CHAPTER 1

Introduction

In this work we study the dynamics of Thurston maps under iteration. A Thurston map is a branched covering map on a 2-sphere S^2 such that each of its critical points has a finite orbit. The most important examples are given by postcritically-finite rational maps on the Riemann sphere $\widehat{\mathbb{C}}$. Most of the time we will also assume that a Thurston map is expanding in a suitable sense. For postcritically-finite rational maps $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ expansion is equivalent to the requirement that f does not have periodic critical points or that its Julia set is equal to $\widehat{\mathbb{C}}$.

These objects were first considered by Thurston as topological model maps in the context of his celebrated characterization of rational maps (see Theorem 2.18). The terminology was introduced by Douady and Hubbard in their proof of this theorem.

Every expanding Thurston map $f\colon S^2 \to S^2$ gives rise to a type of fractal geometry on the underlying sphere S^2. This geometry is represented by a class of *visual metrics* ϱ that are associated with the map. Many dynamical properties of the map are encoded in the geometry of the corresponding *visual sphere*, meaning S^2 equipped with a visual metric ϱ.

For example, we will see that an expanding Thurston map is topologically conjugate to a rational map if and only if (S^2, ϱ) is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$ (see Section 4.1 for the terminology). For us this relation between dynamics and fractal geometry is one of the main motivations for studying expanding Thurston maps.

In order to define a visual metric for a given Thurston map $f\colon S^2 \to S^2$, we will extract some combinatorial data from f. For this we consider a cell decomposition of S^2 and its pull-backs by the iterates f^n. When f is expanding, the diameters of the cells in these decompositions shrink to 0; so we get discrete approximations of S^2 that get finer with larger level n. Given two distinct points in S^2, one can ask at which level the cell decompositions will allow us to distinguish them. Our definition of a visual metric is based on this information.

The visual sphere (S^2, ϱ) of an expanding Thurston map is fractal in the sense that its Hausdorff dimension is typically larger than 2. With a suitable choice of ϱ, the local behavior of f becomes very simple though. Namely, there is a number $\Lambda > 1$ (the *expansion factor* of ϱ) such that f expands ϱ locally by the factor Λ in a sense that will be made precise. So the local behavior of f on (S^2, ϱ) is simplified at the expense of a more complicated geometry of (S^2, ϱ). This point of view is in contrast to the usual setting for complex dynamics, where one studies the action of a rational map on a *smooth* underlying space, namely the Riemann sphere $\widehat{\mathbb{C}}$, considered as a Riemann surface.

It is possible to construct a graph $\mathcal{G}$ that combines the combinatorial data of the cell decompositions on all levels generated by an expanding Thurston map and its iterates. This graph $\mathcal{G}$ is Gromov hyperbolic and its boundary at infinity can naturally be identified with the underlying sphere S^2. Under this identification a metric is a visual metric for the given map f according to our definition if and only if it is a visual metric in the sense of Gromov hyperbolic spaces. This fact relates the study of expanding Thurston maps and of Gromov hyperbolic spaces.

There is an intriguing connection of these ideas to *Cannon's conjecture* in geometric group theory. Roughly speaking, this conjecture predicts that a group G that shares the topological properties of the fundamental group of a closed hyperbolic 3-manifold "is" such a fundamental group (see Section 4.3 for precise statements). In this context one assumes that the group G is Gromov hyperbolic and that its boundary at infinity $\partial_\infty G$ is a 2-sphere. Here $\partial_\infty G$ is naturally equipped with a visual metric that provides $\partial_\infty G$ with a fractal geometry. Then Cannon's conjecture is equivalent to showing that the fractal sphere $\partial_\infty G$ is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$.

So for both types of dynamical systems, namely expanding Thurston maps and Gromov hyperbolic groups G with 2-sphere boundary $\partial_\infty G$, we are led to the investigation of a fractal geometry on the underlying 2-sphere. This analogy can be viewed as an example of *Sullivan's dictionary* which exhibits similarities in complex dynamics and the theory of Kleinian groups. Common to both areas is the desire to characterize conformal dynamical systems in a wider class of dynamical systems characterized by suitable metric-topological conditions. One should not push the analogies too far though: while Cannon's conjecture is generally believed to be true and, accordingly, one expects that the fractal 2-spheres arising from Gromov hyperbolic groups are always quasisymmetrically equivalent to $\widehat{\mathbb{C}}$, this is not always the case for Thurston maps, because not every Thurston map is equivalent to a rational map.

After these remarks about some of the motivations for our investigation, we now state some basic definitions more precisely (more details can be found in Chapter 2). Let $f\colon S^2 \to S^2$ be an (orientation-preserving) branched covering map. As usual, we call a point $c \in S^2$ a *critical point* of f if near c the map f is not a local homeomorphism. A *postcritical point* is any point obtained as an image of a critical point under forward iteration of f. So if we denote by $\operatorname{crit}(f)$ the set of critical points of f and by f^n the n-th iterate of f, then the set of postcritical points of f is given by

$$\operatorname{post}(f) := \bigcup_{n\geq 1} \{f^n(c) : c \in \operatorname{crit}(f)\}.$$

It is a fundamental fact in complex dynamics that much information on the dynamics can be deduced from the structure of the orbits of critical points. A very strong assumption in this respect is that each such orbit is finite, i.e., that $\operatorname{post}(f)$ is a finite set. In this case the map f is called *postcritically-finite*. A *Thurston map* is a (non-homeomorphic) branched covering map $f\colon S^2 \to S^2$ that is postcritically-finite.

