Preface

Кто я? Не каменщик прямой,
Не кровельщик, не корабельщик, –
Двурушник я, с двойной душой,
Я ночи друг, я дня застрельщик.
О. Мандельштам. Грифельная ода.

Who am I? Not a straightforward mason,
Not a roofer, not a shipbuilder, –
I am a double agent, with a duplicitous soul,
I am a friend of the night, a skirmisher of the day.
O. Mandelshtam. The Graphite Ode.

1. What is the object of study in this book?

The main unifying theme of the two volumes of this book is the notion of *ind-coherent sheaf*, or rather, categories of such on various geometric objects. In this section we will try to explain what ind-coherent sheaves are and why we need this notion.

1.1. Who are we? Let us start with a disclosure: this book is *not* really about algebraic geometry.

Or, rather, in writing this book, its authors do not act as real algebraic geometers. This is because the latter are ultimately interested in geometric objects that are constrained/enriched by the algebraicity requirement.

We, however, use algebraic geometry as a tool: this book is written with a view toward applications to *representation theory*.

It just so happens that algebraic geometry is a very (perhaps, even the most) convenient way to formulate representation-theoretic problems of categorical nature. This is not surprising, since, after all, algebraic groups are themselves objects of algebraic geometry.

The most basic example of how one embeds representation theory into algebraic geometry is this: take the category $\mathrm{Rep}(G)$ of algebraic representations of a linear algebraic group G. Algebraic geometry allows us to define/interpret $\mathrm{Rep}(G)$ as the category of quasi-coherent sheaves on the classifying stack BG.

The advantage of this point of view is that many natural constructions associated with the category of representations are already contained in the package of 'quasi-coherent sheaves on stacks'. For example, the functors of restriction and coinduction[1] along a group homomorphism $G' \to G$ are interpreted as the functors of inverse and direct image along the map of stacks

$$BG' \to BG.$$

But what is the advantage of this point of view? Why not stick to the explicit constructions of all the required functors within representation theory?

The reason is that 'explicit constructions' involve 'explicit formulas', and once we move to the world of higher categories (which we inevitably will, in order to meet the needs of modern representation theory), we will find ourselves in trouble: constructions in higher category theory are intolerant of explicit formulas (for an example of a construction that uses formulas see point (III) in Sect. 1.5 below). Rather, when dealing with higher categories, there is a fairly limited package of constructions that we are allowed to perform (see Chapter 1, Sects. 1 and 2 where some of these constructions are listed), and algebraic geometry seems to contain a large chunk (if not all) of this package.

1.2. A stab in the back. Jumping ahead slightly, suppose for example that we want to interpret algebro-geometrically the category $\mathfrak{g}$-mod of modules over a Lie algebra $\mathfrak{g}$.

The first question is: why would one want to do that? Namely, take the universal enveloping algebra $U(\mathfrak{g})$ and interpret $\mathfrak{g}$-mod as modules over $U(\mathfrak{g})$. Why should one mess with algebraic geometry if all we want is the category of modules over an associative algebra?

But let us say that we have already accepted the fact that we want to interpret $\mathrm{Rep}(G)$ as $\mathrm{QCoh}(BG)$. If we now want to consider restriction functor

$$\mathrm{Rep}(G) \to \mathfrak{g}\text{-mod}, \tag{1.1}$$

(where $\mathfrak{g}$ is the Lie algebra of G), we will need to give an algebro-geometric interpretation of $\mathfrak{g}$-mod as well.

If $\mathfrak{g}$ is a usual (=classical) Lie algebra, one can consider the associated formal group, denoted in the book $\exp(\mathfrak{g})$, and one can show (see Volume II, Chapter 7, Sect. 5) that the category $\mathfrak{g}$-mod is canonically equivalent to $\mathrm{QCoh}(B(\exp(\mathfrak{g})))$, the category of quasi-coherent sheaves on the classifying stack[2] of $\exp(\mathfrak{g})$. With this interpretation of $\mathfrak{g}$-mod, the functor (1.1) is simply the pullback functor along the map

$$B(\exp(\mathfrak{g})) \to BG,$$

induced by the (obvious) map $\exp(\mathfrak{g}) \to G$.

Let us now be given a homomorphism of Lie algebras $\alpha : \mathfrak{g}' \to \mathfrak{g}$. The functor of restriction $\mathfrak{g}$-mod $\to \mathfrak{g}'$-mod still corresponds to the pullback functor along the corresponding morphism

$$B(\exp(\mathfrak{g}')) \xrightarrow{f_\alpha} B(\exp(\mathfrak{g})). \tag{1.2}$$

[1] What we call 'coinduction' is the functor right adjoint to restriction, i.e., it is the usual representation-theoretic operation.

[2] One can (reasonably) get somewhat uneasy from the suggestion to consider the category of quasi-coherent sheaves on the *classifying stack of a formal group*, but, in fact, this is a legitimate operation.

Note, however, that when we talk about representations of Lie algebras, the natural functor in the opposite direction is *induction*, i.e., the *left* adjoint to restriction. And being a left adjoint, it *cannot* correspond to the direct image along (1.2) (whatever the functor of direct image is, it is the *right* adjoint of pullback).

This inconsistency leads to the appearance of *ind-coherent sheaves*.

1.3. The birth of IndCoh**.**

What happens is that, although we can interpret $\mathfrak{g}$-mod as $\text{QCoh}(B(\exp(\mathfrak{g})))$, a more natural interpretation is as $\text{IndCoh}(B(\exp(\mathfrak{g})))$. The symbol 'IndCoh' will of course be explained in the sequel. It just so happens that for a classical Lie algebra, the categories $\text{QCoh}(B(\exp(\mathfrak{g})))$ and $\text{IndCoh}(B(\exp(\mathfrak{g})))$ are equivalent (as $\text{QCoh}(BG)$ is equivalent to $\text{IndCoh}(BG)$).

