677.

CHAPTER 17

Quasiconformal measures of geometrically finite groups

In this chapter we investigate the δ-quasiconformal measure or measures associated to a geometrically finite group. Note that since geometrically finite groups are of compact type (Theorem 12.4.5), Theorem 15.4.6 guarantees the existence of a δ-quasiconformal measure μ on Λ. However, this measure is not necessarily unique (Corollary 17.1.8); a sufficient condition for uniqueness is that G is of divergence type (Theorem 1.4.1). In Section 17.1, we generalize a theorem of Dal'bo, Otal, and Peigne [**55**, Théorème A] which shows that "most" geometrically finite groups are of divergence type. In Sections 17.2-17.5 we investigate the geometry of δ-conformal measures; specifically, in Sections 17.2-17.3 we prove a generalization of the Global Measure Formula (Theorem 17.2.2), in Sections 17.4 and 17.5 we investigate the questions of when the δ-conformal measure of a geometrically finite group is doubling and exact dimensional, respectively.

Standing Assumptions 17.0.1. In this chapter, we assume that

(I) X is regularly geodesic and strongly hyperbolic,

(II) $G \leq \mathrm{Isom}(X)$ is nonelementary and geometrically finite, and $\delta < \infty$.[1]

Moreover, we fix a complete set of inequivalent parabolic points $P \subseteq \Lambda_{\mathrm{bp}}$, and for each $p \in P$ we write $\delta_p = \delta(G_p)$, and let $S_p \subseteq \mathcal{E}_p$ be a p-bounded set satisfying (A)-(C) of Lemma 12.3.6. Finally, we choose a number $t_0 > 0$ large enough so that if

$$H_p = H_{p,t_0} = \{x \in X : \mathcal{B}_p(o,x) > t_0\}$$
$$\mathscr{H} = \{g(H_p) : p \in P, g \in G\},$$

then the collection $\mathscr{H}$ is disjoint (cf. Proof of Theorem 12.4.5(B3) $\Rightarrow$ (A)).

17.1. Sufficient conditions for divergence type

In the Standard Case, all geometrically finite groups are of divergence type [**165**, Proposition 2]; however, once one moves to the more general setting of pinched Hadamard manifolds, one has examples of geometrically finite groups of convergence type [**55**, Théorème C]. On the other hand, Proposition 16.6.5 shows that for every δ-conformal measure μ, G is of divergence type if and only if $\mu(\Lambda \setminus \Lambda_{\mathrm{r}}) = 0$. Now by Theorem 12.4.5, $\Lambda \setminus \Lambda_{\mathrm{r}} = \Lambda_{\mathrm{bp}} = G(P)$, so the condition $\mu(\Lambda \setminus \Lambda_{\mathrm{r}}) = 0$ is equivalent to the condition $\mu(P) = 0$. To summarize:

Observation 17.1.1. The following are equivalent:

(A) G is of divergence type.

[1] Note that by Corollary 12.4.17(ii), we have $\delta < \infty$ if and only if $\delta_p < \infty$ for all $p \in P$.

(B) There exists a δ-conformal measure μ on Λ satisfying $\mu(P) = 0$.
(C) Every δ-conformal measure μ on Λ satisfies $\mu(P) = 0$.
(D) There exists a unique δ-conformal measure μ on Λ, and this measure satisfies $\mu(P) = 0$.

In particular, every convex-cobounded group is of divergence type.

It is of interest to ask for sufficient conditions which are not phrased in terms of measures. We have the following:

THEOREM 17.1.2 (Cf. [**165**, Proposition 2], [**55**, Théorème A]). *If $\delta > \delta_p$ for all $p \in P$, then G is of divergence type.*

PROOF. We will demonstrate (B) of Observation 17.1.1. Let μ be the measure constructed in the proof of Theorem 15.4.6, fix $p \in P$, and we will show that $\mu(p) = 0$. In what follows, we use the same notation as in the proof of Theorem 15.4.6. Since G is strongly discrete, we can let ρ be small enough so that $S_\rho = G(o)$. For any neighborhood U of p, we have

$$\mu(p) \leq \liminf_{s \searrow \delta} \mu_s(U) = \liminf_{s \searrow \delta} \frac{1}{\Sigma_{s,k}} \sum_{x \in G(o) \cap U} k(x) e^{-s\|x\|}. \tag{17.1.1}$$

LEMMA 17.1.3.

$$\langle h(o) | x \rangle_o \asymp_+ 0 \;\; \forall x \in S_p.$$

PROOF. Since S_p is p-bounded, Gromov's inequality implies that

$$\langle h(o) | x \rangle_o \wedge \langle h(o) | p \rangle_o \asymp_+ 0$$

for all $h \in G_p$ and $x \in S_p$. Denote the implied constant by σ. For all $h \in G_p$ such that $\langle h(o) | p \rangle_o > \sigma$, we have $\langle h(o) | x \rangle_o \leq \sigma \;\; \forall x \in S_p$. Since this applies to all but finitely many $h \in G_p$, (c) of Proposition 3.3.3 completes the proof. ◁

Let T be a transversal of $G_p \backslash G$ such that $T(o) \subseteq S_p$. Then by Lemma 17.1.3,

$$\|h(x)\| \asymp_+ \|h\| + \|x\| \;\; \forall h \in G_p \;\; \forall x \in T(o).$$

Thus for all $s > \delta$ and $V \subseteq X$,

$$\begin{aligned} \sum_{x \in G(o) \cap U} k(x) e^{-s\|x\|} &= \sum_{h \in G_p} \sum_{x \in hT(o) \cap U} k(e^{\|x\|}) e^{-s\|x\|} \\ &\asymp_\times \sum_{h \in G_p} \sum_{x \in T(o) \cap h^{-1}(U)} k(e^{\|h\| + \|x\|}) e^{-s[\|h\| + \|x\|]}. \end{aligned} \tag{17.1.2}$$

Now fix $0 < \varepsilon < \delta - \delta_p$, and note that by (15.4.1),

$$k(R) \leq k(\lambda R) \lesssim_{\times,\varepsilon} \lambda^\varepsilon k(R) \;\; \forall \lambda > 1 \;\; \forall R \geq 1.$$

Thus setting $V = U$ in (17.1.2) gives

$$\sum_{x \in G(o) \cap U} k(x) e^{-s\|x\|} \lesssim_{\times,\varepsilon} \sum_{\substack{h \in G_p \\ h(S_p) \cap U \neq \varnothing}} e^{-(s-\varepsilon)\|h\|} \sum_{x \in T(o)} k(x) e^{-s\|x\|},$$

while setting $V = X$ gives

$$\Sigma_{s,k} = \sum_{x \in G(o)} k(x) e^{-s\|x\|} \gtrsim_\times \sum_{h \in G_p} e^{-s\|h\|} \sum_{x \in T(o)} k(x) e^{-s\|x\|}.$$

Dividing these inequalities and combining with (17.1.1) gives

$$\mu(p) \lesssim_{\times,\varepsilon} \liminf_{s\searrow\delta} \frac{1}{\Sigma_s(G_p)} \sum_{\substack{h\in G_p\\ h(S_p)\cap U\neq\varnothing}} e^{-(s-\varepsilon)\|h\|} = \frac{1}{\Sigma_\delta(G_p)} \sum_{\substack{h\in G_p\\ h(S_p)\cap U\neq\varnothing}} e^{-(\delta-\varepsilon)\|h\|}.$$

Note that the right hand series converges since $\delta - \varepsilon > \delta_p$ by construction. As the neighborhood U shrinks, the series converges to zero. This completes the proof. $\square$

Combining Theorem 17.1.2 with Proposition 10.3.10 gives the following immediate corollary:

COROLLARY 17.1.4. *If for all $p \in P$, G_p is of divergence type, then G is of divergence type.*

Thus in some sense divergence type can be "checked locally" just like the properties of finite generation and finite Poincaré exponent (cf. Corollary 12.4.17).

COROLLARY 17.1.5. *Every convex-cobounded group is of divergence type.*

REMARK 17.1.6. It is somewhat awkward that it seems to be difficult or impossible to prove Theorem 17.1.2 via any of the equivalent conditions of Observation 17.1.1 other than (B). Specifically, the fact that the above argument works for the measure constructed in Theorem 15.4.6 (the "Patterson–Sullivan measure") but not for other δ-conformal measures seems rather asymmetric. However, after some thought one realizes that it would be impossible for a proof along similar lines to work for every δ-conformal measure. This is because the above proof shows that the Patterson–Sullivan measure μ satisfies

$$\mu(p) = 0 \text{ for all } p \in P \text{ satisfying } \delta > \delta_p, \tag{17.1.3}$$

but there are geometrically finite groups for which (17.1.3) does not hold for all δ-conformal measures μ. Specifically, one may construct geometrically finite groups of convergence type (cf. [**55**, Théorème C]) such that $\delta_p < \delta$ for some $p \in P$; the following proposition shows that there exists a δ-conformal measure for which (17.1.3) fails:

PROPOSITION 17.1.7. *If G is of convergence type, then for each $p \in P$ there exists a δ-conformal measure supported on $G(p)$.*

PROOF. Let

$$\mu = \sum_{g(p)\in G(p)} [g'(p)]^\delta \delta_{g(p)};$$

clearly μ is a δ-conformal measure, but we may have $\mu(\partial X) = \infty$. To prove that this is not the case, as before we let T be a transversal of $G_p\backslash G$ such that $T(o) \subseteq S_p$. Then

$$\mu(\partial X) = \sum_{g(p)\in G(p)} [g'(p)]^\delta = \sum_{g\in T^{-1}} [g'(p)]^\delta \asymp_\times \sum_{g\in T^{-1}} e^{-\delta\|g\|} \leq \Sigma_\delta(G) < \infty. \qquad \square$$

Proposition 17.1.7 yields the following characterization of when there exists a unique δ-conformal measure:

COROLLARY 17.1.8. *The following are equivalent:*

(A) *There exists a unique δ-conformal measure on Λ.*
(B) *Either G is of divergence type, or $\#(P) = 1$.*

17.2. The global measure formula

In this section and the next, we fix a δ-quasiconformal measure μ, and ask the following geometrical question: Given $\eta \in \Lambda$ and $r > 0$, can we estimate $\mu(B(\eta, r))$? If G is convex-cobounded, then we can show that μ is Ahlfors δ-regular (Corollary 17.2.3), but in general the measure $\mu(B(\eta, r))$ will depend on the point η, in a manner described by the *global measure formula*. To describe the global measure formula, we need to introduce some notation:

NOTATION 17.2.1. Given $\xi = g(p) \in \Lambda_{\mathrm{bp}}$, let $t_\xi > 0$ be the unique number such that

$$H_\xi = H_{\xi, t_\xi} = g(H_p) = g(H_{p,t_0}),$$

i.e. $t_\xi = t_0 + \mathcal{B}_\xi(o, g(o))$. (Note that $t_p = t_0$ for all $p \in P$.) Fix $\theta > 0$ large to be determined below (cf. Proposition 17.2.5). For each $\eta \in \Lambda$ and $t > 0$, let $\eta_t = [o, \eta]_t$, and write

$$m(\eta, t) = \begin{cases} e^{-\delta t} & \eta_t \notin \bigcup(\mathscr{H}) \\ e^{-\delta t_\xi}[\mathcal{I}_p(e^{t - t_\xi - \theta}) + \mu(p)] & \eta_t \in H_\xi \text{ and } t \leq \langle \xi | \eta \rangle_o \\ e^{-\delta(2\langle \xi | \eta \rangle_o - t_\xi)} \mathcal{N}_p(e^{2\langle \xi | \eta \rangle_o - t - t_\xi - \theta}) & \eta_t \in H_\xi \text{ and } t > \langle \xi | \eta \rangle_o \end{cases} \tag{17.2.1}$$

(cf. Figure 17.2.1.) Here we use the notation

$$\mathcal{I}_p(R) = \sum_{\substack{h \in G_p \\ \|h\|_p \geq R}} \|h\|_p^{-2\delta}$$

$$\mathcal{N}_p(R) = \mathcal{N}_{\mathcal{E}_p, G_p}(R) = \#\{h \in G_p : \|h\|_p \leq R\}$$

where

$$\|h\|_p = D_p(o, h(o)) = e^{(1/2)\|h\|} \;\; \forall h \in G_p.$$

THEOREM 17.2.2 (Global measure formula; cf. [**160**, Theorem 2] and [**153**, Théorème 3.2]). *For all $\eta \in \Lambda$ and $t > 0$,*

$$m(\eta, t + \sigma) \lesssim_\times \mu(B(\eta, e^{-t})) \lesssim_\times m(\eta, t - \sigma), \tag{17.2.2}$$

where $\sigma > 0$ is independent of η and t (but may depend on θ).