Thurston maps are abundant and include specific *rational Thurston maps* (i.e., rational maps on $\widehat{\mathbb{C}}$ that are postcritically-finite) such as $f(z) = 1 - 2/z^2$ or $f(z) = 1 + (\boldsymbol{i} - 1)/z^4$. More examples can be found in Section 12.3, and a list of examples considered in this book is given in Section 1.9. We will later provide a general

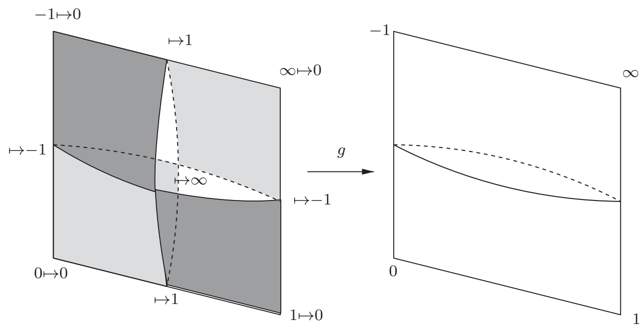

FIGURE 1.1. The Lattès map g.

method for producing Thurston maps (see Proposition 12.3); it follows from one of our main results (Theorem 15.1) that at least some iterate of every expanding Thurston map can be obtained from this construction.

We now turn to the discussion of more specific topics in this introductory chapter. Our main purpose is to give some guidance for the intuition of the reader. We will present some examples and discuss the main concepts and results of this work. Full details can be found in subsequent chapters.

1.1. A Lattès map as a first example

Lattès maps form a large class of well-understood Thurston maps. They are rational maps obtained as quotients of holomorphic torus endomorphisms. They were the first known examples of rational maps whose Julia set is the whole sphere. We will discuss these maps in more detail in Chapter 3; results concerning them will be outlined in Section 1.7. Note that the terminology is not uniform and some authors use the term Lattès map with a slightly different meaning.

We will encounter Lattès maps quite often in this book. On the one hand, they are easy to visualize and construct, and thus often serve as convenient examples to illustrate various phenomena. On the other hand, these maps are quite special and arise in many situations as exceptional cases. In order to introduce some of the main themes of this work, we will now consider a specific Lattès map.

The map is essentially given by Figure 1.1. We will explain this picture in detail momentarily, but we will first define the map by a more standard approach. This may be helpful for readers that are already familiar with Lattès maps.

The square $[0, \frac{1}{2}]^2 \subset \mathbb{R}^2 \cong \mathbb{C}$ can be mapped conformally to the upper half-plane in $\widehat{\mathbb{C}}$ such that the vertices $0, \frac{1}{2}, \frac{1}{2} + \frac{\boldsymbol{i}}{2}, \frac{\boldsymbol{i}}{2}$ of the square are mapped to the points $0, 1, \infty, -1$, respectively. Note that here and in the following a "conformal map" is always bijective. By Schwarz reflection we can extend this to a holomorphic map $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$. Up to postcomposition with a Möbius transformation, this map is a classical *Weierstraß ℘-function*; it is doubly-periodic with respect to the lattice $\Gamma := \mathbb{Z} \oplus \mathbb{Z}\boldsymbol{i}$ and induces a double branched covering map of the torus $\mathbb{T} := \mathbb{C}/\Gamma$ to the sphere $\widehat{\mathbb{C}}$.

Consider the map

$$A\colon \mathbb{C} \to \mathbb{C}, \quad u \mapsto A(u) := 2u.$$

From the properties of the $\wp$-function or directly from the definition of Θ by the reflection process, one can see that $\Theta(v) = \Theta(u)$ for $u, v \in \mathbb{C}$ if and only if $v = \pm u + \gamma$ with $\gamma \in \Gamma$. In this case, $\Theta(2v) = \Theta(2u)$. This implies that there is a well-defined and unique holomorphic map $g \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ such that the diagram

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{A} & \mathbb{C} \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ \widehat{\mathbb{C}} & \xrightarrow{g} & \widehat{\mathbb{C}} \end{array} \tag{1.1}$$

commutes. The map g obtained in this way is a *Lattès map*. It is a rational map. One can show that it is given by

$$g(z) = 4\frac{z(1-z^2)}{(1+z^2)^2} \quad \text{for } z \in \widehat{\mathbb{C}},$$

and that the Julia set of g is the whole sphere.

More relevant for us than this explicit formula for g is that one can describe g geometrically as indicated in Figure 1.1. To explain this, note that there is an essentially unique path metric on $\widehat{\mathbb{C}}$ obtained as a "push-forward" of the Euclidean metric on $\mathbb{C}$ by the map Θ. This metric is in fact the *canonical orbifold metric* of g (see Section A.10 and Section 2.5).