Now, the functor of restriction along the homomorphism α will be given by the functor

$$(f_\alpha)^! : \text{IndCoh}(B(\exp(\mathfrak{g}'))) \to \text{IndCoh}(B(\exp(\mathfrak{g})));$$

this is the !-pullback functor, which is the *raison d'être* for the theory of IndCoh.

However, the functor of induction $\mathfrak{g}'$-mod $\to$ $\mathfrak{g}$-mod will be the functor of *IndCoh direct image*

$$(f_\alpha)_*^{\text{IndCoh}} : \text{IndCoh}(B(\exp(\mathfrak{g}'))) \to \text{IndCoh}(B(\exp(\mathfrak{g}))), \tag{1.3}$$

which is the *left* adjoint of $(f_\alpha)^!$. This adjunction is due to the fact that the morphism f_α is, in an appropriate sense, proper.

Now, even though, as was mentioned above, for a usual Lie algebra $\mathfrak{g}$, the categories

$$\text{QCoh}(B(\exp(\mathfrak{g}))) \text{ and } \text{IndCoh}(B(\exp(\mathfrak{g})))$$

are equivalent, the functor $(f_\alpha)_*^{\text{IndCoh}}$ of (1.3) is as different as can be from the functor

$$(f_\alpha)_* : \text{QCoh}(B(\exp(\mathfrak{g}'))) \to \text{QCoh}(B(\exp(\mathfrak{g})))$$

(the latter is quite ill-behaved).

For an analytically minded reader let us also offer the following (albeit somewhat loose) analogy: $\text{QCoh}(-)$ behaves more like functions on a space, while $\text{IndCoh}(-)$ behaves more like measures on the same space.

1.4. What can we do with ind-coherent sheaves? As we saw in the example of Lie algebras, the kind of geometric objects on which we will want to consider IndCoh (e.g., $B(\exp(\mathfrak{g}))$) are quite a bit more general than the usual objects on which we consider quasi-coherent sheaves, the latter being schemes (or algebraic stacks).

A natural class of algebro-geometric objects for which IndCoh is defined is that of *inf-schemes*, introduced and studied in Volume II, Part I of the book. This class includes all schemes, but also formal schemes, as well as classifying spaces of formal groups, etc. In addition, if X is a scheme, its de Rham prestack[3] X_{dR} is an inf-scheme, and ind-coherent sheaves on X_{dR} will be the same as *crystals* (a.k.a. D-modules) on X.

[3]The de Rham prestack of a given scheme X is obtained by 'modding' out X by the groupoid of its infinitesimal symmetries, see Volume II, Chapter 4, Sect. 1.1.1 for a precise definition.

Thus, for any inf-scheme $\mathcal{X}$ we have a well-defined category $\mathrm{IndCoh}(\mathcal{X})$. For any map of inf-schemes $f : \mathcal{X}' \to \mathcal{X}$ we have functors

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}') \to \mathrm{IndCoh}(\mathcal{X})$$

and

$$f^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X}').$$

Moreover, if f is proper[4], then the functors $(f_*^{\mathrm{IndCoh}}, f^!)$ form an adjoint pair.

Why should we be happy to have this? The reason is that this is exactly the kind of operations one needs in geometric representation theory.

1.5. Some examples of what we can do.

(I) Take $\mathcal{X}'$ to be a scheme X and $\mathcal{X} = X_{\mathrm{dR}}$, with f being the canonical projection $X \to X_{\mathrm{dR}}$. Then the adjoint pair

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(X) \rightleftarrows \mathrm{IndCoh}(X_{\mathrm{dR}}) : f^!$$

identifies with the pair

$$\mathbf{ind}_{\text{D-mod}} : \mathrm{IndCoh}(X) \rightleftarrows \text{D-mod}(X) : \mathbf{ind}_{\text{D-mod}},$$

corresponding to forgetting and inducing the (right) D-module structure (as we shall see shortly in Sect. 2.3, for a scheme X, the category $\mathrm{IndCoh}(X)$ is only slightly different from the usual category of quasi-coherent sheaves $\mathrm{QCoh}(X)$).

(II) Suppose we have a morphism of schemes $g : Y \to X$ and set

$$Y_{\mathrm{dR}} \xrightarrow{f := g_{\mathrm{dR}}} X_{\mathrm{dR}}.$$

The corresponding functors

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Y_{\mathrm{dR}}) \to \mathrm{IndCoh}(X_{\mathrm{dR}}) \text{ and } f^! : \mathrm{IndCoh}(X_{\mathrm{dR}}) \to \mathrm{IndCoh}(Y_{\mathrm{dR}})$$

identify with the functors

$$g_{*,\mathrm{Dmod}} : \mathrm{Dmod}(Y) \to \mathrm{Dmod}(X) \text{ and } g^!_{\mathrm{Dmod}} : \mathrm{Dmod}(X) \to \mathrm{Dmod}(Y)$$

of D-module (a.k.a. de Rham) push-forward and pullback, respectively.

Note that while the operation of pullback of (right) D-modules corresponds to !-pullback on the underlying $\mathcal{O}$-module, the operation of D-module push-forward is less straightforward as it involves taking fiber-wise de Rham cohomology. So, the operation of the IndCoh direct image does something quite non-trivial in this case.

(III) Suppose we have a Lie algebra $\mathfrak{g}$ that acts (by vector fields) on a scheme X. In this case we can create a diagram

$$B(\exp(\mathfrak{g})) \xleftarrow{f_1} B_X(\exp(\mathfrak{g})) \xrightarrow{f_2} X_{\mathrm{dR}},$$

where $B_X(\exp(\mathfrak{g}))$ is an inf-scheme, which is the quotient of X by the action of $\mathfrak{g}$.