COROLLARY 17.2.3. *If G is convex-cobounded, then*

$$\mu(B(\eta, r)) \asymp_\times r^\delta \;\; \forall \eta \in \Lambda \;\; \forall 0 < r \leq 1, \tag{17.2.3}$$

i.e. μ is Ahlfors δ-regular.

PROOF. If G is convex-cobounded then $\mathscr{H} = \varnothing$, so $m(\eta, t) = e^{-\delta t} \;\; \forall \eta, t$, and thus (17.2.2) reduces to (17.2.3). ☐

REMARK 17.2.4. Corollary 17.2.3 can be deduced directly from Lemma 17.3.7, which appears in the next section.

We will prove Theorem 17.2.2 in the next section. For now, we investigate more closely the function $t \mapsto m(\eta, t)$ defined by (17.2.1). The main result of this section is the following proposition, which will be used in the proof of Theorem 17.2.2:

PROPOSITION 17.2.5. *If θ is chosen sufficiently large, then for all $\eta \in \Lambda$ and $0 < t_1 < t_2$,*

$$m(\eta, t_2) \lesssim_{\times, \theta} m(\eta, t_1). \tag{17.2.4}$$

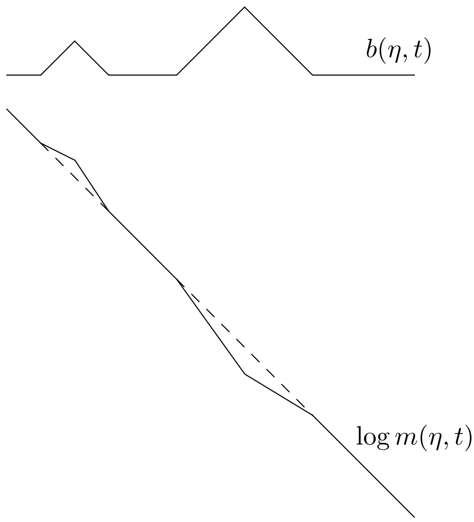

FIGURE 17.2.1. A possible (approximate) graph of the functions $t \mapsto b(\eta, t)$ and $t \mapsto \log m(\eta, t)$ (cf. (17.2.1) and (17.2.6)). The graph indicates that there are at least two inequivalent parabolic points $p_1, p_2 \in P$, which satisfy $\mathcal{N}_{p_i}(R) \asymp_\times R^{2\delta}\mathcal{I}_{p_i}(R) \asymp_\times R^{k_i}$ for some $k_1 < 2\delta < k_2$. The dotted line in the second graph is just the line $y = -\delta t$.
Note the relation between the two graphs, which may be either direct or inverted depending on the functions $\mathcal{N}_p$. Specifically, the relation is direct for the first cusp but inverted for the second cusp.

The proof of Proposition 17.2.5 itself requires several lemmas.

LEMMA 17.2.6. *Fix $\xi, \eta \in \partial X$ and $t > 0$, and let $x = \eta_t$. Then*

$$\mathcal{B}_\xi(o, x) \asymp_+ t \wedge (2\langle \xi|\eta\rangle_o - t). \tag{17.2.5}$$

PROOF. Since $\langle o|\eta\rangle_x = 0$, Gromov's inequality gives $\langle o|\xi\rangle_x \wedge \langle \xi|\eta\rangle_x \asymp_+ 0$.
Case 1: $\langle o|\xi\rangle_x \asymp_+ o$. In this case, by (h) of Proposition 3.3.3,

$$\mathcal{B}_\xi(o, x) = -\mathcal{B}_\xi(x, o) = -[2\langle o|\xi\rangle_x - \|x\|] \asymp_+ \|x\| = t,$$

while (g) of Proposition 3.3.3 gives

$$\langle \xi|\eta\rangle_o = \langle \xi|\eta\rangle_x + \frac{1}{2}[\mathcal{B}_\xi(o, x) + \mathcal{B}_\eta(o, x)] \gtrsim_+ \frac{1}{2}[t + t] = t;$$

thus $\mathcal{B}_\xi(o, x) \asymp_+ t \asymp_+ t \wedge (2\langle \xi|\eta\rangle_o - t)$.
Case 2: $\langle \xi|\eta\rangle_x \asymp_+ o$. In this case, (g) of Proposition 3.3.3 gives

$$\langle \xi|\eta\rangle_o \asymp_+ \frac{1}{2}[\mathcal{B}_\xi(o, x) + \mathcal{B}_\eta(o, x)] = \frac{1}{2}[\mathcal{B}_\xi(o, x) + t] \lesssim_+ \frac{1}{2}[t + t] = t;$$

thus $\mathcal{B}_\xi(o, x) \asymp_+ 2\langle \xi|\eta\rangle_o - t \asymp_+ t \wedge (2\langle \xi|\eta\rangle_o - t)$. □

Corollary 17.2.7. *The function*

$$b(\eta,t) = \begin{cases} 0 & \eta_t \notin \bigcup(\mathscr{H}) \\ t \wedge (2\langle\xi|\eta\rangle_o - t) - t_\xi & \eta_t \in H_\xi \end{cases} \tag{17.2.6}$$

satisfies

$$b(\eta, t+\tau) \asymp_{+,\tau} b(\eta, t-\tau). \tag{17.2.7}$$

Proof. Indeed, by Lemma 17.2.6,

$$\begin{aligned} b(\eta,t) &\asymp_+ \begin{cases} 0 & \eta_t \notin \bigcup(\mathscr{H}) \\ \mathcal{B}_\xi(o,\eta_t) - t_\xi & \eta_t \in H_\xi \end{cases} \\ &= 0 \vee \max_{\xi\in\Lambda_{\mathrm{bp}}} (\mathcal{B}_\xi(o,\eta_t) - t_\xi). \end{aligned}$$

The right hand side is 1-Lipschitz continuous with respect to t. This fact demonstrates (17.2.7). ☐

Lemma 17.2.8. *For all $\xi \in G(p) \subseteq \Lambda_{\mathrm{bp}}$, $p \in P$, there exists $g \in G$ such that*

$$\xi = g(p),\ \|g\| \asymp_+ t_\xi, \text{ and } \{\eta\in\partial X : [o,\eta]\cap H_\xi \neq \varnothing\} \subseteq \mathrm{Shad}(g(o),\sigma), \tag{17.2.8}$$

where $\sigma > 0$ is independent of ξ.

Proof. Write $\xi = g(p)$ for some $g \in G$. Since $x := \xi_{t_\xi} \in \partial H_\xi$, Lemma 12.3.6(D) shows that

$$d(g^{-1}(x), h(o)) \asymp_+ 0$$

for some $h \in G_p$. We claim that gh is the desired isometry. Clearly $\|gh\| \asymp_+ \|x\| = t_\xi$. Fix $\eta \in \partial X$ such that $[o,\eta]\cap H_\xi \neq \varnothing$, say $\eta_t \in H_\xi$. By Lemma 17.2.6, we have

$$\|x\| = t_\xi < \mathcal{B}_\xi(o,\eta_t) \asymp_+ t\wedge(2\langle\xi|\eta\rangle_o - t) \leq \langle\xi|\eta\rangle_o \leq \langle x|\eta\rangle_o,$$

i.e. $\eta \in \mathrm{Shad}(x,\sigma) \subseteq \mathrm{Shad}(g(o),\sigma+\tau)$ for some $\sigma,\tau > 0$. ☐

Proof of Proposition 17.2.5. Fix $\eta \in \Lambda$ and $0 < t_1 < t_2$.

Case 1: $\eta_{t_1}, \eta_{t_2} \in H_\xi$ for some $\xi = g(p) \in \Lambda_{\mathrm{bp}}$, g satisfying (17.2.8). In this case, (17.2.4) follows immediately from (17.2.1) unless $t_1 \leq \langle\xi|\eta\rangle_o < t_2$. If the latter holds, then

$$\begin{aligned} m(\eta,t_1) &\geq \lim_{t\nearrow\langle\xi|\eta\rangle_o} m(\eta,t) = e^{-\delta t_\xi}[\mathcal{I}_p(e^{\langle\xi|\eta\rangle_o - t_\xi - \theta}) + \mu(p)] \\ m(\eta,t_2) &\leq \lim_{t\searrow\langle\xi|\eta\rangle_o} m(\eta,t) = e^{-\delta(2\langle\xi|\eta\rangle_o - t_\xi)}\mathcal{N}_p(e^{\langle\xi|\eta\rangle_o - t_\xi - \theta}). \end{aligned}$$

Consequently, to demonstrate (17.2.4) it suffices to show that

$$\mathcal{N}_p(e^t) \lesssim_{\times,\theta} e^{2\delta t}\mathcal{I}_p(e^t), \tag{17.2.9}$$

where $t := \langle\xi|\eta\rangle_o - t_\xi - \theta > 0$.

To demonstrate (17.2.9), let $\zeta = g^{-1}(\eta) \in \Lambda$. We have

$$\langle p|\zeta\rangle_o = \langle\xi|\eta\rangle_{g(o)} \asymp_+ \langle\xi|\eta\rangle_o - \|g\| \asymp_+ \langle\xi|\eta\rangle_o - t_\xi = t + \theta$$

and thus

$$D_p(o,\zeta) \asymp_\times e^{t+\theta}.$$

Since p is a bounded parabolic point, there exists $h_\zeta \in G_p$ such that $D_p(h_\zeta(o),\zeta) \lesssim_\times 1$. Denoting all implied constants by C, we have

$$C^{-1}e^{t+\theta} - C \leq D_p(o,\zeta) - D_p(h_\zeta(o),\zeta) \leq \|h_\zeta\|_p \leq D_p(o,\zeta) + D_p(h_\zeta(o),\zeta) \leq Ce^{t+\theta} + C.$$

Choosing $\theta \geq \log(4C)$, we have

$$2e^t \leq \|h_\zeta\|_p \leq 2Ce^{t+\theta} \text{ unless } e^{t+\theta} \leq 2C^2.$$

If $2e^t \leq \|h_\zeta\|_p \leq 2Ce^{t+\theta}$, then for all $h \in G_p$ satisfying $\|h\|_p \leq e^t$ we have $e^t \leq \|h_\zeta h\|_p \lesssim_{\times,\theta} e^t$; it follows that

$$\mathcal{I}_p(e^t) \geq \sum_{h\in G_p} \|h_\zeta h\|_p^{-2\delta} \asymp_{\times,\theta} e^{-2\delta t}\mathcal{N}_p(e^t),$$

thus demonstrating (17.2.9). On the other hand, if $e^{t+\theta} \leq 2C^2$, then both sides of (17.2.9) are bounded from above and below independent of t.

Case 2: No such ξ exists. In this case, for each i write $\eta_i \in H_{\xi_i}$ for some $\xi_i = g_i(p_i) \in \Lambda_{\mathrm{bp}}$ if such a ξ_i exists. If ξ_1 exists, let $s_1 > t_1$ be the smallest number such that $\eta_{s_1} \in \partial H_{\xi_1}$, and if ξ_2 exists, let $s_2 < t_2$ be the largest number such that $\eta_{s_2} \in \partial H_{\xi_2}$. If ξ_i does not exist, let $s_i = t_i$. Then $t_1 \leq s_1 \leq s_2 \leq t_2$. Since $m(\eta,s_i) = e^{-\delta s_i}$, we have $m(\eta,s_2) \leq m(\eta,s_1)$, so to complete the proof it suffices to show that

$$m(\eta,s_1) \lesssim_{\times,\theta} m(\eta,t_1) \text{ and}$$
$$m(\eta,s_2) \gtrsim_{\times,\theta} m(\eta,t_2).$$

By Case 1, it suffices to show that

$$m(\eta,s_1) \lesssim_\times \lim_{t\nearrow s_1} m(\eta,t) \text{ if } \xi_1 \text{ exists, and}$$
$$m(\eta,s_2) \gtrsim_\times \lim_{t\searrow s_2} m(\eta,t) \text{ if } \xi_2 \text{ exists.}$$

Comparing with (17.2.1), we see that the desired formulas are

$$e^{-\delta s_1} \lesssim_\times e^{-\delta(2\langle\xi|\eta\rangle_o - t_{\xi_1})}\mathcal{N}_p(e^{2\langle\xi_1|\eta\rangle_o - s_1 - t_{\xi_1}})$$
$$e^{-\delta s_2} \gtrsim_\times e^{-\delta t_{\xi_2}}[\mathcal{I}_p(e^{s_2 - t_{\xi_2}}) + \mu(p)],$$

which follow upon observing that the definitions of s_1 and s_2 imply that $s_1 \asymp_+ 2\langle\xi|\eta\rangle_o - t_{\xi_1}$ and $s_2 \asymp_+ t_{\xi_2}$ (cf. Lemma 17.2.6). □

17.3. Proof of the global measure formula

Although we have finished the proof of Proposition 17.2.5, we still need a few lemmas before we can begin the proof of Theorem 17.2.2. Throughout these lemmas, we fix $p \in P$, and let

$$R_p = \sup_{x\in S_p} D_p(o,x) < \infty.$$

Here $S_p \subseteq \mathcal{E}_p$ is a p-bounded set satisfying $\Lambda \setminus \{p\} \subseteq G_p(S_p)$, as in Standing Assumptions 17.0.1.