Geometrically, the sphere equipped with this metric looks like a pillow. In general, a *pillow* (see Section A.10) is a metric space P obtained from gluing two identical copies $X_{\mathtt{w}}$ and $X_{\mathtt{b}}$ of a (simple and compact) Euclidean polygon $X \subset \mathbb{C}$ together along their boundaries. The pillow is equipped with the induced path metric. Under the given identification, $\partial X_{\mathtt{w}} \cong \partial X_{\mathtt{b}}$ is a Jordan curve in the pillow P called its *equator*.

In our case, the upper and lower half-planes in $\widehat{\mathbb{C}}$ equipped with the canonical orbifold metric are isometric to copies of the square $S = [0, 1/2]^2$. If we glue two copies of S together along their boundaries, then we obtain the pillow P. We color one of these squares, say the one corresponding to the upper half-plane, white, and the other square black.

The square $S = [0, 1/2]^2 \subset \mathbb{R}^2 \cong \mathbb{C}$ (and each of its translates by $\frac{1}{2}(m + n\boldsymbol{i})$ where $m, n \in \mathbb{Z}$) can be subdivided into four squares of side length $1/4$. If S' is such a square, then $A(S')$ is a square of side length $1/2$ that is mapped by Θ to either the upper or the lower half-plane, meaning to either the black or the white face of the pillow P. It follows from (1.1) that g has a very similar mapping behavior on P.

More precisely, we divide each of the two sides of P (each of the two isometric copies of S contained in P) into four smaller squares of half the side length, and color the eight small squares in a checkerboard fashion black and white. If we map one such small white square to the large white square by a Euclidean similarity (that scales by the factor 2), then this map extends by reflection to the whole pillow. There are obviously many different ways to color and map the small squares. If we do this in an appropriate way as indicated in Figure 1.1, then we obtain the map g.

The vertices where four small squares intersect are the critical points of g. They are mapped by g to the set $\{1, \infty, -1\}$, which in turn is mapped to $\{0\}$. The point

0 is a fixed point of g. So g is a postcritically-finite branched covering map on the 2-sphere P with $\mathrm{post}(g) = \{0, 1, \infty, -1\}$, and hence a Thurston map. The postcritical points of g are the vertices of the pillow, which are the conical singularities of our canonical orbifold metric. The extended real line $\mathcal{C} := \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ (corresponding to the equator of the pillow) is a Jordan curve that is invariant under g in the sense that $g(\mathcal{C}) \subset \mathcal{C}$ and contains the set $\{0, 1, \infty, -1\}$ of postcritical points of g. The set $g^{-1}(\mathcal{C})$ is an embedded graph in the pillow consisting of all sides of the small squares on the left hand side of Figure 1.1 as edges and the points in $g^{-1}(\mathrm{post}(g))$, i.e., the corners of these squares, as vertices. This graph $g^{-1}(\mathcal{C})$ determines the tiling in this picture.

The set $g^{-2}(\mathcal{C})$ is obtained by pulling $g^{-1}(\mathcal{C})$ back by the map g. Since g restricted to any small square S' is a homeomorphism onto one of the two large squares S forming the pillow, in this process S' is subdivided in the same way as S was subdivided by the small squares of side length 1/4 (i.e., S' is subdivided into 4 squares). It follows that $g^{-2}(\mathcal{C})$ subdivides the pillow into $4 \times 8 = 32$ squares of side length 1/8. Proceeding in this way inductively, we see that the preimage $g^{-n}(\mathcal{C})$ of $\mathcal{C}$ under the iterate g^n subdivides the pillow into $2 \cdot 4^n$ squares of side length 2^{-n-1} for $n \in \mathbb{N}$.

The complementary components of $g^{-n}(\mathcal{C})$ are the interiors of these squares. In particular, the diameters of these components tend to 0 uniformly as $n \to \infty$. This fact will be the basis of our definition of an *expanding* Thurston map. Accordingly, g is such a map.

For each $n \in \mathbb{N}$ the set $g^{-n}(\mathcal{C})$ forms an embedded graph in the pillow P with the points in $g^{-n}(\mathrm{post}(g))$ as vertices. This is also meaningful for $n = 0$, if we interpret g^0 as the identity map on the pillow P. Then this graph is just the Jordan curve $\mathcal{C}$ with the points in $\mathrm{post}(g)$ as vertices.

The graph $g^{-n}(\mathcal{C})$ is the 1-dimensional skeleton or 1-skeleton of a *cell decomposition* $\mathcal{D}^n = \mathcal{D}^n(g, \mathcal{C})$ of the pillow P generated by g and $\mathcal{C}$ (see Chapter 5 for the terminology that we use here and below). The 2-dimensional cells or *tiles* of the cell decomposition $\mathcal{D}^n$ are squares of side length 2^{-n-1} and are given by the closures of the complementary components of $g^{-n}(\mathcal{C})$ in P. The map g sends each cell in $\mathcal{D}^{n+1}$ homeomorphically to a cell in $\mathcal{D}^n$ (for all $n \in \mathbb{N}_0$); so g is *cellular* for each pair $(\mathcal{D}^{n+1}, \mathcal{D}^n)$ of cell decompositions.