Then the composite functor

$$(f_2)_*^{\mathrm{IndCoh}} \circ (f_1)^! : \mathrm{IndCoh}(B(\exp(\mathfrak{g}))) \to \mathrm{IndCoh}(X_{\mathrm{dR}})$$

identifies with the *localization functor*

$$\mathfrak{g}\text{-mod} \to \mathrm{Dmod}(X).$$

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[4]Properness means the following: to every inf-scheme there corresponds its underlying reduced scheme, and a map between inf-schemes is proper if and only if the map of the underlying reduced schemes is proper in the usual sense.

This third example should be a particularly convincing one: the localization functor, which is usually defined by an explicit formula

$$M \mapsto D_X \underset{U(\mathfrak{g})}{\otimes} M,$$

is given here by the general formalism.

2. How do we do we construct the theory of IndCoh?

Whatever inf-schemes are, for an individual inf-scheme $\mathcal{X}$, the category $\mathrm{IndCoh}(\mathcal{X})$ is bootstrapped from the corresponding categories for schemes by the following procedure:

$$\mathrm{IndCoh}(\mathcal{X}) = \lim_{Z \to \mathcal{X}} \mathrm{IndCoh}(Z). \tag{2.1}$$

Some explanations are in order.

2.1. What do we mean by limit?

(a) In formula (2.1), the symbol 'lim' appears. This is the limit of categories, but not quite. If we were to literally take the limit in the category of categories, we would obtain utter nonsense. This is a familiar phenomenon: the (literally understood) limit of, say, triangulated categories is not well-behaved. A well-known example of this is that the derived category of sheaves on a space cannot be recovered from the corresponding categories on an open cover. However, this can be remedied if instead of the triangulated categories we consider their higher categorical enhancements, i.e., the corresponding ∞-categories.

So, what we actually mean by 'limit', is the limit taken in the *∞-category* of *∞-categories*. That is, in the preceding discussion, all our $\mathrm{IndCoh}(-)$ are actually ∞-categories. In our case, they have a bit more structure: they are k-linear over a fixed ground field k; we call them *DG categories*, and denote the ∞-category of such by DGCat.

Thus, ∞-categories inevitably appear in this book.

(b) The indexing (∞)-category appearing in the expression (2.1) is the (∞)-category opposite to that of schemes Z equipped with a map $Z \to \mathcal{X}$ to our inf-scheme $\mathcal{X}$. The transition functors are given by

$$(Z' \overset{f}{\to} Z) \in \mathrm{Sch}_{/\mathcal{X}} \rightsquigarrow \mathrm{IndCoh}(Z) \overset{f^!}{\longrightarrow} \mathrm{IndCoh}(Z').$$

So, in order for the expression in (2.1) to make sense we need to make the assignment

$$Z \rightsquigarrow \mathrm{IndCoh}(Z), \quad (Z' \overset{f}{\to} Z) \rightsquigarrow (\mathrm{IndCoh}(Z) \overset{f^!}{\longrightarrow} \mathrm{IndCoh}(Z')) \tag{2.2}$$

into a *functor of ∞-categories*

$$\mathrm{IndCoh}^!_{\mathrm{Sch}} : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{DGCat}. \tag{2.3}$$

To that end, before we proceed any further, we need to explain what the DG category $\mathrm{IndCoh}(Z)$ is for a scheme Z.

For a scheme Z, the category $\mathrm{IndCoh}(Z)$ will be almost the same as $\mathrm{QCoh}(Z)$. The former is obtained from the latter by a *renormalization* procedure, whose nature we shall now explain.

2.2. Why renormalize? Keeping in mind the examples of $\mathrm{Rep}(G)$ and $\mathfrak{g}$-mod, it is natural to expect that the assignment (2.2) (for schemes, and then also for inf-schemes) should have the following properties:

(i) For every scheme Z, the DG category $\mathrm{IndCoh}(Z)$ should contain infinite direct sums;

(ii) For a map $Z' \overset{f}{\to} Z$, the functor $\mathrm{IndCoh}(Z) \overset{f^!}{\longrightarrow} \mathrm{IndCoh}(Z')$ should preserve infinite direct sums.

This means that the functor (2.3) takes values in the subcategory of DGCat, where we allow as objects only DG categories satisfying (i)[5] and as 1-morphisms only functors that satisfy (ii)[6].

Let us first try to make this work with the usual QCoh. We refer the reader to Chapter 3, where the DG category $\mathrm{QCoh}(\mathcal{X})$ is introduced for an arbitrary *prestack*, and in particular a scheme. However, for a scheme Z, whatever the DG category $\mathrm{QCoh}(Z)$ is, its homotopy category (which is a triangulated category) is the usual (unbounded) derived category of quasi-coherent sheaves on Z.

Suppose we have a map of schemes $Z' \overset{f}{\to} Z$. The construction of the !-pullback functor

$$f^! : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(Z')$$

is quite complicated, except when f is proper. In the latter case, $f^!$, which from now on we will denote by $f^{!,\mathrm{QCoh}}$, is defined to be the *right adjoint* of

$$f_* : \mathrm{QCoh}(Z') \to \mathrm{QCoh}(Z).$$

The only problem is that the above functor $f^{!,\mathrm{QCoh}}$ *does not* preserve infinite direct sums. The simplest example of a morphism for which this happens is

$$f : \mathrm{Spec}(k) \to \mathrm{Spec}(k[t]/t^2)$$

(or the embedding of a *singular* point into any scheme).

The reason for the failure to preserve infinite direct sums is this: the left adjoint of $f^{!,\mathrm{QCoh}}$, i.e., f_*, *does not* preserve compactness. Indeed, f_* does not necessarily send *perfect complexes* on Z' to perfect complexes on Z, unless f is of finite Tor-dimension[7].