LEMMA 17.3.1. *For all $A \subseteq G_p$,*

$$\mu\left(\bigcup_{h\in A} h(S_p)\right) \asymp_\times \sum_{h\in A} e^{-\delta\|h\|} = \sum_{h\in A} \|h\|_p^{-2\delta}. \tag{17.3.1}$$

PROOF. As the equality follows from Observation 6.2.10, we proceed to demonstrate the asymptotic. By Lemma 17.1.3, there exists $\sigma > 0$ such that $S_p \subseteq \mathrm{Shad}_{h^{-1}(o)}(o,\sigma)$ for all $h \in G_p$. Then by the Bounded Distortion Lemma 4.5.6,

$$\mu(h(S_p)) = \int_{S_p} (\overline{h}')^\delta \, \mathrm{d}\mu \asymp_{\times,\sigma} e^{-\delta\|h\|}\mu(S_p) \asymp_\times e^{-\delta\|h\|}.$$

(In the last asymptotic, we have used the fact that $\mu(S_p) > 0$, which follows from the fact that $\Lambda \setminus \{p\} \subseteq G_p(S_p)$ together with the fact that μ is not a pointmass (Corollary 15.4.3).) Combining with the subadditivity of μ gives the $\lesssim$ direction of the first asymptotic of (17.3.1). To get the $\gtrsim$ direction, we observe that since S_p is p-bounded, the strong discreteness of G_p implies that $S_p \cap h(S_p) \neq \varnothing$ for only finitely many $h \in G_p$; it follows that the function $\eta \mapsto \#\{h \in G_p : \eta \in h(S_p)\}$ is bounded, and thus

$$\begin{aligned}\mu\left(\bigcup_{h\in A} h(S_p)\right) &\asymp_\times \int \#\{h \in G_p : \eta \in h(S_p)\} \, \mathrm{d}\mu(\eta)\\ &= \sum_{h\in A} \mu(h(S_p))\\ &\asymp_\times \sum_{h\in A} e^{-\delta\|h\|}\end{aligned}$$

□

COROLLARY 17.3.2. *For all $r > 0$,*

$$\mathcal{I}_p\left(\frac{2}{r}\right) \lesssim_\times \mu\big(B(p,r)\setminus\{p\}\big) \lesssim_\times \mathcal{I}_p\left(\frac{1}{2r}\right) \tag{17.3.2}$$

PROOF. Since

$$\bigcup_{\substack{h\in G_p\\ \|h\|_p \geq R+R_p}} h(S_p) \subseteq B(p,1/R)\setminus\{p\} = \mathcal{E}_p \setminus B_p(o,R) \subseteq \bigcup_{\substack{h\in G_p\\ \|h\|_p \geq R-R_p}} h(S_p),$$

Lemma 17.3.1 gives

$$\mathcal{I}_p\left(\frac{1}{r} + R_p\right) \lesssim_\times \mu(B(p,r)) \lesssim_\times \mathcal{I}_p\left(\frac{1}{r} - R_p\right),$$

thus proving the lemma if $r \leq 1/(2R_p)$. But when $r > 1/(2R_p)$, all terms of (17.3.2) are bounded from above and below independent of r. □

Adding $\mu(p)$ to all sides of (17.3.2) gives

$$\mathcal{I}_p\left(\frac{2}{r}\right) + \mu(p) \lesssim_\times \mu(B(p,r)) \lesssim_\times \mathcal{I}_p\left(\frac{1}{2r}\right) + \mu(p). \tag{17.3.3}$$

COROLLARY 17.3.3 (Cf. Figure 17.3.1). *Fix $\eta \in \Lambda$ and $t > 0$ such that $\eta_t \in H_\xi$ for some $\xi = g(p) \in \Lambda_{\mathrm{bp}}$ satisfying $t \leq \langle\xi|\eta\rangle_o - \log(2)$. Then*

$$e^{-\delta t_\xi}[\mathcal{I}_p(e^{t-t_\xi+\sigma}) + \mu(p)] \lesssim_\times \mu\big(B(\eta,e^{-t})\big) \lesssim_\times e^{-\delta t_\xi}[\mathcal{I}_p(e^{t-t_\xi-\sigma}) + \mu(p)],$$

where $\sigma > 0$ is independent of η and t.

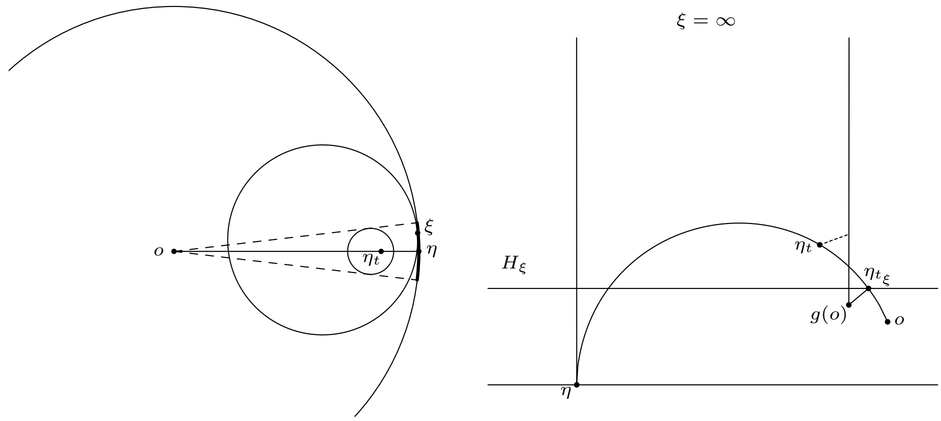

FIGURE 17.3.1. Cusp excursion in the ball model (left) and upper half-space model (right). Since $\xi = g(p) \in B(\eta, e^{-t})$, our estimate of $\mu(B(\eta, e^{-t}))$ is based on the function $\mathcal{I}_p$, which captures information "at infinity" about the cusp p. In the right-hand picture, the measure of $B(\eta, e^{-t})$ can be estimated by considering the measure from the perspective of $g(o)$ of a small ball around ξ.

PROOF. The inequality $\langle \xi|\eta\rangle_o \geq t + \log(2)$ implies that

$$B(\xi, e^{-t}/2) \subseteq B(\eta, e^{-t}) \subseteq B(\xi, 2e^{-t}).$$

Without loss of generality suppose that g satisfies (17.2.8). Since $t > t_\xi$, (4.5.9) guarantees that $B(\xi, 2e^{-t}) \subseteq \mathrm{Shad}(g(o), \sigma_0)$ for some $\sigma_0 > 0$ independent of η and t. Then by the Bounded Distortion Lemma 4.5.6, we have

$$\begin{aligned} B\left(p, e^{-(t-t_\xi)}/(2C)\right) &\subseteq g^{-1}(B(\xi, e^{-t}/2)) \\ &\subseteq g^{-1}(B(\eta, e^{-t})) \\ &\subseteq g^{-1}(B(\xi, 2e^{-t})) \\ &\subseteq B\left(p, 2Ce^{-(t-t_\xi)}\right) \end{aligned}$$

for some $C > 0$, and thus

$$e^{-\delta t_\xi}\mu\left(B\left(p, e^{-(t-t_\xi)}/(2C)\right)\right) \lesssim_\times \mu(B(\eta, e^{-t})) \lesssim_\times e^{-\delta t_\xi}\mu\left(B\left(p, 2Ce^{-(t-t_\xi)}\right)\right).$$

Combining with (17.3.3) completes the proof. □

LEMMA 17.3.4. *For all $\eta \in \Lambda \setminus \{p\}$ and $3R_p \leq R \leq D_p(o, \eta)/2$,*

$$D_p(o, \eta)^{-2\delta}\mathcal{N}_p(R/2) \lesssim_\times \mu(B_p(\eta, R)) \lesssim_\times D_p(o, \eta)^{-2\delta}\mathcal{N}_p(2R).$$

Proof. Since $\eta \in \Lambda\setminus\{p\} \subseteq G_p(S_p)$, there exists $h_\eta \in G_p$ such that $\eta \in h_\eta(S_p)$. Since

$$\bigcup_{\substack{h\in G_p\\ \|h\|_p \leq R-R_p}} h_\eta h(S_p) \subseteq B_p(\eta,R) \subseteq \bigcup_{\substack{h\in G_p\\ \|h\|_p \leq R+R_p}} h_\eta h(S_p),$$

Lemma 17.3.1 gives

$$\sum_{\substack{h\in G_p\\ \|h\|_p \leq R-R_p}} \|h_\eta h\|_p^{-2\delta} \lesssim_\times \mu(B_p(\eta,R)) \lesssim_\times \sum_{\substack{h\in G_p\\ \|h\|_p \leq R+R_p}} \|h_\eta h\|_p^{-2\delta}.$$

The proof will be complete if we can show that for each $h \in G_p$ such that $\|h\|_p \leq R + R_p$, we have

$$\|h_\eta h\|_p \asymp_\times D_p(o,\eta). \tag{17.3.4}$$

And indeed,

$$D_p(\eta, h_\eta h(o)) \leq D_p(\eta, h_\eta(o)) + \|h\|_p \leq R_p + (R+R_p) \leq \frac{5}{6} D_p(o,\eta),$$

demonstrating (17.3.4) with an implied constant of 6. ☐

Corollary 17.3.5. *For all $\eta \in \Lambda\setminus\{p\}$ and $6R_pD(p,\eta)^2 \leq r \leq D(p,\eta)/4$, we have*

$$D(p,\eta)^{2\delta}\mathcal{N}_p\left(\frac{r}{4D(p,\eta)^2}\right) \lesssim_\times \mu(B(\eta,r)) \lesssim_\times D(p,\eta)^{2\delta}\mathcal{N}_p\left(\frac{4r}{D(p,\eta)^2}\right).$$

Proof. By (4.2.2), for every $\zeta \in B_p\left(\eta, \frac{r}{D(p,\eta)(D(p,\eta)+r)}\right)$ we have that

$$\begin{aligned} D(\eta,\zeta) = \frac{D_p(\eta,\zeta)}{D_p(o,\eta)D_p(o,\zeta)} &\leq \frac{D_p(\eta,\zeta)}{D_p(o,\eta)(D_p(o,\eta) - D_p(\eta,\zeta))} \\ &\leq \frac{\frac{r}{D(p,\eta)(D(p,\eta)+r)}}{D_p(o,\eta)\left(D_p(o,\eta) - \frac{r}{D(p,\eta)(D(p,\eta)+r)}\right)} \\ &= r. \end{aligned}$$

Analogously, (4.2.2) also implies that for every $\zeta \in B(\eta,r)$ we have

$$\begin{aligned} D_p(\eta,\zeta) &= \frac{D(\eta,\zeta)}{D(p,\eta)D(p,\zeta)} \\ &\leq \frac{r}{D(p,\eta)\left(D(p,\eta)-r\right)}. \end{aligned}$$

Combining these inequalities gives us that

$$B_p\left(\eta, \frac{r}{D(p,\eta)(D(p,\eta)+r)}\right) \subseteq B(\eta,r) \subseteq B_p\left(\eta, \frac{r}{D(p,\eta)(D(p,\eta)-r)}\right)$$