Since $\mathcal{C}$ is g-invariant in the sense that $g(\mathcal{C}) \subset \mathcal{C}$, we have $g^{-n}(\mathcal{C}) \subset g^{-(n+1)}(\mathcal{C})$ for each $n \in \mathbb{N}_0$. This inclusion for 1-skeleta implies that the cell decomposition $\mathcal{D}^{n+1}$ is a *refinement* of $\mathcal{D}^n$. On a more intuitive level, this means that the tiles in $\mathcal{D}^n$ are subdivided by the tiles in $\mathcal{D}^{n+1}$.

The tiles in $\mathcal{D}^0$ are the two initial squares of side length 1/2 forming the pillow, and $\mathcal{D}^1$ is formed by squares of side length 1/4 subdividing these squares. Since we repeat the same subdivision procedure in the passage from $\mathcal{D}^n$ to $\mathcal{D}^{n+1}$, this whole sequence of cell decompositions $\mathcal{D}^n$ is essentially generated by the initial pair $(\mathcal{D}^1, \mathcal{D}^0)$. This pair $(\mathcal{D}^1, \mathcal{D}^0)$ is a *cellular Markov partition* for g (see Definition 5.8). The map g sends each cell in $\mathcal{D}^1$ to a cell in $\mathcal{D}^0$. The cellular Markov partition $(\mathcal{D}^1, \mathcal{D}^0)$ together with this information completely determines the map g (up to conjugation). In this sense, the dynamics of g is described by finite combinatorial data.

In fact, one can turn this process around and construct g from this combinatorial data, meaning essentially from the information encoded in Figure 1.1. A related discussion can be found in Section 1.5.

1.2. Cell decompositions

The previous example motivates several concepts for a general Thurston map $f\colon S^2 \to S^2$, in particular the combinatorial description of f that we will employ.

We choose a Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$, and consider the preimages $f^{-n}(\mathcal{C})$. Then for each $n \in \mathbb{N}_0$ one obtains an associated cell decomposition $\mathcal{D}^n = \mathcal{D}^n(f, \mathcal{C})$ of S^2. Its vertices are the points in $f^{-n}(\operatorname{post}(f))$, and its 1-skeleton the set $f^{-n}(\mathcal{C})$. The condition $\operatorname{post}(f) \subset \mathcal{C}$ ensures that the closure of each complementary component of $f^{-n}(\mathcal{C})$ is a closed Jordan region. These sets are the 2-dimensional cells in $\mathcal{D}^n$. We call each such set a *tile of level* n or an n-*tile* (of the cell decomposition). Similarly, we call any point $v \in f^{-n}(\operatorname{post}(f))$ a *vertex* of level n or an n-*vertex*; then $\{v\}$ is a 0-dimensional cell in $\mathcal{D}^n$. Finally, the closure e of a component of $f^{-n}(\mathcal{C}) \setminus f^{-n}(\operatorname{post}(f))$ is called an *edge* of level n or an n-*edge*; then e is a 1-dimensional cell of $\mathcal{D}^n$. The cells in $\mathcal{D}^n(f, \mathcal{C})$ of any dimension are called the n-*cells* for given f and $\mathcal{C}$. Note that here n always refers to the level of the cell and not to its dimension.

The cell decomposition $\mathcal{D}^0$ contains two tiles (the two closed Jordan regions in S^2 bounded by $\mathcal{C}$), $k = \#\operatorname{post}(f)$ vertices (the points $p \in \operatorname{post}(f)$), and k edges (the closed arcs into which the points in $\operatorname{post}(f)$ divide $\mathcal{C}$). We will study cell decompositions and their relation to Thurston maps in more detail in Chapter 5. Various examples for the cell decompositions $\mathcal{D}^n$ generated in this way can be found in Figures 8.1, 12.1, 12.7, and 15.1.

We say that a Thurston map $f\colon S^2 \to S^2$ is *expanding* if there exists a Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$ such that the complementary components of $f^{-n}(\mathcal{C})$ become uniformly small in diameter as $n \to \infty$. Here S^2 is equipped with any metric inducing the topology on S^2. It is easy to see that this condition is independent of the choice of this base metric. Later we will show that it is also independent of the choice of $\mathcal{C}$ and will give other characterizations of expansion (see Chapter 6, in particular Proposition 6.4). A rational Thurston map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is expanding precisely if it does not have periodic critical points or if its Julia set is equal to $\widehat{\mathbb{C}}$ (see Proposition 2.3). Note that in general a (non-rational) expanding Thurston map may have periodic critical points (see Example 12.21).

Put differently, a Thurston map f is expanding if and only if the tiles in $\mathcal{D}^n = \mathcal{D}^n(f, \mathcal{C})$ shrink to 0 in diameter uniformly as $n \to \infty$. This allows us to describe points in S^2 by suitable sequences of tiles. So we can think of $\mathcal{D}^n$ as a discrete approximation of S^2 that becomes finer with larger n.

We have seen that for the example g discussed in the previous section the n-tiles become uniformly small in diameter as $n \to \infty$; so we conclude that g is an expanding Thurston map.