So, our attempt with QCoh fails (ii) above.

2.3. Ind-coherent sheaves on a scheme. The nature of the renormalization procedure that produces $\mathrm{IndCoh}(Z)$ out of $\mathrm{QCoh}(Z)$ is to force (ii) from Sect. 2.2 'by hand'.

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[5] Such DG categories are called *cocomplete*.

[6] Such functors are called *continuous*.

[7] We remark that a similar phenomenon, where instead of the category $\mathrm{QCoh}(\mathrm{Spec}(k[t]/t^2)) = k[t]/t^2$-mod we have the category of representations of a finite group, leads to the notion of Tate cohomology: the trivial representation on $\mathbb{Z}$ is *not* a compact object in the category of representations.

As we just saw, the problem with $f^{!,\mathrm{QCoh}}$ was that its left adjoint f_* did not send the corresponding subcategories of perfect complexes to one another. However, f_* sends the subcategory

$$\mathrm{Coh}(Z') \subset \mathrm{QCoh}(Z')$$

to

$$\mathrm{Coh}(Z) \subset \mathrm{QCoh}(Z),$$

where Coh(−) denotes the subcategory of bounded complexes, whose cohomology sheaves are *coherent* (as opposed to quasi-coherent).

The category $\mathrm{IndCoh}(Z)$ is defined as the *ind-completion* of $\mathrm{Coh}(Z)$ (see Chapter 1, Sect. 7.2 for what this means). The functor f_* gives rise to a functor $\mathrm{Coh}(Z') \to \mathrm{Coh}(Z)$, and ind-extending we obtain a functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z).$$

Its right adjoint, denoted $f^! : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Z')$ satisfies (ii) from Sect. 2.2.

Are we done? Far from it. First, we need to define the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z)$$

for a morphism f that is not necessarily proper. This will not be difficult, and will be done by appealing to t-structures, see Sect. 2.4 below.

What is much more serious is to define $f^!$ for any f. More than that, we need $f^!$ not just for an individual f, but we need the data of (2.2) to be a functor of ∞-categories as in (2.3). Roughly a third of the work in this book goes into the construction of the functor (2.3); we will comment on the nature of this work in Sect. 2.5 and then in Sect. 3 below.

2.4. In what sense is IndCoh **a 'renormalization' of** QCoh**?** The tautological embedding $\mathrm{Coh}(Z) \hookrightarrow \mathrm{QCoh}(Z)$ induces, by ind-extension, a functor

$$\Psi_Z : \mathrm{IndCoh}(Z) \to \mathrm{QCoh}(Z).$$

The usual t-structure on the DG category $\mathrm{Coh}(Z)$ induces one on $\mathrm{IndCoh}(Z)$. The key feature of the functor Ψ_Z is that it is *t-exact.* Moreover, for every fixed n, the resulting functor

$$\mathrm{IndCoh}(Z)^{\geq -n} \to \mathrm{QCoh}(Z)^{\geq -n}$$

is an *equivalence*[8]. The reason for this is that any coherent complex can be approximated by a perfect one up to something in $\mathrm{Coh}(Z)^{<-n}$ for any given n.

In other words, the difference between $\mathrm{IndCoh}(Z)$ and $\mathrm{QCoh}(Z)$ occurs 'somewhere at $-\infty$'. So, this difference can only become tangible in the finer questions of homological algebra (such as convergence of spectral sequences).

However, we do need to address such questions adequately if we want to have a functioning theory, and for the kind of applications we have in mind (see Sect. 1.5 above) this necessitates working with IndCoh rather than QCoh.

As an illustration of how the theory of IndCoh takes something very familiar and unravels it to something non-trivial, consider the IndCoh direct image functor.

[8]But the functor Ψ_Z is an equivalence on all of $\mathrm{IndCoh}(Z)$ if *and only if* Z is smooth.

In the case of schemes, for a morphism $f : Z' \to Z$, the functor

$$f_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(Z') \to \mathrm{IndCoh}(Z)$$

does 'little new' as compared to the usual

$$f_* : \mathrm{QCoh}(Z') \to \mathrm{QCoh}(Z).$$

Namely, f_*^{IndCoh} is the unique functor that preserves infinite direct sums and makes the diagram

$$\begin{array}{ccc} \mathrm{IndCoh}(Z')^{\geq -n} & \xrightarrow[\sim]{\Psi_{Z'}} & \mathrm{QCoh}(Z')^{\geq -n} \\ {\scriptstyle f_*^{\mathrm{IndCoh}}}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathrm{IndCoh}(Z)^{\geq -n} & \xrightarrow[\sim]{\Psi_{Z}} & \mathrm{QCoh}(Z)^{\geq -n} \end{array}$$

commute for every n.

However, as was already mentioned, once we extend the formalism of IndCoh direct image to inf-schemes, we will in particular obtain the de Rham direct image functor. So, it is in the world of inf-schemes that IndCoh shows its full strength.

2.5. Construction of the !-pullback functor. As has been mentioned already, a major component of work in this book is the construction of the functor

$$\mathrm{IndCoh}^!_{\mathrm{Sch}} : (\mathrm{Sch})^{\mathrm{op}} \to \mathrm{DGCat}$$

of (2.3).

We already know what $\mathrm{IndCoh}(Z)$ is for an individual scheme. We now need to extend it to morphisms.

For a morphism $f : Z' \to Z$, we can factor it as

$$Z' \overset{f_1}{\to} \overline{Z}' \overset{f_2}{\to} Z, \tag{2.4}$$

where f_1 is an open embedding and f_2 is proper. We then define

$$f^! : \mathrm{IndCoh}(Z) \to \mathrm{IndCoh}(Z')$$

to be

$$f_1^! \circ f_2^!,$$

where

(i) $f_2^!$ is the *right adjoint* of $(f_2)_*^{\mathrm{IndCoh}}$;

(ii) $f_1^!$ is the *left adjoint* of $(f_1)_*^{\mathrm{IndCoh}}$.