Now since $r \leq D(p,\eta)/4$, we have

$$B_p\left(\eta, \frac{r}{2D(p,\eta)^2}\right) \subseteq B(\eta,r) \subseteq B_p\left(\eta, \frac{2r}{D(p,\eta)^2}\right).$$

On the other hand, since $6R_pD(p,\eta)^2 \leq r \leq D(p,\eta)/4$, we have

$$3R_p \leq \frac{r}{2D(p,\eta)^2} \leq \frac{2r}{D(p,\eta)^2} \leq \frac{D(p,\eta)}{2}$$

whereupon Lemma 17.3.4 completes the proof. ☐

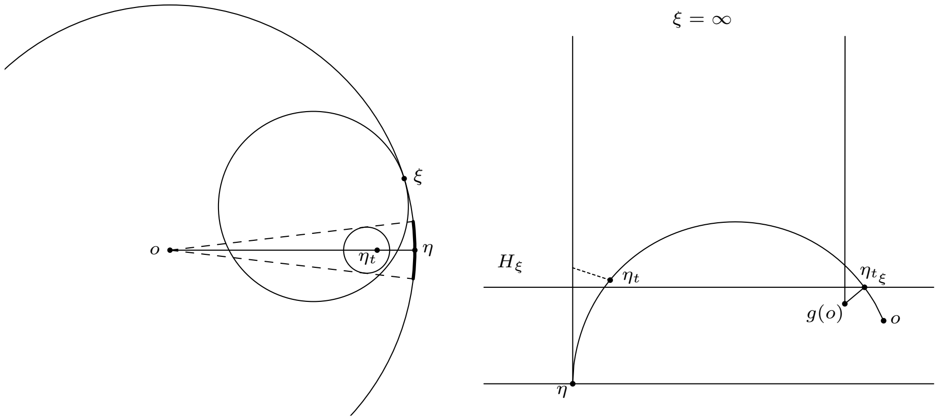

FIGURE 17.3.2. Cusp excursions in the ball model (left) and upper half-space model (right). Since $\xi = g(p) \notin B(\eta, e^{-t})$, our estimate of $\mu(B(\eta, e^{-t}))$ is based on the function $\mathcal{N}_p$, which captures "local" information about the cusp p. In the right-hand picture, the measure of $B(\eta, e^{-t})$ can be estimated by considering the measure from the perspective of $g(o)$ of a large ball around η taken with respect to the D_ξ-metametric.

COROLLARY 17.3.6 (Cf. Figure 17.3.2). *Fix* $\eta \in \Lambda$ *and* $t > 0$ *such that* $\eta_t \in H_\xi$ *for some* $\xi = g(p) \in \Lambda_{\mathrm{bp}}$. *If*

$$\langle \xi | \eta \rangle_o + \tau \leq t \leq 2\langle \xi | \eta \rangle_o - t_\xi - \tau, \tag{17.3.5}$$

then

$$\begin{aligned} e^{-\delta(2\langle \xi|\eta\rangle_o - t_\xi)} \mathcal{N}_p(e^{2\langle \xi|\eta\rangle_o - t_\xi - t - \sigma}) &\lesssim_\times \mu\big(B(\eta, e^{-t})\big) \\ &\lesssim_\times e^{-\delta(2\langle \xi|\eta\rangle_o - t_\xi)} \mathcal{N}_p(e^{2\langle \xi|\eta\rangle_o - t_\xi - t + \sigma}) \end{aligned} \tag{17.3.6}$$

where $\sigma, \tau > 0$ *are independent of* η *and* t.

PROOF. Without loss of generality suppose that g satisfies (17.2.8), and write $\zeta = g^{-1}(\eta)$. Since $t > t_\xi$, (4.5.9) guarantees that $B(\eta, e^{-t}) \subseteq \mathrm{Shad}(g(o), \sigma_0)$ for some $\sigma_0 > 0$ independent of η and t. Then by the Bounded Distortion Lemma 4.5.6, we have

$$B(\zeta, e^{-(t - t_\xi)}/C) \subseteq g^{-1}(B(\eta, e^{-t})) \subseteq B(\zeta, Ce^{-(t - t_\xi)})$$

for some $C > 0$, and thus

$$e^{-\delta t_\xi} \mu(B(\zeta, e^{-(t-t_\xi)}/C)) \lesssim_\times \mu(B(\eta, e^{-t})) \lesssim_\times e^{-\delta t_\xi} \mu(B(\zeta, Ce^{-(t-t_\xi)})).$$

If

$$6R_p D(p, \eta)^2 \leq \frac{e^{-(t-t_\xi)}}{C} \leq Ce^{-(t-t_\xi)} \leq \frac{D(p, \zeta)}{4}, \tag{17.3.7}$$

then Corollary 17.3.5 guarantees that

$$e^{-\delta t_\xi} D(p,\zeta)^{2\delta} \mathcal{N}_p\left(\frac{e^{-(t-t_\xi)}}{4CD(p,\zeta)^2}\right) \lesssim_\times \mu(B(\eta,e^{-t}))$$
$$\lesssim_\times e^{-\delta t_\xi} D(p,\zeta)^{2\delta} \mathcal{N}_p\left(\frac{4Ce^{-(t-t_\xi)}}{D(p,\zeta)^2}\right).$$

On the other hand, since $\xi,\eta \in \mathrm{Shad}(g(o),\sigma_0)$, the Bounded Distortion Lemma 4.5.6 guarantees that

$$D(p,\zeta) \asymp_\times e^{t_\xi} D(\xi,\eta) = e^{-(\langle\xi|\eta\rangle_o - t_\xi)}.$$

Denoting the implied constant by K, we deduce (17.3.6) with $\sigma = \log(4CK^2)$. The proof is completed upon observing that if $\tau = \log(4CK \vee 6R_pCK^2)$, then (17.3.5) implies (17.3.7). ☐

Lemma 17.3.7 (Cf. Lemma 15.4.1). *Fix $\eta \in \Lambda$ and $t > 0$ such that $\eta_t \notin \bigcup(\mathscr{H})$. Then*

$$\mu(B(\eta,e^{-t})) \asymp_\times e^{-\delta t}.$$

Proof. By (12.4.2), there exists $g \in G$ such that $d(g(o),\eta_t) \asymp_+ 0$. By (4.5.9), we have $B(\eta,e^{-t}) \subseteq \mathrm{Shad}(g(o),\sigma)$ for some $\sigma > 0$ independent of η, t. It follows that

$$\mu(B(\eta,e^{-t})) \asymp_\times e^{-\delta t}\mu(g^{-1}(B(\eta,e^{-t}))).$$

To complete the proof it suffices to show that $\mu(g^{-1}(B(\eta,e^{-t})))$ is bounded from below. By the Bounded Distortion Lemma 4.5.6,

$$g^{-1}(B(\eta,e^{-t})) \supseteq B(g^{-1}(\eta),\varepsilon)$$

for some $\varepsilon > 0$ independent of η, t. Now since G is of compact type, we have

$$\inf_{x\in\Lambda} \mu(B(x,\varepsilon)) \geq \min_{x\in S_{\varepsilon/2}} \mu(B(x,\varepsilon/2)) > 0$$

where $S_{\varepsilon/2}$ is a maximal $\varepsilon/2$-separated subset of Λ. This completes the proof. ☐

We are now ready to prove Theorem 17.2.2:

Proof of Theorem 17.2.2. Let σ_0 denote the implied constant of (17.2.5). Then by (17.2.1), for all $\eta \in \Lambda$, $t > 0$, and $\xi \in \Lambda_{\mathrm{bp}}$,

(17.3.8)
$$m(\eta,t) = \begin{cases} e^{-\delta t_\xi}[\mathcal{I}_p(e^{t-t_\xi-\theta}) + \mu(p)] & t_\xi + \sigma_0 \leq t \leq \langle\xi|\eta\rangle_o \\ e^{-\delta(2\langle\xi|\eta\rangle_o - t_\xi)}\mathcal{N}_p(e^{2\langle\xi|\eta\rangle_o - t - t_\xi - \theta}) & \langle\xi|\eta\rangle_o < t \leq 2\langle\xi|\eta\rangle_o - t_\xi - \sigma_0 \\ \text{unknown} & \text{otherwise} \end{cases}$$

Applying this formula to Corollaries 17.3.3 and 17.3.6 yields the following:

Lemma 17.3.8. *There exists $\tau \geq \sigma_0$ such that for all $\eta \in \Lambda$ and $t > 0$.*

(i) *If for some ξ, $t_\xi + \tau \leq t \leq \langle\xi|\eta\rangle_o - \tau$, then* (17.2.2) *holds.*
(ii) *If for some ξ, $\langle\xi|\eta\rangle_o + \tau \leq t \leq 2\langle\xi|\eta\rangle_o - t_\xi - \tau$, then* (17.2.2) *holds.*

Now fix $\eta \in \Lambda$, and let

$$A = \left\{t > 0 : \eta_t \notin \bigcup(\mathscr{H})\right\} \cup \bigcup_{\xi\in\Lambda_{\mathrm{bp}}} [t_\xi + \tau, \langle\xi|\eta\rangle_o - \tau] \cup \bigcup_{\xi\in\Lambda_{\mathrm{bp}}} [\langle\xi|\eta\rangle_o + \tau, 2\langle\xi|\eta\rangle_o - t_\xi - \tau].$$

Then by Lemmas 17.3.7 and 17.3.8, $(17.2.2)_{\sigma=\tau}$ holds for all $t \in A$.

Claim 17.3.9. *Every interval of length* 2τ *intersects* A.

Proof. If $[s-\tau, s+\tau]$ does not intersect A, then by connectedness, there exists $\xi \in \Lambda_{\mathrm{bp}}$ such that $\eta_t \in H_\xi$ for all $t \in [s-\tau, s+\tau]$. By Lemma 17.2.6, the fact that $\eta_{s\pm\tau} \in H_\xi$ implies that $t_\xi \leq s \leq 2\langle\xi|\eta\rangle_o - t_\xi$ (since $\tau \geq \sigma_0$). If $s \leq \langle\xi|\eta\rangle_o$, then $[s-\tau, s+\tau] \cap [t_\xi + \tau, \langle\xi|\eta\rangle_o - \tau] \neq \varnothing$, while if $s \geq \langle\xi|\eta\rangle_o$, then $[s-\tau, s+\tau] \cap [\langle\xi|\eta\rangle_o + \tau, 2\langle\xi|\eta\rangle_o - t_\xi - \tau] \neq \varnothing$. ◁

Thus for all $t > 0$, there exist $t_\pm \in A$ such that $t - 2\tau \leq t_- \leq t \leq t_+ \leq t - 2\tau$; then

$$\begin{aligned} m(\eta, t+3\tau) \lesssim_\times m(\eta, t_+ + \tau) &\lesssim_\times \mu\big(B(\eta, e^{-t_+})\big) \\ &\leq \ \mu\big(B(\eta, e^{-t})\big) \\ &\leq \ \mu\big(B(\eta, e^{-t_-})\big) \lesssim_\times m(\eta, t_- - \tau) \lesssim_\times m(\eta, t - 3\tau), \end{aligned}$$

i.e. $(17.2.2)_{\sigma=3\tau}$ holds. □

17.4. Groups for which μ is doubling

Recall that a measure μ is said to be *doubling* if for all $\eta \in \mathrm{Supp}(\mu)$ and $r > 0$, $\mu(B(\eta, 2r)) \asymp_\times \mu(B(\eta, r))$. In the Standard Case, the Global Measure Formula implies that the δ-conformal measure of a geometrically finite group is always doubling (Example 17.4.11). However, in general there are geometrically finite groups whose δ-conformal measures are not doubling (Example 17.4.12). It is therefore of interest to determine necessary and sufficient conditions on a geometrically finite group for its δ-conformal measure to be doubling. The Global Measure Formula immediately yields the following criterion:

Lemma 17.4.1. μ *is doubling if and only if the function* m *satisfies*

$$m(\eta, t+\tau) \asymp_{\times,\tau} m(\eta, t-\tau) \ \ \forall \eta \in \Lambda \ \ \forall t, \tau > 0. \tag{17.4.1}$$

Proof. If (17.4.1) holds, then (17.2.2) reduces to

$$\mu(B(\eta, e^{-t})) \asymp_\times m(\eta, t), \tag{17.4.2}$$

and then (17.4.1) shows that μ is doubling. On the other hand, if μ is doubling, then (17.2.2) implies that

$$m(\eta, t-\tau) \lesssim_\times \mu(B(\eta, e^{-(t-\tau-\sigma)})) \asymp_\times \mu(B(\eta, e^{-(t+\tau+\sigma)})) \lesssim_\times m(\eta, t+\tau);$$

combining with Proposition 17.2.5 shows that (17.4.1) holds. □

Of course, the criterion (17.4.1) is not very useful by itself, since it refers to the complicated function m. In what follows we find more elementary necessary and sufficient conditions for doubling. First we must introduce some terminology.