Often the precise choice of the Jordan curve $\mathcal{C}$ with $\operatorname{post}(f) \subset \mathcal{C}$ will play no essential role, meaning that we may chose any such curve for our considerations. If the curve $\mathcal{C}$ is not f-invariant, then in general the cell decompositions $\mathcal{D}^n$ will not be compatible for different levels n. Without precise knowledge of the behavior of the map we will then not have any information on how $(n+1)$-cells and n-cells intersect. The situation changes when the Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$

is f-invariant. We will discuss existence of such invariant Jordan curves and the resulting combinatorial description of Thurston maps in Section 1.5.

1.3. Fractal spheres

We want to motivate other important concepts of our investigation, in particular the concept of visual metrics. To do this, we will discuss another Thurston map and an associated fractal 2-sphere. As our main purpose here is to provide the reader with some intuition on the definition of a visual metric ϱ and on the fractal nature of the sphere (S^2, ϱ), we will omit the justification of some details.

The map arises from a geometric construction that is similar to the one used to describe the Lattès map in Section 1.1. Again we start with a pillow obtained by gluing together two squares along their boundaries; see the top right of Figure 1.2. This pillow is a polyhedral surface $\mathcal{S}^0$ homeomorphic to the 2-sphere. It carries a natural cell decomposition $\mathcal{D}^0$ with the two squares as tiles, the four sides of the common boundary of the squares as edges, and the four common corners of the squares as vertices. To distinguish them from other topological cells that we will introduce momentarily, we consider them as cells of level 0 and accordingly call them 0-tiles, 0-edges, and 0-vertices. As in the example of Section 1.1, we assign colors to the tiles; say the top square of $\mathcal{S}^0$ as shown in Figure 1.2 is white, and the bottom square is black.

To obtain cells on the next level 1, each of the two squares, or more precisely 0-tiles, is divided into four squares of half the side length. We call these eight smaller squares tiles of level 1, or simply 1-tiles. The edges of these squares are 1-edges. We slit the sphere along one such 1-edge in the white 0-tile and glue in two small squares at the slit, as indicated on the upper left in Figure 1.2. This gives two additional 1-tiles and we obtain a polyhedral surface $\mathcal{S}^1$ homeomorphic to the 2-sphere. The surface $\mathcal{S}^1$ carries a cell decomposition given by the topological cells of level 1 as described. We color the 1-tiles black and white in a checkerboard fashion so that 1-tiles sharing an edge have different color, as indicated in Figure 1.2.

To define a Thurston map based on our construction, we choose an identification of the polyhedral surface $\mathcal{S}^1$ with $\mathcal{S}^0$. To do this, we represent the six 1-tiles that replaced the white 0-tile topologically as subsets of this tile, and similarly the other four 1-tiles as subsets of the black 0-tile. So the 0-tiles are "subdivided" by 1-tiles. This is indicated on the lower left in Figure 1.2. Under this identification the cell decomposition of $\mathcal{S}^1$ gives a cell decomposition $\mathcal{D}^1$ on $\mathcal{S}^0$ that is a refinement of the cell decomposition $\mathcal{D}^0$.

Now we can define a Thurston map as follows. We map each white 1-tile on the polyhedral surface $\mathcal{S}^1$ to the white 0-tile in $\mathcal{S}^0$, and each black 1-tile in $\mathcal{S}^1$ to the black 0-tile in $\mathcal{S}^0$ by a similarity map (preserving orientation). This is a well-defined and uniquely determined map on $\mathcal{S}^1$ if we do this so that the 1-vertices marked by a black or a white dot on the upper left in Figure 1.2 are sent to 0-vertices in the upper right of the picture with the same markings.

If we identify $\mathcal{S}^1$ with $\mathcal{S}^0$ as discussed, we get a map $h\colon S^2 \to S^2$ on the 2-sphere $S^2 := \mathcal{S}^0$. Since h restricted to each 1-tile is a homeomorphism onto a 0-tile, h is a branched covering map. The critical points of h are the 1-vertices where at least four 1-tiles intersect. These critical points are all mapped to vertices of the pillow, i.e., to 0-vertices. All 0-vertices in turn are mapped to the 0-vertex marked black. Thus h is a postcritically-finite branched covering map on S^2, i.e., a Thurston map.