Of course, in order to have $f^!$ as a well-defined functor, we need to show that its definition is independent of the factorization of f as in (2.4). Then we will have to show that the definition is compatible with compositions of morphisms. But this is only the tip of the iceberg.

Since we want to have a functor between ∞-categories, we need to supply the assignment

$$f \rightsquigarrow f^!$$

with a *homotopy-coherent* system of compatibilities for n-fold compositions of morphisms, a task which appears infeasible to do 'by hand'.

What we do instead is we prove an existence and uniqueness theorem... not for (2.3), but rather for a more ambitious piece of structure. We refer the reader to Chapter 5, Proposition 2.1.4 for the precise formulation. Here we will only say that, in addition to (2.3), this structure contains the data of a functor

$$\text{IndCoh} : \text{Sch} \to \text{DGCat}, \tag{2.5}$$

$$Z \rightsquigarrow \text{IndCoh}(Z), \quad (Z' \xrightarrow{f} Z) \rightsquigarrow (\text{IndCoh}(Z') \xrightarrow{f_*^{\text{IndCoh}}} \text{IndCoh}(Z)),$$

as well as *compatibility* between (2.3) and (2.5).

The latter means that whenever we have a Cartesian square

$$\begin{array}{ccc} Z_1' & \xrightarrow{g'} & Z' \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ Z_1 & \xrightarrow{g} & Z \end{array} \tag{2.6}$$

there is a canonical isomorphism of functors, called base change:

$$(f_1)_*^{\text{IndCoh}} \circ (g')^! \simeq g^! \circ f_*^{\text{IndCoh}}. \tag{2.7}$$

2.6. Enter DAG. The appearance of the Cartesian square (2.6) heralds another piece of 'bad news'. Namely, Z_1' must be the fiber product

$$Z_1 \underset{Z}{\times} Z'.$$

But what category should we take this fiber product in? If we look at the example

$$\begin{array}{ccc} \text{pt} \underset{\mathbb{A}^1}{\times} \text{pt} & \longrightarrow & \text{pt} \\ \downarrow & & \downarrow \\ \text{pt} & \longrightarrow & \mathbb{A}^1, \end{array}$$

(here pt = Spec(k), $\mathbb{A}^1$ = Spec($k[t]$)), we will see that the fiber product $\text{pt} \underset{\mathbb{A}^1}{\times} \text{pt}$ *cannot* be taken to be the point-scheme, i.e., it cannot be the fiber product in the category of usual (=classical) schemes. Rather, we need to take

$$\text{pt} \underset{\mathbb{A}^1}{\times} \text{pt} = \text{Spec}(k \underset{k[t]}{\otimes} k),$$

where the tensor product is understood in the *derived* sense, i.e.,

$$k \underset{k[t]}{\otimes} k = k[\epsilon], \quad \deg(\epsilon) = -1.$$

This is to say that in building the theory of IndCoh, we cannot stay with classical schemes, but rather need to enlarge our world to that of *derived algebraic geometry.*

So, unless the reader has already guessed this, in all the previous discussion, the word 'scheme' had to be understood as 'derived scheme'[9] (although in the main body of the book we say just 'scheme', because everything is derived).

However, this is not really 'bad news'. Since we are already forced to work with ∞-categories, passing from classical algebraic geometry to DAG does not add

[9]Technically, for whatever has to do with IndCoh, we need to add the adjective 'laft'='locally almost of finite type', see Chapter 2, Sect. 3.5 for what this means.

a new level of complexity. But it does add *a lot* of new techniques, for example in anything that has to do with deformation theory (see Volume II, Chapter 1).

Moreover, many objects that appear in geometric representation theory naturally belong to DAG (e.g., Springer fibers, moduli of local systems on a curve, moduli of vector bundles on a surface). That is, these objects are *not* classical, i.e., we cannot ignore their derived structure if we want to study their scheme-theoretic (as opposed to topological) properties. So, we would have wanted to do DAG in any case.

Here are two particular examples:

(I) Consider the category of D-modules (resp., perverse) sheaves on the double quotient

$$I\backslash G((t))/I,$$

where G is a connected reductive group, $G((t))$ is the corresponding loop group (considered as an ind-scheme) and $I \subset G((t))$ is the Iwahori subgroup. Then Bezrukavnikov's theory (see [**Bez**]) identifies this category with the category of ad-equivariant ind-coherent (resp., coherent) sheaves on the *Steinberg scheme* (for the Langlands dual group). But what do we mean by the Steinberg scheme? By definition, this is the fiber product

$$\widetilde{\mathcal{N}} \underset{\mathfrak{g}}{\times} \widetilde{\mathcal{N}}, \tag{2.8}$$

where $\widetilde{\mathcal{N}}$ is the Springer resolution of the nilpotent cone. However, in order for this equivalence to hold, the fiber product in (2.8) needs be understood in the *derived* sense.

(II) Let X be a smooth and complete curve. Let $\mathrm{Pic}(X)$ be the *Picard stack* of X, i.e., the stack parameterizing line bundles on X. Let $\mathrm{LocSys}(X)$ be the stack parameterizing 1-dimensional local systems on X. The Fourier-Mukaï-Laumon transform defines an equivalence

$$\mathrm{Dmod}(\mathrm{Pic}(X)) \simeq \mathrm{QCoh}(\mathrm{LocSys}(X)).$$

However, in order for this equivalence to hold, we need to understand $\mathrm{LocSys}(X)$ as a *derived stack.*

2.7. Back to inf-schemes. The above was a somewhat lengthy detour into the constructions of the theory of IndCoh on schemes. Now, if $\mathcal{X}$ is an inf-scheme, the category $\mathrm{IndCoh}(\mathcal{X})$ is defined by the formula (2.1).