Definition 17.4.2. A function $f : [1,\infty) \to [1,\infty)$ is called *doubling* if there exists $\beta > 1$ such that

$$f(\beta R) \lesssim_{\times,\beta} f(R) \ \ \forall R \geq 1, \tag{17.4.3}$$

and *codoubling* if there exists $\beta > 1$ such that

$$f(\beta R) - f(R) \gtrsim_{\times,\beta} f(R) \ \ \forall R \geq 1. \tag{17.4.4}$$

OBSERVATION 17.4.3. If there exists $\beta > 1$ such that

$$\mathcal{N}_p(\beta R) > \mathcal{N}_p(R) \;\; \forall R \geq 1,$$

then $\mathcal{N}_p$ is codoubling.

PROOF. Fix $R \geq 1$; there exists $h \in G_p$ such that $2R < \|h\|_p \leq 2\beta R$. We have

$$h\{j \in G_p : \|j\|_p \leq R\} \subseteq \{j \in G_p : R < \|j\|_p \leq (2\beta + 1)R\},$$

and taking cardinalities gives

$$\mathcal{N}_p(R) \leq \mathcal{N}_p\big((2\beta+1)R\big) - \mathcal{N}_p(R). \qquad \square$$

We are now ready to state a more elementary characterization of when μ is doubling:

PROPOSITION 17.4.4. *μ is doubling if and only if all of the following hold:*

(I) *For all $p \in P$, $\mathcal{N}_p$ is both doubling and codoubling.*

(II) *For all $p \in P$ and $R \geq 1$,*

$$\mathcal{I}_p(R) \asymp_\times R^{-2\delta}\mathcal{N}_p(R). \tag{17.4.5}$$

(III) *G is of divergence type.*

Moreover, (II) *can be replaced by*

(II′) *For all $p \in P$ and $R \geq 1$,*

$$\widetilde{\mathcal{I}}_p(R) := \sum_{k=0}^{\infty} e^{-2\delta k}\mathcal{N}_p(e^k R) \asymp_\times \mathcal{N}_p(R). \tag{17.4.6}$$

PROOF THAT (I)-(III) IMPLY μ DOUBLING. Fix $\eta \in \Lambda$ and $t, \tau > 0$, and we will demonstrate (17.4.1). By (II), (III), and Observation 17.1.1, we have

$$\begin{aligned} m(\eta,t) &\asymp_\times \begin{cases} e^{-\delta t} & \eta_t \notin \bigcup(\mathscr{H}) \\ e^{-\delta t_\xi} e^{-2\delta(t - t_\xi - \theta)}\mathcal{N}_p(e^{t-t_\xi-\theta}) & \eta_t \in H_\xi \text{ and } t \leq \langle \xi|\eta\rangle_o \\ e^{-\delta(2\langle \xi|\eta\rangle_o - t_\xi)}\mathcal{N}_p(e^{2\langle\xi|\eta\rangle_o - t - t_\xi - \theta}) & \eta_t \in H_\xi \text{ and } t > \langle \xi|\eta\rangle_o \end{cases} \\ &\asymp_\times e^{-\delta t}\begin{cases} 1 & \eta_t \notin \bigcup(\mathscr{H}) \\ e^{-\delta b(\eta,t)}\mathcal{N}_p(e^{b(\eta,t)-\theta}) & \eta_t \in H_{g(p)} \end{cases} \end{aligned} \tag{17.4.7}$$

where $b(\eta,t)$ is as in (17.2.6). Let $t_\pm = t \pm \tau$. We split into two cases:

Case 1: $\eta_{t_+}, \eta_{t_-} \in H_{g(p)}$ for some $g(p) \in \Lambda_{\text{bp}}$. In this case, (17.4.1) follows from Corollary 17.2.7 together with the fact that $\mathcal{N}_p$ is doubling.

Case 2: $\eta_{t+s} \notin \bigcup(\mathscr{H})$ for some $s \in [-\tau, \tau]$. In this case, Corollary 17.2.7 shows that $b(\eta, t_\pm) \asymp_{+,\tau} 0$ and thus

$$m(\eta, t_+) \asymp_{\times,\tau} e^{-\delta t} \asymp_{\times,\tau} m(\eta, t_-). \qquad \square$$

Before continuing the proof of Proposition 17.4.4, we observe that

$$\begin{aligned} \mathcal{I}_p(R) + R^{-2\delta}\mathcal{N}_p(R) \asymp_\times \sum_{h\in G_p}(R \vee \|h\|_p)^{-2\delta} &\asymp_\times \sum_{h \in G_p}\sum_{k=1}^{\infty}(e^k R)^{-2\delta}[e^k R \geq \|h\|_p] \\ &= \sum_{k=1}^{\infty}(e^k R)^{-2\delta}\mathcal{N}_p(e^k R) \\ &= R^{-2\delta}\widetilde{\mathcal{I}}_p(R). \end{aligned}$$

In particular, it follows that (17.4.6) is equivalent to

$$\mathcal{I}_p(R) \lesssim_\times R^{-2\delta}\mathcal{N}_p(R). \tag{17.4.8}$$

PROOF THAT (I) and (II′) IMPLY (II). Since $\mathcal{N}_p$ is codoubling, let $\beta > 1$ be as in (17.4.4). Then

$$\mathcal{I}_p(R) \geq \sum_{\substack{h\in G_p \\ R<\|h\|_p\leq \beta R}} (\beta R)^{-2\delta} = (\beta R)^{-2\delta}(\mathcal{N}_p(\beta R) - \mathcal{N}_p(R)) \gtrsim_{\times,\beta} R^{-2\delta}\mathcal{N}_p(R).$$

Combining with (17.4.8) completes the proof. □

PROOF THAT μ DOUBLING IMPLIES (I)-(III) AND (II′). Since a doubling measure whose topological support is a perfect set cannot have an atomic part, we must have $\mu(P) = 0$ and thus by Observation 17.1.1, (III) holds. Since

$$m(p,t) \asymp_{\times,p} \mathcal{I}_p(e^{t-t_0-\theta}) + \mu(p) = \mathcal{I}_p(e^{t-t_0-\theta})$$

for all sufficiently large t, setting $\eta = p$ in (17.4.1) shows that the function $\mathcal{I}_p$ is doubling.

Fix $\eta \in \Lambda \setminus \{p\}$. Let $\sigma_0 > 0$ denote the implied constant of (17.2.5). For $s \in [t_0 + \sigma_0 + \tau, \langle p|\eta\rangle_o - \tau]$, plugging $t = 2\langle p|\eta\rangle_o - s$ into (17.4.1) and simplifying using $(17.3.8)_{\xi=p}$ shows that

$$\mathcal{N}_p(e^{s-\tau-t_0-\theta}) \asymp_{\times,\tau} \mathcal{N}_p(e^{s+\tau-t_0-\theta}). \tag{17.4.9}$$

Since $\langle p|\eta\rangle_o$ can be made arbitrarily large, (17.4.9) holds for all $s \geq t_0 + \sigma_0 + \tau$. It follows that $\mathcal{N}_p$ is doubling.

Next, we compare the values of $m(\eta, \langle p|\eta\rangle_o \pm \tau)$. This gives (assuming $\langle p|\eta\rangle_o > t_0 + \sigma_0 + \tau$)

$$e^{-\delta t_0}\mathcal{I}_p(e^{\langle p|\eta\rangle_o-\tau-t_0-\theta}) \asymp_\times e^{-\delta(2\langle p|\eta\rangle_o-t_0)}\mathcal{N}_p(e^{\langle p|\eta\rangle_o-\tau-t_0-\theta}).$$

Letting $R_\eta = \exp(\langle p|\eta\rangle_o - \tau - t_0 - \theta)$, we have

$$\mathcal{I}_p(R_\eta) \asymp_\times R_\eta^{-2\delta}\mathcal{N}_p(R_\eta). \tag{17.4.10}$$

Now fix $\zeta \in \Lambda \setminus \{p\}$ and $h \in G_p$, and let $\eta = h(\zeta)$. Then $D_p(h(o), \eta) \asymp_{+,\zeta} 0$, and thus the triangle inequality gives

$$1 \leq D_p(o, \eta) \asymp_{+,\zeta} \|h\|_p \geq 1,$$

and so $R_\eta \asymp_\times D_p(o, \eta) \asymp_{\times,\zeta} \|h\|_p$. Combining with (17.4.10) and the fact that the functions $\mathcal{I}_p$ and $\mathcal{N}_p$ are doubling, we have

$$\mathcal{I}_p(\|h\|_p) \asymp_\times \|h\|_p^{-2\delta}\mathcal{N}_p(\|h\|_p) \tag{17.4.11}$$

for all $h \in G_p$.

Now fix $1 \leq R_1 < R_2$ such that $\|h_i\|_p = R_i$ for some $h_1, h_2 \in G_p$, but such that the formula $R_1 < \|h\|_p < R_2$ is not satisfied for any $h \in G_p$. Then

$$\lim_{R\searrow R_1} \mathcal{I}_p(R) = \lim_{R\nearrow R_2} \mathcal{I}_p(R) \text{ and } \lim_{R\searrow R_1} \mathcal{N}_p(R) = \lim_{R\nearrow R_2} \mathcal{N}_p(R).$$

On the other hand, applying (17.4.11) with $h = h_1, h_2$ gives

$$\mathcal{I}_p(R_i) \asymp_\times R_i^{-2\delta}\mathcal{N}_p(R_i).$$

Since $\mathcal{I}_p$ and $\mathcal{N}_p$ are doubling, we have

$$R_1^{-2\delta} \asymp_\times \frac{\mathcal{I}_p(R_1)}{\mathcal{N}_p(R_1)} \asymp_\times \frac{\lim_{R\searrow R_1}\mathcal{I}_p(R)}{\lim_{R\searrow R_1}\mathcal{N}_p(R)} = \frac{\lim_{R\nearrow R_2}\mathcal{I}_p(R)}{\lim_{R\nearrow R_2}\mathcal{N}_p(R)} \asymp_\times \frac{\mathcal{I}_p(R_2)}{\mathcal{N}_p(R_2)} \asymp_\times R_2^{-2\delta}$$

and thus $R_1 \asymp_\times R_2$. Since R_1, R_2 were arbitrary, Observation 17.4.3 shows that $\mathcal{N}_p$ is codoubling. This completes the proof of (I).

It remains to demonstrate (II) and (II′). Given any $R \geq 1$, since $\mathcal{N}_p$ is codoubling, we may find $h \in G_p$ such that $\|h\|_p \asymp_\times R$; combining with (17.4.11) and the fact that $\mathcal{I}_p$ and $\mathcal{N}_p$ are doubling gives (17.4.5) and (17.4.8), demonstrating (II) and (II′). □

We note that the proof actually shows the following (cf. (17.4.7)):

Corollary 17.4.5. *If μ is doubling, then*

$$\mu(B(\eta, e^{-t})) \asymp_\times e^{-\delta t}\begin{cases}1 & \eta_t \notin \bigcup(\mathscr{H})\\ e^{-\delta b(\eta,t)}\mathcal{N}_p(e^{b(\eta,t)}) & \eta_t \in H_{g(p)}\end{cases}$$

for all $\eta \in \Lambda$, $t > 0$. Here $b(\eta,t)$ is as in (17.2.6).

Although Proposition 17.4.4 is the best necessary and sufficient condition we can give for doubling, in what follows we give necessary conditions and sufficient conditions which are more elementary (Proposition 17.4.8), although the necessary conditions are not the same as the sufficient conditions. In practice these conditions are usually powerful enough to determine whether any given measure is doubling.