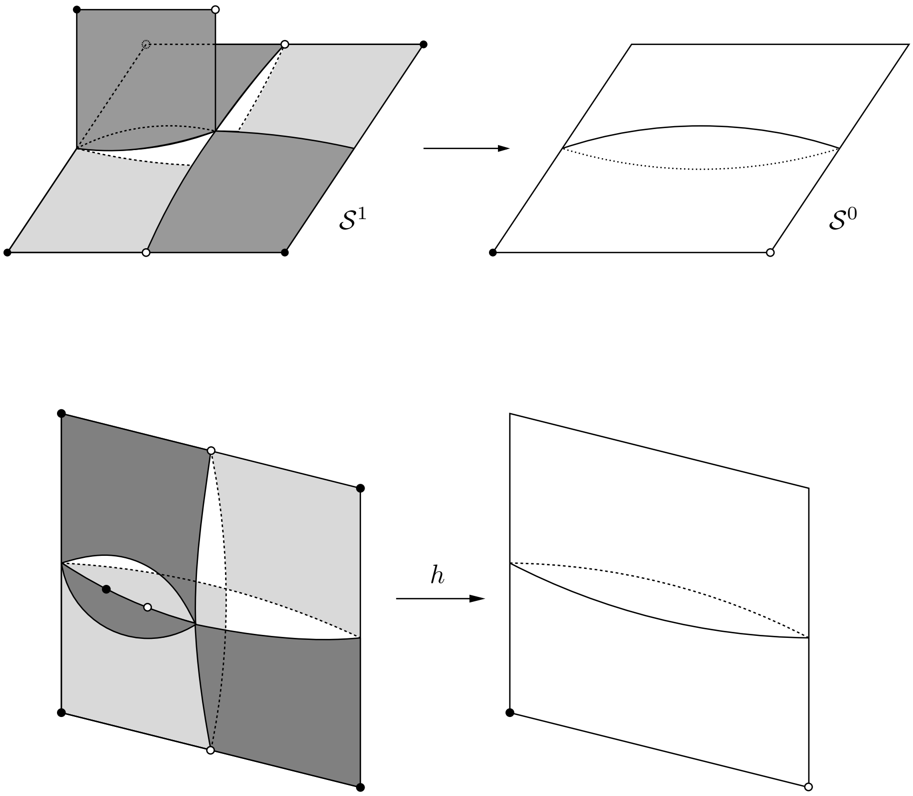

FIGURE 1.2. The map h.

Note that the equator $\mathcal{C}$ of the pillow is an h-invariant Jordan curve (i.e., $h(\mathcal{C}) \subset \mathcal{C}$) and that the cell decomposition $\mathcal{D}^1$ on $\mathcal{S}^1 \cong \mathcal{S}^0$ is determined by $h^{-1}(\mathcal{C})$. Namely, $h^{-1}(\mathcal{C})$ is a topological graph that gives the 1-skeleton of this cell decomposition, and each 1-tile is the closure of a complementary component of $h^{-1}(\mathcal{C})$.

The relevant information on the map h is contained in the combinatorics of the cell decompositions $\mathcal{D}^0$ and $\mathcal{D}^1$ and a map $L\colon \mathcal{D}^1 \to \mathcal{D}^0$ that records how h associates the cells in $\mathcal{D}^1$ with cells in $\mathcal{D}^0$. This triple $(\mathcal{D}^1, \mathcal{D}^0, L)$ forms a *two-tile subdivision rule* that is *realized* by h. We will give precise definitions of these concepts in Chapter 12. The map h depends on choices and is not uniquely determined, but another map realizing the same subdivision rule $(\mathcal{D}^1, \mathcal{D}^0, L)$ is *Thurston equivalent* to h (see Definition 2.4 for the terminology).

There is no rational map that realizes this combinatorial picture as our map h. More precisely, h is not Thurston equivalent to a rational map, because h has a *Thurston obstruction*. This, together with the terminology, will be explained in Section 2.6.

We will now describe a fractal sphere $\mathcal{S}$ that is associated with our construction and gives an alternative way to view our map h. The sphere $\mathcal{S}$ is obtained similarly as the well-known snowflake curve. We will also define a metric ϱ on $\mathcal{S}$.

To construct the space $\mathcal{S}$, we do not identify the surfaces $\mathcal{S}^0$ and $\mathcal{S}^1$. Instead, we consider the passage from $\mathcal{S}^0$ to $\mathcal{S}^1$ as a replacement procedure. The white

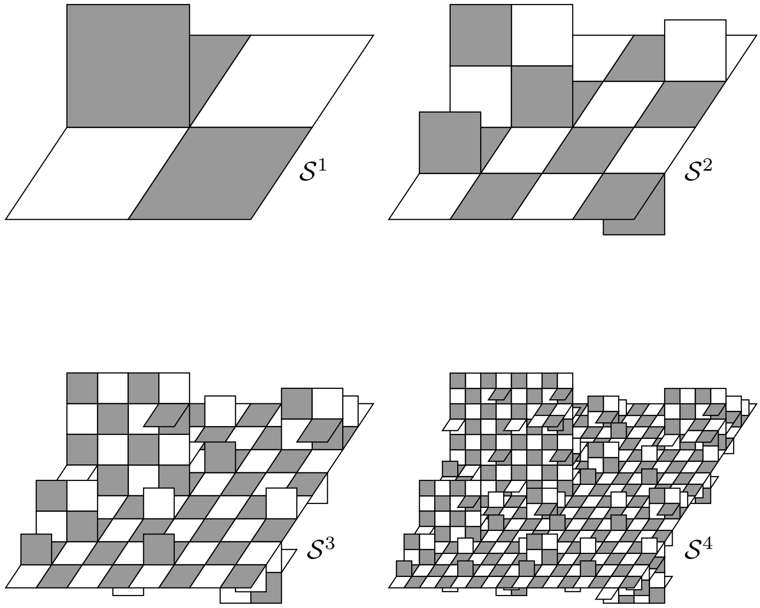

FIGURE 1.3. Polyhedral surfaces obtained from the replacement rule.