Thus, informally, an object $\mathcal{F} \in \mathrm{IndCoh}(\mathcal{X})$ is a family of assignments

$$(Z \overset{x}{\to} \mathcal{X}) \rightsquigarrow \mathcal{F}_{Z,x} \in \mathrm{IndCoh}(Z)$$

(here Z is a scheme) plus

$$(Z' \overset{f}{\to} Z) \in \mathrm{Sch}_{/\mathcal{X}} \rightsquigarrow f^!(\mathcal{F}_{Z,x}) \simeq \mathcal{F}_{Z',x'},$$

along with a homotopy-coherent compatibility data for compositions of morphisms.

For a map $g : \mathcal{X}' \to \mathcal{X}$, the functor

$$g^! : \mathrm{IndCoh}(\mathcal{X}) \to \mathrm{IndCoh}(\mathcal{X}')$$

is essentially built into the construction. Recall, however, that our goal is to also have the functor

$$g_*^{\mathrm{IndCoh}} : \mathrm{IndCoh}(\mathcal{X}') \to \mathrm{IndCoh}(\mathcal{X}).$$

The construction of the latter requires some work (which occupies most of Volume II, Chapter 3). What we show is that there exists a unique system of such functors such that for every commutative (but not necessarily Cartesian) diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'} & \mathcal{X}' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Z & \xrightarrow{i} & \mathcal{X} \end{array}$$

with Z, Z' being schemes and the morphisms i, i' proper, we have an isomorphism

$$g_*^{\mathrm{IndCoh}} \circ (i')_*^{\mathrm{IndCoh}} \simeq i_*^{\mathrm{IndCoh}} \circ f^{\mathrm{IndCoh}},$$

where i_*^{IndCoh} (resp., $(i')_*^{\mathrm{IndCoh}}$) is the left adjoint of $i^!$ (resp., $(i')^!$).

Amazingly, this procedure contains the de Rham push-forward functor as a particular case.

3. What is actually done in this book?

This book consists of two volumes. The first volume consists of three parts and an appendix and the second volume consists of two parts. Each part consists of several chapters. The chapters are designed so that they can be read independently from one another (in a sense, each chapter is structured as a separate paper with its own introduction that explains what this particular chapter does).

Below we will describe the contents of the different parts and chapters from several different perspectives: (a) goals and role in the overall project; (b) practical implications; (c) nature of work; (d) logical dependence.

3.1. The contents of the different parts.

Volume I, Part I is called 'preliminaries', and it is really preliminaries.

Volume I, Part II builds the theory of IndCoh on schemes.

Volume I, Part III develops the formalism of *categories of correspondences*; it is used as a 'black box' in the key constructions in Volume I, Part II and Volume II, Part I: this is our tool of bootstrapping the theory of IndCoh out of a much smaller amount of data.

Volume I, Appendix provides a sketch of the theory of $(\infty, 2)$-categories, which, in turn, is crucially used in Volume I, Part III.

Volume II, Part I defines the notion of *inf-scheme* and extends the formalism of IndCoh from schemes to inf-schemes, and in that it achieves one of the two main goals of this book.

Volume II, Part II consists of applications of the theory of IndCoh: we consider formal moduli problems, Lie theory and infinitesimal differential geometry; i.e., exactly the things one needs for geometric representation theory. Making these constructions available is the second of our main goals.

3.2. Which chapters should a practically minded reader be interested in? Not all the Chapters in this book make an enticing read; some are downright technical and tedious. Here is, however, a description of the 'cool' things that some of the Chapters do:

None of the material in Volume I, Part I alters the pre-existing state of knowledge.

Volume I, Chapters 4 and 5 should not be a difficult read. They construct the theory of IndCoh on schemes (the hard technical work is delegated to Volume I, Chapter 7). The reader cannot avoid reading these chapters if he/she is interested in the applications of IndCoh: one has to have an idea of what IndCoh is in order to use it.

Volume I, Chapters 6 is routine. The only really useful thing from it is the functor

$$\Upsilon_Z : \mathrm{QCoh}(Z) \to \mathrm{IndCoh}(Z),$$

given by tensoring an object of QCoh(Z) with the dualizing complex

$$\omega_Z \in \mathrm{IndCoh}(Z).$$

Extract this piece of information from Sects. 3.2-3.3 and move on.

Volume I, Chapter 7 introduces the formalism of correspondences. The idea of the category of correspondences is definitely something worth knowing. We recommend the reader to read Sect. 1 in its entirety, then understand the universal property stated in Sect. 3, and finally get an idea about the two extension theorems, proved in Sects. 4 and 5, respectively. These extension theorems are the mechanism by means of which we construct IndCoh as a functor out of the category of correspondences in Volume I, Chapter 5.

Volume I, Chapter 8 proves a rather technical extension theorem, stated in Sect. 1; we do not believe that the reader will gain much by studying its proof. This theorem is key to the extension of IndCoh from schemes to inf-schemes in Volume II, Chapter 3.

Volume I, Chapter 9 is routine, except for one observation, contained in Sects. 2.2-2.3: the natural involution on the category of correspondences encodes *duality*. In fact, this is how we construct Serre duality on IndCoh(Z) and Verdier duality on Dmod(Z) where Z is a scheme (or inf-scheme), see Chapter 5, Sect. 4.2, Volume II, Chapter 3, Sect. 6.2, and Volume II, Chapter 4, Sect. 2.2, respectively.

Volume I, Chapter 10 introduces the notion of $(\infty,2)$-category and some basic constructions in the theory of $(\infty,2)$-categories. This Chapter is not very technical (mainly because it omits most proofs) and might be of independent interest.