To state the result, we need the concept of the *polynomial growth rate* of a function:

Definition 17.4.6 (Cf. (11.2.4)). The *(polynomial) growth rate* of a function $f : [1,\infty) \to [1,\infty)$ is the limit

$$\alpha(f) := \lim_{\lambda,R\to\infty} \frac{\log f(\lambda R) - \log f(R)}{\log(\lambda)}$$

if it exists. If the limit does not exist, then the numbers

$$\alpha^*(f) := \limsup_{\lambda,R\to\infty} \frac{\log f(\lambda R) - \log f(R)}{\log(\lambda)}$$

$$\alpha_*(f) := \liminf_{\lambda,R\to\infty} \frac{\log f(\lambda R) - \log f(R)}{\log(\lambda)}$$

are the *upper* and *lower polynomial growth rates* of f, respectively.

Lemma 17.4.7. *Let $f : [1,\infty) \to [1,\infty)$.*

(i) *f is doubling if and only if $\alpha^*(f) < \infty$.*
(ii) *f is codoubling if and only if $\alpha_*(f) > 0$.*
(iii)

$$\alpha_*(f) \leq \liminf_{\lambda\to\infty} \frac{\log f(\lambda)}{\log(\lambda)} \leq \limsup_{\lambda\to\infty} \frac{\log f(\lambda)}{\log(\lambda)} \leq \alpha^*(f).$$

In particular, $\alpha_*(\mathcal{N}_p) \leq 2\delta_p \leq \alpha^*(\mathcal{N}_p)$.

Proof of (i). Suppose that f is doubling, and let $C > 1$ denote the implied constant of (17.4.3). Iterating gives

$$f(\beta^n R) \leq C^n f(R) \;\; \forall n \in \mathbb{N} \;\; \forall R \geq 1$$

and thus

$$f(\lambda R) \lesssim_\times \lambda^{\log_\beta(C)} f(R) \;\; \forall \lambda > 1 \;\; \forall R \geq 1.$$

It follows that $\alpha^*(f) \leq \log_\beta(C) < \infty$. The converse direction is trivial. □

Proof of (ii). Suppose that f is codoubling, and let $C > 1$ denote the implied constant of (17.4.4). Then

$$f(\beta R) \geq (1 + C^{-1}) f(R) \;\; \forall R \geq 1.$$

Iterating gives

$$f(\beta^n R) \geq (1 + C^{-1})^n f(R) \;\; \forall n \in \mathbb{N} \;\; \forall R \geq 1$$

and thus

$$f(\lambda R) \gtrsim_\times \lambda^{\log_\beta(1+C^{-1})} f(R) \;\; \forall \lambda > 1 \;\; \forall R \geq 1.$$

It follows that $\alpha_*(f) \geq \log_\beta(1 + C^{-1}) > 0$. The converse direction is trivial. □

Proof of (iii). Let $R_n \to \infty$. For each $n \in \mathbb{N}$,

$$\limsup_{\lambda \to \infty} \frac{\log f(\lambda R_n) - \log f(R_n)}{\log(\lambda)} = \overline{s} := \limsup_{\lambda \to \infty} \frac{\log f(\lambda)}{\log(\lambda)}.$$

Thus given $s < \overline{s}$, we may find a large number $\lambda_n > 1$ such that

$$\frac{\log f(\lambda_n R_n) - \log f(R_n)}{\log(\lambda_n)} > s.$$

Since $\lambda_n, R_n \to \infty$ as $n \to \infty$, it follows that $\alpha^*(f) \geq s$; since s was arbitrary, $\alpha^*(f) \geq \overline{s}$. A similar argument shows that $\alpha_*(f) \leq \underline{s}$.

Finally, when $f = \mathcal{N}_p$, the equality $\overline{s} = \underline{s} = 2\delta_p$ is a consequence of (8.1.2) and Observation 6.2.10. □

We can now state our final result regarding criteria for doubling:

Proposition 17.4.8. *In the following list,* (A) ⇒ (B) ⇒ (C):

(A) *For all* $p \in P$, $0 < \alpha_*(\mathcal{N}_p) \leq \alpha^*(\mathcal{N}_p) < 2\delta$.
(B) μ *is doubling.*
(C) *For all* $p \in P$, $0 < \alpha_*(\mathcal{N}_p) \leq \alpha^*(\mathcal{N}_p) \leq 2\delta$.

Proof of (A) ⇒ (B). Suppose that (A) holds. Then by Lemma 17.4.7, (I) of Proposition 17.4.4 holds. Since $\delta_p \leq \alpha^*(\mathcal{N}_p)/2 < \delta$ for all $p \in P$, Theorem 17.1.2 implies that (III) of Proposition 17.4.4 holds. To complete the proof, we need to show that (II′) of Proposition 17.4.4 holds. Fix $s \in (\alpha^*(\mathcal{N}_p), 2\delta)$. Since $s > \alpha^*(\mathcal{N}_p)$, we have

$$\mathcal{N}_p(\lambda R) \lesssim_{\times,s} \lambda^s \mathcal{N}_p(R) \;\; \forall \lambda > 1, \; R \geq 1$$

and thus

$$\mathcal{N}_p(R) \leq \widetilde{\mathcal{I}}_p(R) \lesssim_\times \sum_{k=0}^{\infty} e^{-2\delta k} e^{sk} \mathcal{N}_p(R) \asymp_\times \mathcal{N}_p(R),$$

demonstrating (17.4.6) and completing the proof. □

Proof of (B) ⇒ (C). Suppose μ is doubling. By (I) of Proposition 17.4.4, $\alpha_*(\mathcal{N}_p) > 0$. On the other hand, by (17.4.6) we have

$$\lambda^{-2\delta}\mathcal{N}_p(\lambda R) \lesssim_\times \mathcal{N}_p(R) \;\; \forall \lambda > 1, \; R \geq 1$$

and thus $\alpha^*(\mathcal{N}_p) \leq 2\delta$. □

Proposition 17.4.4 shows that if G is a geometrically finite group with δ-conformal measure μ, then the question of whether μ is doubling is determined entirely by its parabolic subgroups $(G_p)_{p\in P}$ and its Poincaré set Δ_G. A natural question is when the second input can be removed, that is: if we are told what the parabolic subgroups $(G_p)_{p\in P}$ are, can we sometimes determine whether μ is doubling without looking at Δ_G? A trivial example is that if $\alpha_*(\mathcal{N}_p) = 0$ or $\alpha^*(\mathcal{N}_p) = \infty$ for some $p \in P$, then we automatically know that μ is not doubling. Conversely, the following definition and proposition describe when we can deduce that μ is doubling:

Definition 17.4.9. A parabolic group $H \leq \mathrm{Isom}(X)$ with global fixed point $p \in \partial X$ is *pre-doubling* if

$$0 < \alpha_*(\mathcal{N}_{\mathcal{E}_p,H}) \leq \alpha^*(\mathcal{N}_{\mathcal{E}_p,H}) = 2\delta_H < \infty \tag{17.4.12}$$

and H is of divergence type.

Proposition 17.4.10.

(i) *If G_p is pre-doubling for every $p \in P$, then μ is doubling.*
(ii) *Let $H \leq \mathrm{Isom}(X)$ be a parabolic subgroup, and let $g \in \mathrm{Isom}(X)$ be a loxodromic isometry such that $\langle g, H\rangle$ is a strongly separated Schottky product. Then the following are equivalent:*
   (A) *H is pre-doubling.*
   (B) *For every $n \in \mathbb{N}$, the δ_n-quasiconformal measure μ_n of $G_n = \langle g^n, H\rangle$ is doubling. Here we assume that $\delta_n := \delta(G_n) < \infty$.*

Proof of (i). For all $p \in P$, the fact that G_p is of divergence type implies that $\delta > \delta_p$ (Proposition 10.3.10); combining with (17.4.12) gives $0 < \alpha_*(\mathcal{N}_p) \leq \alpha^*(\mathcal{N}_p) < 2\delta$. Proposition 17.4.8 completes the proof. □

Proof of (ii). Since (up to equivalence) the only parabolic point of G_n is the global fixed point of H (Proposition 12.4.19), the implication (A) ⇒ (B) follows from part (i). Conversely, suppose that (B) holds. Then by Proposition 17.4.8, we have

$$0 < \alpha_*(\mathcal{N}_{\mathcal{E}_p,H}) \leq \alpha^*(\mathcal{N}_{\mathcal{E}_p,H}) \leq 2\delta_n < \infty.$$

Since $\delta_n \to \delta_H$ as $n \to \infty$ (Proposition 10.3.7(iv)), taking the limit and combining with the inequality $2\delta_H \leq \alpha^*(\mathcal{N}_{\mathcal{E}_p,H})$ yields (17.4.12). On the other hand, by Proposition 17.4.4, for each n, G_n is of divergence type, so applying Proposition 10.3.7(iv) again, we see that H is of divergence type. □

Example 17.4.11. If

$$\mathcal{N}_p(R) \asymp_\times R^{2\delta_p} \;\; \forall p \in P, \tag{17.4.13}$$

then the groups $(G_p)_{p\in P}$ are pre-doubling, and thus by Proposition 17.4.10(i), μ is doubling. Combining with Corollary 17.4.5 gives

$$\mu(B(\eta, e^{-t})) \asymp_\times e^{-\delta t}\begin{cases} 1 & \eta_t \notin \bigcup(\mathscr{H}) \\ e^{(2\delta_\xi - \delta)b(\eta,t)} & \eta_t \in H_\xi \end{cases}.$$

This generalizes B. Schapira's global measure formula [**153**, Théorème 3.2] to the setting of regularly geodesic strongly hyperbolic metric spaces.

We remark that the asymptotic (17.4.13) is satisfied whenever X is a finite-dimensional algebraic hyperbolic space; see e.g. [**137**, Lemma 3.5]. In particular, specializing Schapira's global measure formula to the settings of finite-dimensional algebraic hyperbolic spacess and finite-dimensional real hyperbolic spaces give the global measure formulas of Newberger [**137**, Main Theorem] and Stratmann–Velani–Sullivan [**160**, Theorem 2], [**165**, Theorem on p.271], respectively.

By contrast, when $X = \mathbb{H} = \mathbb{H}^\infty$, the asymptotic (17.4.13) is usually not satisfied. Let us summarize the various behaviors that we have seen for the orbital counting functions of parabolic groups acting on $\mathbb{H}^\infty$, and their implications for doubling:

Examples 17.4.12 (Examples of doubling and non-doubling Patterson–Sullivan measures of geometrically finite subgroups of $\mathrm{Isom}(\mathbb{H}^\infty)$).

1. In the proof of Theorem 11.2.11 (cf. Remark 11.2.12), we saw that if Γ is a finitely generated virtually nilpotent group and if $f : [1, \infty) \to [1, \infty)$ is a function satisfying
$$\alpha_\Gamma < \alpha_*(f) \leq \alpha^*(f) < \infty,$$
then there exists a parabolic group $H \leq \mathrm{Isom}(\mathbb{H}^\infty)$ isomorphic to Γ whose orbital counting function is asymptotic to f. Now, a group H constructed in this way may or may not be pre-doubling; it depends on the chosen function f. We note that by applying Proposition 17.4.10(ii) to such a group, one can construct examples of geometrically finite subgroups of $\mathrm{Isom}(\mathbb{H}^\infty)$ whose Patterson–Sullivan measures are not doubling. On the other hand, for any parabolic group H constructed in this way, if H is embedded into a geometrically finite group G with sufficiently large Poincaré exponent (namely $2\delta_G > \alpha^*(f)$), then the Patterson–Sullivan measure of G may be doubling (assuming that no other parabolic subgroups of G are causing problems).
2. In Theorem 14.1.5, we showed that if $f : [0, \infty) \to \mathbb{N}$ satisfies the condition
$$\forall 0 \leq R_1 \leq R_2 \; f(R_1) \text{ divides } f(R_2),$$
then there exists a parabolic subgroup of $\mathrm{Isom}(\mathbb{H}^\infty)$ whose orbital counting function is equal to f. This provides even more examples of parabolic groups which are not pre-doubling. In particular, it provides examples of parabolic groups H which satisfy either $\alpha_*(\mathcal{N}_H) = 0$ or $\alpha^*(\mathcal{N}_H) = \infty$ (cf. Example 11.2.18); such groups cannot be embedded into any geometrically finite group with a doubling Patterson–Sullivan measure.

Note that example 2 can be used to construct a geometrically finite group acting isometrically on an $\mathbb{R}$-tree which does not have a doubling Patterson–Sullivan measure. On the other hand, example 1 has no analogue in $\mathbb{R}$-trees by Remark 6.1.8.