0-tile is replaced with the top part of $\mathcal{S}^1$, consisting of six 1-tiles, i.e., squares of side length 1/2; we call this top part of $\mathcal{S}^1$ the *white generator*. Similarly, the four 1-tiles subdividing the black 0-tile form the *black generator*. The polyhedral surface $\mathcal{S}^1$ consisting of ten squares is the first approximation of the fractal space $\mathcal{S}$ that we are about to construct by iterating this procedure.

Namely, the 1-tiles of the black and white generators are also colored as indicated in Figure 1.2. So if we replace each black or white 1-tile with a suitably scaled copy of the black or white generators, then we obtain a polyhedral surface $\mathcal{S}^2$ glued together from squares of side length 1/4 as 2-tiles. Here we have to be careful about how precisely a tile is replaced with an appropriate generator, because the generators with their given colorings of tiles are not symmetric with respect to rotations. To specify the replacement rule uniquely, we use the additional markings of some points. Each generator carries two points corresponding to the points on $\mathcal{S}^0$ marked black or white. In the replacement process we require that these points match the corresponding points with the same markings on 1-tiles.

If we iterate the replacement procedure in this way, we obtain polyhedral surfaces $\mathcal{S}^n$ for all levels $n \in \mathbb{N}_0$ glued together from squares of side length $1/2^n$. Each surface $\mathcal{S}^n$ carries a natural cell decomposition $\mathcal{D}^n$ given by these squares as tiles. Some iterates of this construction are shown in Figure 1.3. The pictures essentially indicate the gluing pattern of the squares which give the surfaces. One should view them as abstract polyhedral surfaces, and not confuse them with the underlying subsets of $\mathbb{R}^3$ in these pictures. Each surface $\mathcal{S}^n$ is a topological 2-sphere and carries a piecewise Euclidean path metric ϱ_n with conical singularities.

One can now extract a self-similar "fractal" space $\mathcal{S}$ as a limit $\mathcal{S}^n \to \mathcal{S}$ for $n \to \infty$ in several ways. One possibility is to pass to a Gromov-Hausdorff limit of the sequence $(\mathcal{S}^n, \varrho_n)$ of metric spaces. We will discuss a different method that is closer in spirit to our general definition of a visual metric (see Chapter 8 and Chapter 10; similar considerations appear in Chapter 14).

Namely, given an n-tile $\mathcal{X}^n \subset \mathcal{S}^n$ (which is a square of side length 2^{-n}), and an $(n+1)$-tile $\mathcal{X}^{n+1} \subset \mathcal{S}^{n+1}$, we write $\mathcal{X}^n \sqsupset \mathcal{X}^{n+1}$ if $\mathcal{X}^{n+1}$ is contained in the scaled copy of a generator that replaced $\mathcal{X}^n$ in the construction of $\mathcal{S}^{n+1}$ from $\mathcal{S}^n$. We now consider descending sequences $\mathcal{X}^0 \sqsupset \mathcal{X}^1 \sqsupset \mathcal{X}^2 \sqsupset \dots$. On an intuitive level the squares in such a sequence should shrink to a point in our desired limit space $\mathcal{S}$ represented by the sequence. Here we consider two sequences $\{\mathcal{X}^n\}$ and $\{\mathcal{Y}^n\}$ as equivalent and representing the same point if $\mathcal{X}^n \cap \mathcal{Y}^n \neq \emptyset$ for all $n \in \mathbb{N}_0$. It is not hard to see that this indeed defines an equivalence relation for descending sequences. By definition our limit space $\mathcal{S}$ is now the set of all equivalence classes.

For $x, y \in \mathcal{S}$ we set

$$\varrho(x,y) := \limsup_{n\to\infty} \operatorname{dist}_{\varrho_n}(\mathcal{X}^n, \mathcal{Y}^n), \tag{1.2}$$

where $\{\mathcal{X}^n\}$ and $\{\mathcal{Y}^n\}$ are sequences representing x and y, respectively. Then ϱ is well-defined and one can show that this is a metric on $\mathcal{S}$.

For $x, y \in \mathcal{S}$, $x \neq y$, we define

$$m(x,y) := \inf\{n \in \mathbb{N} : \mathcal{X}^n \cap \mathcal{Y}^n = \emptyset\}, \tag{1.3}$$

where the infimum is taken over all sequences $\{\mathcal{X}^n\}$ and $\{\mathcal{Y}^n\}$ representing x and y, respectively. Then

$$\varrho(x,y) \asymp 2^{-m(x,y)} \tag{1.4}$$

for $x, y \in \mathcal{S}$, $x \neq y$. This notation (which will be used frequently) means that there is a constant $C \geq 1$ such that

$$\frac{1}{C}\varrho(x,y) \leq 2^{-m(x,y)} \leq C\varrho(x,y).$$

We refer to the constant C as $C(\asymp)$ in such inequalities. In the present case, $C(\asymp)$ does not depend on x, y, or n. So roughly speaking, the distance of two distinct points in $\mathcal{S}$ is given in terms of the minimal level on which two descending sequences representing the points can distinguish them.

