Preface

Aim of the book. The aim of this book is to introduce and develop an arithmetic analogue of *classical differential geometry*; this analogue will be referred to as *arithmetic differential geometry*. In this new geometry the ring of integers $\mathbb{Z}$ will play the role of a ring of functions on an infinite dimensional manifold. The role of coordinate functions on this manifold will be played by the prime numbers $p \in \mathbb{Z}$. The role of partial derivatives of functions with respect to the coordinates will be played by the *Fermat quotients*, $\delta_p n := \frac{n-n^p}{p} \in \mathbb{Z}$, of integers $n \in \mathbb{Z}$ with respect to the primes p. The role of metrics (respectively 2-forms) will be played by symmetric (respectively anti-symmetric) matrices with coefficients in $\mathbb{Z}$. The role of connections (respectively curvature) attached to metrics or 2-forms will be played by certain adelic (respectively global) objects attached to matrices as above. One of the main conclusions of our theory will be that (the "manifold" corresponding to) $\mathbb{Z}$ is "intrinsically curved;" this curvature of $\mathbb{Z}$ (and higher versions of it) will be encoded into a $\mathbb{Q}$-Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$, which we refer to as the *holonomy algebra*, and the study of this algebra is, essentially, the main task of the theory.

Needless to say, *arithmetic differential geometry* is still in its infancy. However, its foundations, which we present here, seem to form a solid platform upon which one could further build. Indeed, the main differential geometric concepts of this theory turn out to be related to classical number theoretic concepts (e.g., Christoffel symbols are related to Legendre symbols); existence and uniqueness results for the main objects (such as the arithmetic analogues of Ehresmann, Chern, Levi-Cività, and Lax connections) are being proved; the problem of defining curvature (which in arithmetic turns out to be non-trivial) is solved in some important cases (via our method of analytic continuation between primes and, alternatively, via algebraization by correspondences); and some basic vanishing/non-vanishing theorems are being proved for various types of curvature. It is hoped that all of the above will create a momentum for further investigation and further discovery.

Immediate context. A starting point for this circle of ideas can be found in our paper [**23**] where we showed how to construct arithmetic analogues of the classical jet spaces of Lie and Cartan; these new spaces were referred to as *arithmetic jet spaces*. The main idea, in this construction, was to replace classical derivation operators acting on functions with *Fermat quotient operators* acting on numbers and to develop an *arithmetic differential calculus* that parallels classical calculus. There were two directions of further development: one towards a theory of *arithmetic differential equations* and another one towards an *arithmetic differential geometry*. A theory of *arithmetic differential equations* was developed in a series of papers [**23**]-[**42**], [**7**] and was partly summarized in our monograph [**35**] (cf. also the survey papers [**43, 102**]); this theory led to a series of applications to invariant theory [**27, 7, 28, 35**], congruences between modular forms [**27, 37, 38**], and Diophantine geometry of Abelian and Shimura varieties [**24, 36**]. On the other hand an *arithmetic differential geometry* was developed in a series of papers [**42**]-[**47**], [**8**]; the present book follows, and further develops, the theory in this latter series of papers.

We should note that our book [**35**] on *arithmetic differential equations* and the present book on *arithmetic differential geometry*, although both based on the same conceptual framework introduced in [**23**], are concerned with rather different objects. In particular the two books are independent of each other and the overlap between them is minimal. Indeed the book [**35**] was mainly concerned with arithmetic differential calculus on Abelian and Shimura varieties. By contrast, the present book is concerned with arithmetic differential calculus on the classical groups

$$GL_n, SL_n, SO_n, Sp_n,$$

and their corresponding symmetric spaces.

Of course, the world of Abelian/Shimura varieties and the world of classical groups, although not directly related within *abstract algebraic geometry*, are closely related through *analytic* concepts such as *uniformization* and *representation theory.* The prototypical example of this relation is the identification

$$M_1(\mathbb{C}) \simeq SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

where M_1 is the coarse moduli space of Abelian varieties

$$\mathbb{C}/\text{lattice}$$

of dimension one; the curve M_1 is, of course, one of the simplest Shimura varieties. It is conceivable that the analytic relation between the two worlds referred to above could be implemented via certain *arithmetic differential correspondences* (correspondences between arithmetic jet spaces); in particular it is conceivable that the subject matter of the present book and that of our previous book [**35**] might be related in ways that go beyond what can be seen at this point. A suggestion for such a possible relation comes from the theory of *δ-Hodge structures* developed in [**22**] and from a possible arithmetic analogue of the latter that could emerge from [**7, 35**].

There are at least two major differential geometric themes that are missing from the present book: *geodesics* and the *Laplacian.* It is unclear at this point what the arithmetic analogues of geodesics could be. On the other hand, for first steps towards an arithmetic Laplacian (and, in particular, for a concept of curvature based on it), we refer to [**34**]. But note that the flavor of [**34**] is quite different from that of the present book. Again, it is conceivable that the arithmetic Laplacian theory in [**34**] could be related, in ways not visible at this point, to our theory here.