17.5. Exact dimensionality of μ

We now turn to the question of the fractal dimensions of the measure μ. We recall that the *Hausdorff dimension* and *packing dimension* of a measure μ on ∂X

are defined by the formulas

$$\dim_H(\mu) = \inf\{\dim_H(A) : \mu(\partial X \setminus A) = 0\}$$
$$\dim_P(\mu) = \inf\{\dim_P(A) : \mu(\partial X \setminus A) = 0\}.$$

If G is of convergence type, then μ is atomic, so $\dim_H(\mu) = \dim_P(\mu) = 0$. Consequently, for the remainder of this chapter we make the

STANDING ASSUMPTION 17.5.1. G is of divergence type.

Given this assumption, it is natural to expect that $\dim_H(\mu) = \dim_P(\mu) = \delta$. Indeed, the inequality $\dim_H(\mu) \leq \delta$ follows immediately from Theorems 1.2.1 and 12.4.5, and in the Standard Case equality holds [**160**, Proposiiton 4.10]. Even stronger than the equalities $\dim_H(\mu) = \dim_P(\mu) = \delta$, it is natural to expect that μ is *exact dimensional*:

DEFINITION 17.5.2. A measure μ on a metric space (Z, D) is called *exact dimensional of dimension s* if the limit

$$d_\mu(\eta) := \lim_{t\to\infty} \frac{1}{t} \log \frac{1}{\mu(B(\eta, e^{-t}))} \tag{17.5.1}$$

exists and equals s for μ-a.e. $\eta \in Z$.

For example, every Ahlfors s-regular measure is exact dimensional of dimension s.

If the limit in (17.5.1) does not exist, then we denote the lim inf by $\underline{d}_\mu(\eta)$ and the lim sup by $\overline{d}_\mu(\eta)$.

PROPOSITION 17.5.3 ([**127**, §8]). *For any measure μ on a metric space (Z, D),*

$$\dim_H(\mu) = \operatorname*{ess\,sup}_{\eta\in Z} \underline{d}_\mu(\eta)$$
$$\dim_P(\mu) = \operatorname*{ess\,sup}_{\eta\in Z} \overline{d}_\mu(\eta).$$

In particular, if μ is exact dimensional of dimension s, then

$$\dim_H(\mu) = \dim_P(\mu) = s.$$

Combining Proposition 17.5.3 with Lemma 17.3.7 and Observation 17.1.1 immediately yields the following:

OBSERVATION 17.5.4. If μ is the Patterson–Sullivan measure of a geometrically finite group of divergence type, then

$$\dim_H(\mu) \leq \delta \leq \dim_P(\mu).$$

In particular, if μ is exact dimensional, then μ is exact dimensional of dimension δ.

It turns out that μ is not necessarily exact dimensional (Example 17.5.14), but counterexamples to exact dimensionality must fall within a very narrow window (Theorem 17.5.9), and in particular if μ is doubling then μ is exact dimensional (Corollary 17.5.12). As a first step towards these results, we will show that exact dimensionality is equivalent to a certain Diophantine condition. For this, we need to recall some results from [**73**].

17.5.1. Diophantine approximation on Λ. Classically, Diophantine approximation is concerned with the approximation of a point $x \in \mathbb{R} \setminus \mathbb{Q}$ by a rational number $p/q \in \mathbb{Q}$. The two important quantities are the *error term* $|x - p/q|$ and the *height* q. Given a function $\Psi : \mathbb{N} \to [0, \infty)$, the point $x \in \mathbb{R} \setminus \mathbb{Q}$ is said to be Ψ-*approximable* if

$$\left| x - \frac{p}{q} \right| \leq \Psi(q) \text{ for infinitely many } p/q \in \mathbb{Q}.$$

In the setting of a group acting on a hyperbolic metric space, we can instead talk about *dynamical* Diophantine approximation, which is concerned with the approximation of a point $\eta \in \Lambda$ by points $g(\xi) \in G(\xi)$, where $\xi \in \Lambda$ is a distinguished point. For this to make sense, one needs a new definition of error and height: the error term is defined to be $D(g(\xi), \eta)$, and the height is defined to be $b^{\|g\|}$. (If there is more than one possibility for g, it may be chosen so as to minimize the height.) Some motivation for these definitions comes from considering classical Diophantine approximation as a special case of dynamical Diophantine approximation which occurs when $X = \mathbb{H}^2$ and $G = \mathrm{SL}_2(\mathbb{Z})$; see e.g. [**73**, Observation 1.15] for more details. Given a function $\Phi : [0, \infty) \to (0, \infty)$, the point $\eta \in \Lambda$ is said to be Φ, ξ-*well approximable* if for every $K > 0$ there exists $g \in G$ such that

$$D(g(\xi), \eta) \leq \Phi(K b^{\|g\|}) \text{ for infinitely many } g \in G$$

(cf. [**73**, Definition 1.36]). Moreover, η is said to be ξ-*very well approximable* if

$$\omega_\xi(\eta) := \limsup_{\substack{g \in G \\ g(\xi) \to \eta}} \frac{-\log_b D(g(\xi), \eta)}{\|g\|} > 1$$

(cf. [**73**, p.9]). The set of Φ, ξ-well approximable points is denoted $\mathrm{WA}_{\Phi,\xi}$, while the set of ξ-very well approximable points is denoted VWA_ξ. Finally, a point η is said to be *Liouville* if $\omega_\xi(\eta) = \infty$; the set of Liouville points is denoted $\mathrm{Liouville}_\xi$.

In the following theorems, we return to the setting of Standing Assumptions 17.0.1 and 17.5.1.

Theorem 17.5.5 (Corollary of [**73**, Theorem 8.1]). *Fix $p \in P$, and let $\Phi : [0, \infty) \to (0, \infty)$ be a function such that the function $t \mapsto t\Phi(t)$ is nonincreasing. Then*

(i) $\mu(\mathrm{WA}_{\Phi,p}) = 0$ or 1 *according to whether the series*

$$\sum_{g \in G} e^{-\delta\|g\|} \mathcal{I}_p \left(\frac{1}{e^{\|g\|} \Phi(K e^{\|g\|})} \right) \tag{17.5.2}$$

converges for some $K > 0$ or diverges for all $K > 0$, respectively.

(ii) $\mu(\mathrm{VWA}_p) = 0$ or 1 *according to whether the series*

$$\Sigma_{\mathrm{div}}(p, \kappa) := \sum_{g \in G} e^{-\delta\|g\|} \mathcal{I}_p(e^{\kappa\|g\|}) \tag{17.5.3}$$

converges for all $\kappa > 0$ or diverges for some $\kappa > 0$, respectively.

(iii) $\mu(\mathrm{Liouville}_p) = 0$ or 1 *according to whether the series $\Sigma_{\mathrm{div}}(p, \kappa)$ converges for some $\kappa > 0$ or diverges for all $\kappa > 0$, respectively.*

PROOF. Standing Assumption 17.5.1, Theorem 1.4.1, and Observation 17.1.1 imply that μ is ergodic and that $\mu(p) = 0$, thus verifying the hypotheses of [**73**, Theorem 8.1]. Theorem 17.2.2 shows that

$$\mathcal{I}_p(C_1/r) \lesssim_{\times,p} \mu(B(p,r)) \lesssim_{\times,p} \mathcal{I}_p(C_2/r)$$

for some constants $C_1 \geq 1 \geq C_2 > 0$. Thus for all $K > 0$,

$$\begin{aligned}
&\sum_{g\in G} e^{-\delta\|g\|}\mathcal{I}_p\left(\frac{1}{e^{\|g\|}\Phi(KC_1 e^{\|g\|})}\right)\\
&\leq \sum_{g\in G} e^{-\delta\|g\|}\mathcal{I}_p\left(\frac{C_1}{e^{\|g\|}\Phi(K e^{\|g\|})}\right)\\
&\lesssim_\times [\mathbf{73}, (8.1)]\\
&\lesssim_\times \sum_{g\in G} e^{-\delta\|g\|}\mathcal{I}_p\left(\frac{C_2}{e^{\|g\|}\Phi(K e^{\|g\|})}\right)\\
&\leq \sum_{g\in G} e^{-\delta\|g\|}\mathcal{I}_p\left(\frac{1}{e^{\|g\|}\Phi((K/C_1) e^{\|g\|})}\right).
\end{aligned}$$

Thus, [**73**, (8.1)] diverges for all $K > 0$ if and only if (17.5.2) diverges for all $K > 0$. This completes the proof of (i). To demonstrate (ii) and (iii), simply note that $\mathrm{VWA}_p = \bigcup_{c>0} \mathrm{WA}_{\Phi_c,p}$ and $\mathrm{Liouville}_p = \bigcap_{c>0} \mathrm{WA}_{\Phi_c,p}$, where $\Phi_c(t) = t^{-(1+c)}$, and apply (i). The constant K may be absorbed by a slight change of κ. □

THEOREM 17.5.6 (Corollary of [**73**, Theorem 7.1]). *For all $\xi \in \Lambda$ and $c > 0$,*

$$\dim_H(\mathrm{WA}_{\Phi_c,\xi}) \leq \frac{\delta}{1+c},$$

where $\Phi_c(t) = t^{-(1+c)}$ as above. In particular, $\dim_H(\mathrm{Liouville}_\xi) = 0$, and VWA_ξ can be written as the countable union of sets of Hausdorff dimension strictly less than δ.

(No proof is needed as this follows directly from [**73**, Theorem 7.1].)

There is a relation between dynamical Diophantine approximation by the orbits of parabolic points and the lengths of cusp excursions along geodesics. A well-known example is that a point $\eta \in \Lambda$ is dynamically badly approximable with respect to every parabolic point if and only if the geodesic $[o,\eta]$ has bounded cusp excursion lengths [**73**, Proposition 1.21]. The following observation is in a similar vein:

OBSERVATION 17.5.7. For $\eta \in \Lambda$, we have:

$$\eta \in \bigcup_{p\in P} \mathrm{VWA}_p \;\Leftrightarrow\; \limsup_{\substack{\xi\in\Lambda_{\mathrm{bp}}\\ t_\xi\to\infty}} \frac{\langle\xi|\eta\rangle - t_\xi}{t_\xi} > 0 \;\Leftrightarrow\; \limsup_{t\to\infty} \frac{b(\eta,t)}{t} > 0$$

$$\eta \in \bigcup_{p\in P} \mathrm{Liouville}_p \;\Leftrightarrow\; \limsup_{\substack{\xi\in\Lambda_{\mathrm{bp}}\\ t_\xi\to\infty}} \frac{\langle\xi|\eta\rangle - t_\xi}{t_\xi} = \infty \;\Leftrightarrow\; \limsup_{t\to\infty} \frac{b(\eta,t)}{t} = 1.$$

PROOF. If $\xi = g(p) \in \Lambda_{\mathrm{bp}}$, then $\|g\| \gtrsim_+ t_\xi$, with $\asymp_+$ for at least one value of g (Lemma 17.2.8). Thus

$$\max_{p\in P} \omega_p(\eta) = \max_{p\in P} \limsup_{\substack{g\in G\\ g(p)\to\eta}} \frac{\log D(g(p),\eta)}{\|g\|} = \limsup_{\substack{\xi\in\Lambda_{\mathrm{bp}}\\ \xi\to\eta}} \frac{\langle \xi|\eta\rangle}{t_\xi},$$

so

$$\limsup_{\substack{\xi\in\Lambda_{\mathrm{bp}}\\ t_\xi\to\infty}} \frac{\langle \xi|\eta\rangle - t_\xi}{t_\xi} = \max_{p\in P} \omega_p(\eta) - 1. \tag{17.5.4}$$

On the other hand, it is readily verified that if $[o,\eta]$ intersects H_ξ, then the function $f(t) = b(\eta,t)/t$ attains its maximum at $t = \langle \xi|\eta\rangle_o$, at which $f(t) = \langle \xi|\eta\rangle_o - t_\xi$. Thus we have that

$$\begin{aligned} \limsup_{t\to\infty} \frac{b(\eta,t)}{t} &= \limsup_{\substack{\xi\in\Lambda_{\mathrm{bp}}\\ t_\xi\to\infty}} \sup_{\substack{t>0\\ \eta_t\in H_\xi}} \frac{b(\eta,t)}{t} = \limsup_{\substack{\xi\in\Lambda_{\mathrm{bp}}\\ t_\xi\to\infty}} \frac{\langle \xi|\eta\rangle_o - t_\xi}{\langle \xi|\eta\rangle_o} \\ &= 1 - \frac{1}{\max_{p\in P}\omega_p(\eta)} \end{aligned} \tag{17.5.5}$$

Since

$$\max_{p\in P} \omega_p(\eta) \begin{cases} = \infty & \eta \in \bigcup_{p\in P} \mathrm{Liouville}_p \\ \in (1,\infty) & \eta \in \bigcup_{p\in P} \mathrm{VWA}_p \setminus \bigcup_{p\in P} \mathrm{Liouville}_p \\ = 1 & \eta \notin \bigcup_{p\in P} \mathrm{VWA}_p \end{cases}$$

applying (17.5.4) and (17.5.5) completes the proof. □

We are now ready to state our main theorem regarding the relation between exact dimensionality and dynamical Diophantine approximation:

THEOREM 17.5.8. *The following are equivalent:*

(A) $\mu(\mathrm{VWA}_p) = 0\ \ \forall p \in P$.
(B) μ *is exact dimensional.*
(C) $\dim_H(\mu) = \delta$.
(D) $\mu(\mathrm{VWA}_\xi) = 0\ \ \forall \xi \in \Lambda$.