Volume I, Chapter 11 does a few more technical things in the theory of $(\infty,2)$-categories. It introduces the $(\infty,2)$-*category of* $(\infty,2)$-*categories*, denoted 2-Cat. We then discuss the straightening/unstraightening procedure in the $(\infty,2)$-categorical context and the $(\infty,2)$-categorical Yoneda lemma. The statements of the results from this Chapter may be of independent interest.

Volume I, Chapter 12 discusses the notion of adjunction in the context of $(\infty,2)$-categories. The main theorem in this Chapter explicitly constructs the *universal adjointable functor* (and its variants), and we do believe that this is of interest beyond the particular goals of this book.

Volume II, Chapter 1 is background on deformation theory. The reason it is included in the book is that the notion of inf-scheme is based on deformation theory. However, the reader may find the material in Sects. 1-7 of this Chapter useful without any connection to the contents of the rest of the book.

Volume II, Chapter 2 introduces inf-schemes. It is quite technical. So, the practically minded reader should just understand the definition (Sect. 3.1) and move on.

Volume II, Chapter 3 bootstraps the theory of IndCoh from schemes to inf-schemes. It is not too technical, and should be read (for the same reason as Volume I, Chapters 4 and 5). The hard technical work is delegated to Volume I, Chapter 8.

Volume II, Chapter 4 explains how the theory of crystals/D-modules follows from the theory of IndCoh on inf-schemes. Nothing in this Chapter is very exciting, but it should not be a difficult read either.

Volume II, Chapter 5 is about formal moduli problems. It proves a pretty strong result, namely, the equivalence of categories between formal groupoids acting on a given prestack $\mathcal{X}$ (assumed to admit deformation theory) and formal moduli problems *under* $\mathcal{X}$.

Volume II, Chapter 6 is a digression on the general notion of Lie algebra and Koszul duality in a symmetric monoidal DG category. It gives a nice interpretation of the universal enveloping algebra of a Lie algebra of $\mathfrak{g}$ as the homological Chevalley complex of the Lie algebra obtained by *looping* $\mathfrak{g}$. The reader may find this Chapter useful and independently interesting.

Volume II, Chapter 7 develops Lie theory in the context of inf-schemes. Namely, it establishes an equivalence of categories between group inf-schemes (over a given base $\mathcal{X}$) and Lie algebras in IndCoh($\mathcal{X}$). One can regard this result as one of the main applications of the theory developed hereto.

Volume II, Chapters 8 and 9 use the theory developed in the preceding Chapters for 'differential calculus' in the context of DAG. We discuss Lie algebroids and their universal envelopes, the procedure of deformation to the normal cone, etc. For example, the notion of n-th infinitesimal neighborhood developed in Volume II, Chapter 9 gives rise to the Hodge filtration.

3.3. The nature of the technical work. The substance of mathematical thought in this book can be roughly split into three modes of cerebral activity: (a) making constructions; (b) overcoming difficulties of homotopy-theoretic nature; (c) dealing with issues of *convergence*.

Mode (a) is hard to categorize or describe in general terms. This is what one calls 'the fun part'.

Mode (b) is something much better defined: there are certain constructions that are obvious or easy for ordinary categories (e.g., define categories or functors by an explicit procedure), but require some ingenuity in the setting of higher categories. For many readers that would be the least fun part: after all it is clear that the thing should work, the only question is how to make it work without spending another 100 pages.

Mode (c) can be characterized as follows. In low-tech terms it consists of showing that certain spectral sequences converge. In a language better adapted for our needs, it consists of proving that in some given situation we can swap a limit and a colimit (the very idea of IndCoh was born from this mode of thinking). One can say that mode (c) is a sort of analysis within algebra. Some people find it fun.

Here is where the different Chapters stand from the point of view of the above classification:

Volume I, Chapter 1 is (b) and a little of (c).

Volume I, Chapter 2 is (a) and a little of (c).

Volume I, Chapter 3 is (c).

Volume I, Chapter 4 is (a) and (c).

Volume I, Chapter 5 is (a).

Volume I, Chapter 6 is (b).

Volume I, Chapters 7-9 are (b).

Volume I, Chapters 10-12 are (b).

Volume II, Chapter 1 is (a) and a little of (c).

Volume II, Chapter 2 is (a) and a little of (c).

Volume II, Chapter 3 is (a).

Volume II, Chapter 4 is (a).

Volume II, Chapter 5 is (a).

Volume II, Chapter 6 is (c) and a little of (b).

Volume II, Chapter 7 is (c) and a little of (a).

Volume II, Chapters 8 and 9 are (a).

3.4. Logical dependence of chapters. This book is structured so that Volume I prepares the ground and Volume II reaps the fruit. However, below is a scheme of the logical dependence of chapters, where we allow a 5% *skip margin* (by which we mean that the reader skips certain things[10] and comes back to them when needed).

3.4.1. Volume I, Chapter 1 reviews ∞-categories and higher algebra. Read it only if you have no prior knowledge of these subjects. In the latter case, here is what you will need in order to understand the constructions in the main body of the book:

Read Sects. 1-2 to get an idea of how to operate with ∞-categories (this is a basis for everything else in the book).

Read Sects. 5-7 for a summary of *stable* ∞-categories: this is what our QCoh($-$) and IndCoh($-$) are; forget on the first pass about the additional structure of k-linear DG category (the latter is discussed in Sect. 10).

[10]These are things that can be taken on faith without compromising the overall understanding of the material.

Read Sects. 3-4 for a summary of monoidal structures and duality in the context of higher category theory. You will need it for this discussion of Serre duality and for Volume I, Chapter 6.

Sects. 8-9 are about algebra in (symmetric) monoidal stable ∞-categories. You will need it for Volume II, Part II of the book.