Larger context. By what has been said so far this book is devoted to unveiling a new type of "geometric" structures on $\mathbb{Z}$. This is, of course, in line with the classical effort of using the analogy between numbers and functions to the advantage of number theory; this effort played a key role in the development of number theory throughout the 20th century up to the present day, as reflected in the work of Dedekind, Kronecker, Hilbert, Artin, Weil, Iwasawa, Grothendieck, Manin, Parshin, Arakelov (to name just a few of the early contributors). However, as we shall explain in the Introduction, this classical theory (which views number fields as analogous to Riemann surfaces) differs in essential ways from the theory of the present book (which views number fields as analogous to infinite dimensional manifolds). Our theory should also be contrasted with more recent geometric theories of "the discrete" such as: arithmetic topology [**104**]; the theory around the p-curvature conjecture of Grothendieck [**87**]; the Ihara differential [**78**]; the Fontaine-Colmez calculus [**64**]; discrete differential geometry [**10**]; and the geometries over the field with one element, $\mathbb{F}_1$, proposed by a number of people including: Tits [**118**], Kapranov-Smirnov [**86**], Soulé [**115**], Deitmar [**56**], Kurokawa et.al. [**93**], Connes-Consani [**55**], Manin-Marcolli [**117**], Lorscheid [**117**], Haran [**70**], etc. The non-vanishing curvature in our theory also prevents our arithmetic differential geometry from directly fitting into Borger's λ-ring framework [**13**] for $\mathbb{F}_1$; indeed, roughly speaking, λ-ring structure leads to zero curvature. For each individual prime, though, our theory is consistent with Borger's philosophy of $\mathbb{F}_1$; cf. [**13, 102**] and the Introduction to [**35**]. In spite of the differences between these theories and ours we expect interesting analogies and interactions.

This being said what is, after all, the position of our theory among more established mathematical theories? The answer we would like to suggest is that our curvature of $\mathbb{Z}$, encoded in the Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$, could be viewed as an infinitesimal counterpart of the absolute Galois group $\Gamma_{\mathbb{Q}} := Gal(\mathbb{Q}^a/\mathbb{Q})$ of $\mathbb{Q}$: our Lie algebra $\mathfrak{hol}_{\mathbb{Q}}$ should be to the absolute Galois group $\Gamma_{\mathbb{Q}}$ what the identity component Hol^0 of the holonomy group Hol is to the monodromy group Hol/Hol^0 in classical differential geometry. As such our $\mathfrak{hol}_{\mathbb{Q}}$ could be viewed as an object of study in its own right, as fundamental, perhaps, as the absolute Galois group $\Gamma_{\mathbb{Q}}$ itself. A "unification" of $\mathfrak{hol}_{\mathbb{Q}}$ and $\Gamma_{\mathbb{Q}}$ may be expected in the same way in which Hol^0 and Hol/Hol^0 are "unified" by Hol. Such a unification of $\mathfrak{hol}_{\mathbb{Q}}$ with $\Gamma_{\mathbb{Q}}$ might involve interesting Galois representations in the same way in which the unification of $\Gamma_{\mathbb{Q}}$ with the geometric fundamental group via the arithmetic fundamental group in Grothendieck's theory yields interesting Galois representations [**77, 57**].

Organization of the book. The book starts with an Introduction in which we give an outline of our theory and we briefly compare our theory with other theories.

In Chapter 1 we briefly review the basic algebra and superalgebra concepts that we are going to use throughout the book. Most of this material is standard so the reader may choose to skip this chapter and consult it later as needed.

Chapter 2 will be devoted to revisiting classical differential geometry from an algebraic standpoint. We will be using the language of differential algebra [**91, 112**], i.e., the language of rings with derivations. We call the attention upon the fact that some of the classical differential geometric concepts will be presented in a somewhat non-conventional way in order to facilitate and motivate the transition to the arithmetic setting. This chapter plays no role in our book other than that of a motivational and referential framework; so, again, the reader may choose to skip this chapter and then go back to it as needed or just in order to compare the arithmetic theory with (the algebraic version of) the classical one.

Chapter 3 is where the exposition of our theory properly begins; here we present the basic notions of arithmetic differential geometry and, in particular, we introduce our arithmetic analogues of connection and curvature. The theory will be presented in the framework of arbitrary (group) schemes.

Chapter 4 specializes the theory in Chapter 3 to the case of the group scheme GL_n; here we prove, in particular, our main existence results for Ehresmann, Chern, Levi-Cività, and Lax connections respectively.

Chapters 5, 6, 7, 8 are devoted to the in-depth analysis of these connections; in particular we prove here the existence of the analytic continuation between primes necessary to define curvature for these connections and we give our vanishing/non-vanishing results for these curvatures. In Chapter 5, we also take first steps towards a corresponding (arithmetic differential) Galois theory.

The last Chapter 9 lists some of the natural problems, both technical and conceptual, that one faces in the further development of the theory.

Chapters 1, 2, 3, 4 should be read in a sequence (with Chapters 1 and 2 possibly skipped and consulted later as needed); Chapters 5, 6, 7, 8 depend on Chapters 1, 2, 3, 4 but are essentially independent of each other. Chapter 9 can be read right after the Introduction.

Cross references are organized in a series of sequences as follows. Sections are numbered in one sequence and will be referred to as "section $x.y$." Definitions, Theorems, Propositions, Lemmas, Remarks, and Examples are numbered in a separate sequence and are referred to as "Theorem $x.y$, Example $x.y$," etc. Finally equations are numbered in yet another separate sequence and are referred to simply as "$x.y$." For all three sequences x denotes the number of the chapter.

Readership and prerequisites. The present book addresses graduate students and researchers interested in algebra, number theory, differential geometry, and the analogies between these fields. The only prerequisites are some familiarity with commutative algebra (cf., e.g., the Atiyah-MacDonald book [**4**] or, for more specialized material, Matsumura's book [**103**]) and with foundational scheme theoretic algebraic geometry (e.g., the first two chapters of Hartshorne's book [**72**]). The text also contains a series of remarks that assume familiarity with basic concepts of classical differential geometry (as presented in [**89**], for instance); but these remarks are not essential for the understanding of the book and can actually be skipped.

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