The implication (B) ⇒ (C) is part of Proposition 17.5.3, while (C) ⇒ (D) is an immediate consequence of Theorem 17.5.6, and (D) ⇒ (A) is trivial. Thus we demonstrate (A) ⇒ (B):

PROOF OF (A) ⇒ (B). Fix $\eta \in \Lambda \setminus \bigcup_{p\in P} \mathrm{VWA}_p$ and $t > 0$. Suppose that $\eta_t \in H_\xi$ for some $\xi \in \Lambda_{\mathrm{bp}}$. Let $t_- < t < t_+$ satisfy

$$t_- \asymp_+ t_\xi,\ t_+ \asymp_+ 2\langle \xi|\eta\rangle_o - t_\xi,\ \text{and } \eta_{t_\pm} \notin \bigcup(\mathscr{H}).$$

Then by Lemma 17.3.7,

$$\mu(B(\eta, e^{-t_\pm})) \asymp_\times e^{-\delta t_\pm}.$$

In particular

$$\delta t_- \lesssim_+ \log \frac{1}{\mu(B(\eta,e^{-t}))} \lesssim_+ \delta t_+. \tag{17.5.6}$$

Now, by Observation 17.5.7, we have

$$\frac{t_+ - t_-}{t} \leq \frac{2(\langle \xi|\eta\rangle_o - t_\xi + (\text{constant}))}{t_\xi} \to 0 \text{ as } t\to\infty.$$

Since $t_- < t < t_+$, it follows that $t_-/t, t_+/t \to 1$ as $t \to \infty$. Combining with (17.5.6) gives $d_\mu(\eta) = \delta$ (cf. (17.5.1)). But by assumption (A), this is true for μ-a.e. $\eta \in \Lambda$. Thus μ is exact dimensional. □

17.5.2. Examples and non-examples of exact dimensional measures. Combining Theorems 17.5.8 and 17.5.5 gives a necessary and sufficient condition for μ to be exact dimensional in terms of the convergence or divergence of a family of series. We can ask how often this condition is satisfied. Our first result shows that it is almost always satisfied:

THEOREM 17.5.9. *If for all $p \in P$, the series*

$$\sum_{h \in G_p} e^{-\delta\|h\|}\|h\| \asymp_\times \sum_{h \in G_p} \|h\|_p^{-2\delta} \log \|h\|_p \asymp_\times \sum_{k=0}^{\infty} e^{-2\delta k} k \mathcal{N}_p(e^k) \tag{17.5.7}$$

converges, then μ is exact dimensional.

PROOF. Fix $p \in P$ and $\kappa > 0$. We have

$$\begin{aligned}
\Sigma_{\mathrm{div}}(p,\kappa) &= \sum_{g \in G} e^{-\delta\|g\|} \sum_{\substack{h \in G_p \\ \|h\| \geq \kappa\|g\|/2}} e^{-\delta\|h\|} \\
&= \sum_{h \in G_p} e^{-\delta\|h\|} \sum_{\substack{g \in G \\ \|g\| \leq 2\|h\|/\kappa}} e^{-\delta\|g\|} \\
&\asymp_\times \sum_{h \in G_p} e^{-\delta\|h\|} \sum_{k \leq 2\|h\|/\kappa+1} e^{-\delta k} \#\{g \in G : k-1 \leq \|g\| < k\} \\
&\leq \sum_{h \in G_p} e^{-\delta\|h\|} \sum_{k \leq 2\|h\|/\kappa+1} e^{-\delta k} \mathcal{N}_{X,G}(k) \\
&\lesssim_\times \sum_{h \in G_p} e^{-\delta\|h\|} \sum_{k \leq 2\|h\|/\kappa+1} 1 \qquad \text{(by Corollary 16.7.1)} \\
&\asymp_\times \sum_{h \in G_p} e^{-\delta\|h\|}\|h\|.
\end{aligned}$$

So if (17.5.7) converges, so does $\Sigma_{\mathrm{div}}(p,\kappa)$, and thus by Theorems 17.5.5 and 17.5.8, μ is exact dimensional. □

COROLLARY 17.5.10. *If for all $p \in P$, $\delta_p < \delta$, then μ is exact dimensional.*

PROOF. In this case, the series (17.5.7) converges, as it is dominated by $\Sigma_s(G_p)$ for any $s \in (\delta_p, \delta)$. □

REMARK 17.5.11. Combining with Proposition 10.3.10 shows that if μ is not exact dimensional, then

$$\sum_{h \in G_p} e^{-\delta\|h\|} < \infty = \sum_{h \in G_p} e^{-\delta\|h\|}\|h\|$$

for some $p \in P$. Equivalently,

$$\sum_{k=0}^{\infty} e^{-2\delta k} \mathcal{N}_p(e^k) < \infty = \sum_{k=0}^{\infty} e^{-2\delta k} k \mathcal{N}_p(e^k).$$

This creates a very "narrow window" for the orbital counting function $\mathcal{N}_p$.

COROLLARY 17.5.12. *If μ is doubling, then μ is exact dimensional.*

PROOF. If μ is doubling, then

$$\begin{aligned}\sum_{k=0}^{\infty} e^{-2\delta k} k\mathcal{N}_p(e^k) &= \sum_{k=1}^{\infty}\sum_{\ell=0}^{\infty} e^{-2\delta(k+\ell)}\mathcal{N}_p(e^{k+\ell}) \\ &= \sum_{k=1}^{\infty} e^{-2\delta k}\widetilde{\mathcal{I}}_p(e^k) \\ &\asymp_\times \sum_{k=1}^{\infty} e^{-2\delta k}\mathcal{N}_p(e^k). \qquad \text{(by Proposition 17.4.4)}\end{aligned}$$

Remark 17.5.11 completes the proof. □

Our next theorem shows that in certain circumstances, the converse holds in Theorem 17.5.9. Specifically:

THEOREM 17.5.13. *Suppose that X is an $\mathbb{R}$-tree and that G is the pure Schottky product (cf. Definition* 14.5.7*) of a parabolic group H with a lineal group J. Let p be the global fixed point of H, so that $P = \{p\}$ is a complete set of inequivalent parabolic points for G (Proposition* 12.4.19*). Suppose that the series* (17.5.7) *diverges. Then μ is not exact dimensional; moreover, $\mu(\text{Liouville}_p) = 1$ and $\dim_H(\mu) = 0$.*

EXAMPLE 17.5.14. To see that the hypotheses of this theorem are not vacuous, fix $\delta > 0$ and let

$$f(R) = \frac{R^{2\delta}}{\log^2(R)},$$

or more generally, let f be any increasing function such that $\sum_1^\infty e^{-2\delta k} k f(e^k)$ diverges but $\sum_1^\infty e^{-2\delta k} f(e^k)$ converges. By Theorem 14.1.5, there exists an $\mathbb{R}$-tree X and a parabolic group $H \leq \text{Isom}(X)$ such that $\mathcal{N}_{\mathcal{E}_p,H} \asymp_\times f$. Then the series (17.5.7) diverges, but $\Sigma_\delta(H) < \infty$. Thus, there exists a unique $r > 0$ such that

$$2\sum_{n=1}^{\infty} e^{-\delta r} = \frac{1}{\Sigma_\delta(H) - 1}.$$

Let $J = r\mathbb{Z}$, interpreted as a group acting by translations on the $\mathbb{R}$-tree $\mathbb{R}$, and let G be the pure Schottky product of H and J. Then $\Sigma_\delta(J) - 1 = 2\sum_{n=1}^\infty e^{-\delta r}$, so $(\Sigma_\delta(H) - 1)(\Sigma_\delta(J) - 1) = 1$. Since the map $s \mapsto (\Sigma_s(H) - 1)(\Sigma_s(J) - 1)$ is decreasing, it follows from Proposition 14.5.8 that $\Delta(G) = [0, \delta]$. In particular, G is of divergence type, so Standing Assumption 17.5.1 is satisfied.

REMARK 17.5.15. Applying a BIM embedding allows us to construct an example acting on $\mathbb{H}^\infty$.

PROOF OF THEOREM 17.5.13. As in the proof of Proposition 14.5.8 we let

$$E = (H \setminus \{\text{id}\})(J \setminus \{\text{id}\}),$$

so that

$$G = \bigcup_{n \geq 0} JE^n H.$$

Define a measure θ on E via the formula

$$\theta = \sum_{g \in E} e^{-\delta\|g\|}\delta_g.$$

By Proposition 14.5.8, the fact that G is of divergence type (Standing Assumption 17.5.1), and the fact that $\Sigma_\delta(J), \Sigma_\delta(H) < \infty$ (Proposition 10.3.10), θ is a probability measure. The Patterson–Sullivan measure of G is related to θ by the formula

$$\mu = \frac{1}{\Sigma_\delta(J) - 1} \sum_{j \in J} e^{-\delta \|j\|} j_* \pi_*[\theta^{\mathbb{N}}],$$

where $\pi : E^{\mathbb{N}} \to \Lambda_G$ is the coding map.

Next, we use a theorem proven independently by H. Kesten and A. Raugi,[2] which we rephrase here in the language of measure theory:

Theorem 17.5.16 ([**111**]; see also [**149**]). *Let θ be a probability measure on a set E, and let $f : E \to \mathbb{R}$ be a function such that*

$$\int |f(x)| \, \mathrm{d}\theta(x) = \infty.$$

Then for $\theta^{\mathbb{N}}$-a.e. $(x_n)_1^\infty \in \mathbb{R}^{\mathbb{N}}$,

$$\limsup_{n \to \infty} \frac{|f(x_{n+1})|}{|\sum_1^n f(x_i)|} = \infty.$$

Letting $f(g) = \|g\|$, the theorem applies to our measure θ, because our assumption that (17.5.7) diverges is equivalent to the assertion that $\int f(x) \, \mathrm{d}\theta(x) = \infty$. Now fix $j \in J$, and let $(g_n)_1^\infty \in E^{\mathbb{N}}$ be a $\theta^{\mathbb{N}}$-typical point. Then the limit point

$$\eta = \lim_{n \to \infty} j g_1 \cdots g_n(o)$$

represents a typical point with respect to the Patterson–Sullivan measure μ. By Theorem 17.5.16, we have

$$\limsup_{n \to \infty} \frac{\|g_{n+1}\|}{\sum_1^n \|g_i\|} = \infty.$$

Write $g_i = h_i j_i$ for each i. Then $\|g_i\| = \|h_i\| + \|j_i\|$. Since $\int \|j\| \mathrm{d}\theta(hj) < \infty$, the law of large numbers implies that $\lim_{n \to \infty} \frac{\|j_{n+1}\|}{\sum_1^n \|j_i\|} < \infty$, so

$$\limsup_{n \to \infty} \frac{\|h_{n+1}\|}{\sum_1^n \|g_i\|} = \infty.$$

But $\|h_{n+1}\|$ represents the length of the excursion of the geodesic $[o, \eta]$ into the cusp corresponding to the parabolic point $g_1 \cdots g_n(p)$. Combining with Observation 17.5.7 shows that $\eta \in \text{Liouville}_p$. Since η was a μ-typical point, this shows that $\mu(\text{Liouville}_p) = 1$. By Theorem 17.5.6, this implies that $\dim_H(\mu) = 0$. By Observation 17.5.4, μ is not exact dimensional. $\square$

[2] We are grateful to "cardinal" of `http://mathoverflow.net` and J. P. Conze, respectively, for these references.