Volume I, Chapter 2 introduces DAG proper. If you have not seen any of it before, read Sect. 1 for the (shockingly general, yet useful) notion of prestack. Every category of geometric objects we will encounter in this book (e.g., (derived) schemes, Artin stacks, inf-schemes, etc.) will be a full subcategory of the ∞-category of prestacks. Proceed to Sect. 3.1 for the definition of derived schemes. Skip all the rest.

Volume I, Chapter 3 introduces QCoh on prestacks. Even though the main focus of this book is the theory of ind-coherent sheaves, the latter theory takes a significant input and interacts with that of quasi-coherent sheaves. If you have not seen this before, read Sect. 1 and then Sects. 3.1-3.2.

3.4.2. In Volume I, Chapter 4 we develop the elementary aspects of the theory of IndCoh on schemes: we define the DG category $\text{IndCoh}(Z)$ for an individual scheme Z, construct the IndCoh direct image functor, and also the !-pullback functor for proper morphisms. This Chapter uses the material from Volume I, Part I mentioned above. You will need the material from this chapter in order to proceed with the reading of the book.

Volume I, Chapter 5 builds on Volume 1, Chapter 4, and accomplishes (modulo the material delegated to Volume I, Chapter 7) one of the main goals of this book. We construct IndCoh as a *functor out of the category of correspondences*. In particular, we construct the functor (2.3). The material from this Chapter is also needed for the rest of the book.

In Volume I, Chapter 6 we study the interaction between IndCoh and QCoh. For an individual scheme Z we have an action of $\text{QCoh}(Z)$ (viewed as a monoidal category) on $\text{IndCoh}(Z)$. We study how this action interacts with the formalism of correspondences from Volume I, Chapter 5, and in particular with the operation of !-pullback. The material in this Chapter uses the formalism of monoidal categories and modules over them from Volume I, Chapter 1, as well as the material from Volume I, Chapter 5. Skipping Volume I, Chapter 6 will not impede your understanding of the rest of the book, so it might be a good idea to do so on the first pass.

3.4.3. Volume I, Part II develops the theory of categories of correspondences. It plays a service role for Volume I, Chapter 6 and Volume II, Chapter 3, and relies on the theory of $(\infty, 2)$-categories, developed in Volume I, Appendix.

3.4.4. Volume I, Appendix develops the theory of $(\infty, 2)$-categories. It plays a service role for Volume I, Part III.

Volume I, Chapters 11 and 12 rely on Volume I, Chapter 10, but can be read independently of one another.

3.4.5. Volume II, Chapter 1 introduces deformation theory. It is needed for the definition of inf-schemes and, therefore, for proofs of any results about inf-schemes (that is, for Volume II, Chapter 2). We will also need it for the discussion of formal moduli problems in Volume II, Chapter 5. The prerequisites for Volume II, Chapter 1 are Volume I, Chapters 2 and 3, so it is (almost)[11] independent of the material from Volume I, Part II.

In Volume II, Chapter 2 we introduce inf-schemes and some related notions (ind-schemes, ind-inf-schemes). The material here relies in that of Volume II, Chapter 1, and will be needed in Volume II, Chapter 3.

In Volume II, Chapter 3 we construct the theory of IndCoh on inf-schemes. The material here relies on that from Volume I, Chapter 5 and Volume II, Chapter 2 (and also a tedious general result about correspondences from Volume I, Chapter 8). Thus, Volume II, Chapter 3 achieves one of our goals, the later being making the theory of IndCoh on inf-schemes available. The material from Volume II, Chapter 3 will (of course) be used when will apply the theory of IndCoh, in Volume II, Chapter 4 and 7–9.

In Volume II, Chapter 4 we apply the material from Volume II, Chapter 3 in order to develop a proper framework for crystals (=D-modules), together with the forgetful/induction functors that related D-modules to $\mathcal{O}$-modules. The material from this Chapter will not be used later, except for the extremely useful notion of the de Rham prestack construction $\mathcal{X} \rightsquigarrow \mathcal{X}_{\mathrm{dR}}$.

3.4.6. In Volume II, Chapter 5 we prove a key result that says that in the category of prestacks that admit deformation theory, the operation of taking the quotient with respect to a formal groupoid is well-defined. The material here relies on that from Volume II, Chapter 1 (at some point we appeal to a proposition from Volume II, Chapter 3, but that can be avoided). So, the main result from Volume II, Chapter 5 is independent of the discussion of IndCoh.

Volume II, Chapter 6 is about Lie algebras (or more general operad algebras) in symmetric monoidal DG categories. It only relies on the material from Volume I, Chapter 1, and is independent of the preceding Chapters of the book (no DAG, no IndCoh). The material from this Chapter will be used for the subsequent Chapters in Volume II, Part II.

3.4.7. *A shortcut.* As has been mentioned earlier, Volume II, Chapters 7–9 are devoted to applications of IndCoh to 'differential calculus'. This 'differential calculus' occurs on prestacks that admit deformation theory.

If one really wants to use arbitrary such prestacks, one needs the entire machinery of IndCoh provided by Volume II, Chapter 3. However, if one is content with

[11]Whenever we want to talk about *tangent* (as opposed to *cotangent*) spaces, we have to use IndCoh rather than QCoh, and these parts in Volume II, Chapter 1 use the material from Volume I, Chapter 5.

working with inf-schemes (which would suffice for the majority of applications), much less machinery would suffice:

The cofinality result from Volume II, Chapter 3, Sect. 4.3 implies that we can bypass the entire discussion of correspondences, and only use the material from Volume I, Chapter 4, i.e., IndCoh on schemes and !-pullbacks for proper (in fact, finite) morphisms.

3.4.8. Volume II, Chapters 7-9 form a logical succession. As input from the preceding chapters they use Volume II, Chapter 3 (resp., Volume I, Chapter 5 (see Sect. 3.4.7 above), Volume II, Chapter 1 and Volume II, Chapters 5–6.