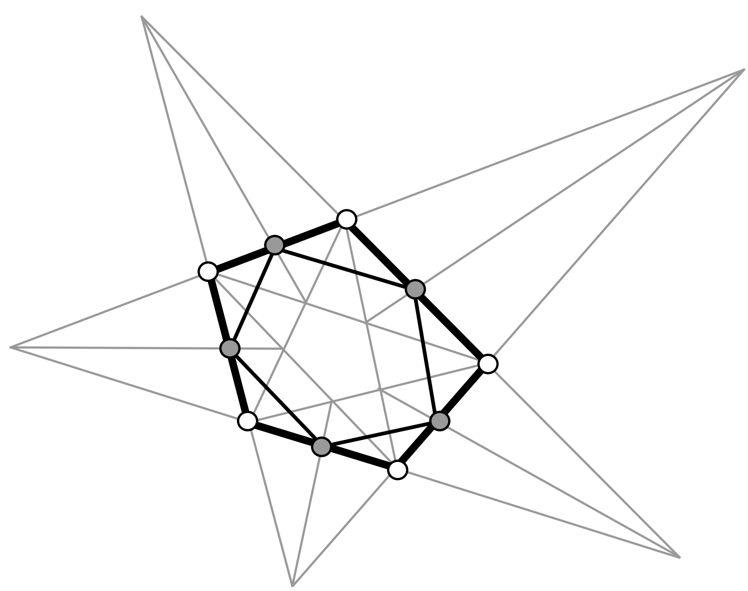
Preface

There is a simple and well-known construction which starts with one polygon and returns a new polygon whose vertices are the midpoints of the edges of the original. *The midpoint map* is a good name for this construction, and we consider it as a mapping on the space of polygons with a fixed number of vertices. When written in coordinates, the midpoint map is a linear transformation that is closely related to the heat equation: The vertex coordinates of the new polygon are averages of the vertex coordinates of the original. The midpoint map commutes with affine transformations of the plane. If you move the original polygon by an affine transformation, then the new one goes along for the ride.

In this monograph I will study a non-linear construction that is somewhat like the midpoint map but which commutes with projective transformations. I think of the construction as something like a cross between the midpoint map and the so-called pentagram map. I call the construction the *projective heat map* because I imagine – perhaps with scant justification – that the construction models how heat might flow in a world governed by projective geometry.

The projective heat map starts with a polygon P and returns a new polygon $P' = H(P)$. The figure illustrates the construction when the polygons involved are pentagons. P is the outer black one with white vertices, and P' is the inner black polygon with grey vertices. The auxiliary grey lines are just scaffolding for the construction.



The main purpose of this monograph is to answer the question: *What does the projective heat map do to pentagons?* That is, what happens when the construction is iterated; what is the sequence $\{H^n(P)\}$ like? The question leads naturally to the study of a certain 2-dimensional real rational map. This rational map has surprisingly intricate behavior and a beautiful "Julia set". I will give rigorous, computer-assisted proofs of structural results which capture the main features of the projective heat map as it acts on pentagons, and this will give a pretty complete answer to the original question. I wrote an extensive graphical user interface which illustrates almost all the constructions of the monograph. This program is intended as a companion for the monograph.

To broaden the scope of this monograph, I will also discuss (in less depth) some other interesting polygon iterations such as the midpoint map, the map derived from Napoleon's Theorem, and the pentagram map. I will also include a numerical study of what the projective heat map does to N-gons in general, and how it interacts with the pentagram map.

This book is suitable for graduate students interested in projective geometry, rational maps, and polygon iterations. Most of the introductory chapters could also be read by advanced undergraduates. To make this work accessible to a broader audience, I have included some expository material on projective geometry, elementary algebraic geometry, and basic dynamical systems such as the one-sided shift and the Smale horseshoe. All these things play a role in the analysis of the projective heat map.

In his forthcoming 2017 Brown University Ph.D. thesis, Quang Nhat Le explores a 1-parameter family of maps generalizing the projective heat map. This monograph does not discuss these matters, but the interested reader might like to know about the existence of Nhat's thesis.

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