It is intuitively clear that $(\mathcal{S}, \varrho)$ is a topological 2-sphere. To outline a rigorous proof for this fact, we return to the Thurston map h defined above. Recall that $\mathcal{C} \subset S^2$ is the h-invariant Jordan curve given by the common boundary of the 0-tiles. The curve $\mathcal{C}$ contains the vertices of the pillow, which are the postcritical points of h. We consider the cell decompositions $\mathcal{D}^n(h, \mathcal{C})$ as discussed in the previous section. Note that the 1-tiles (i.e., the tiles in $\mathcal{D}^1(h, \mathcal{C})$) are exactly the 1-tiles in $\mathcal{S}^1$ under the identification $\mathcal{S}^1 \cong \mathcal{S}^0 = S^2$ (see the bottom left in Figure 1.2).

The map h^n sends each n-tile to a 0-tile homeomorphically, and we can assign colors to n-tiles so that h^n sends an n-tile to the 0-tile of the same color. In the passage from $\mathcal{D}^n(h, \mathcal{C})$ to $\mathcal{D}^{n+1}(h, \mathcal{C})$ each n-tile is subdivided by $(n+1)$-tiles in the same way as the 0-tile of the same color is subdivided by 1-tiles. From this it is clear that there is a one-to-one correspondence between n-tiles in $\mathcal{S}^n$ and n-tiles for the pair $(h, \mathcal{C})$. Moreover, these tiles realize identical combinatorics. More precisely, we have

$$\begin{aligned} \mathcal{X}^{n+1} \sqsupset \mathcal{X}^n \quad &\Leftrightarrow \quad X^{n+1} \supset X^n, \quad \text{and} \\ \mathcal{X}^n \cap \mathcal{Y}^n \neq \emptyset \quad &\Leftrightarrow \quad X^n \cap Y^n \neq \emptyset, \end{aligned}$$

where the n-tiles $\mathcal{X}^n$ and $\mathcal{Y}^n$ in $\mathcal{S}^n$ and the $(n+1)$-tile $\mathcal{X}^{n+1}$ in $\mathcal{S}^{n+1}$ correspond to the n-tiles X^n and Y^n and the $(n+1)$-tile X^{n+1} for $(h, \mathcal{C})$, respectively.

Recall from Section 1.2 that h is expanding if the diameters of n-tiles for $(h, \mathcal{C})$ tend to 0 uniformly as $n \to \infty$ (with respect to some fixed base metric on S^2 representing the topology). In this case one obtains a well-defined map $\varphi\colon \mathcal{S} \to S^2$ by sending a point in $\mathcal{S}$ represented by a descending sequence $\{\mathcal{X}^n\}$ to the unique point in the intersection $\bigcap_{n\in\mathbb{N}_0} X^n$ of the corresponding n-tiles for $(h, \mathcal{C})$. In general, our map h need not be expanding, but we may assume this if we choose the identification of $\mathcal{S}^1$ with $\mathcal{S}^0$ carefully (this hinges on the fact that h is "combinatorially expanding" and so the map can be corrected if necessary to make it expanding; see Theorem 14.2 for details). It is then not hard to see that φ is a homeomorphism, and so $\mathcal{S}$ is a 2-sphere.

Though $(\mathcal{S}, \varrho)$ is a topological 2-sphere, it is not a *quasisphere*. This means that this space is not quasisymmetrically equivalent to the standard 2-sphere (i.e., the unit sphere in $\mathbb{R}^3$, or equivalently the Riemann sphere $\widehat{\mathbb{C}}$ equipped with the chordal metric; see Section 4.1 for the definition of a quasisymmetry). This is closely related to the fact that h is not (equivalent to) a rational map. One can deduce that $(\mathcal{S}, \varrho)$ is not a quasisphere from a general result (see Theorem 18.1 (ii) mentioned in the next section), but one can also show this directly (we will outline an argument in Section 4.4).

The fractal sphere $(\mathcal{S}, \varrho)$ is in a sense the natural domain for our map h; namely, we can conjugate our original map $h\colon S^2 \to S^2$ by the homeomorphism $\varphi\colon \mathcal{S} \to S^2$ to obtain a map on $\mathcal{S}$, also denoted by h. We also obtain n-tiles in $\mathcal{S}$ corresponding to the n-tiles in S^2 under the homeomorphism φ. Roughly speaking, an n-tile in $\mathcal{S}$ is the part of $\mathcal{S}$ that "sits on top" of an n-tile in $\mathcal{S}^n$. The new map $h\colon \mathcal{S} \to \mathcal{S}$ then behaves locally like a similarity map: it scales each $(n+1)$-tile in $\mathcal{S}$ by a factor 2 and matches it with the corresponding n-tile.

1.4. Visual metrics and the visual sphere

After this example we return to the general setting. Let $f\colon S^2 \to S^2$ be an expanding Thurston map. We fix a Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset S^2$. Then for $n \in \mathbb{N}_0$ we have cell decompositions $\mathcal{D}^n = \mathcal{D}^n(f, \mathcal{C})$ with 1-skeleton $f^{-n}(\mathcal{C})$ as defined in Section 1.2.

Since f is expanding, the diameters of n-tiles (i.e., tiles in $\mathcal{D}^n$) shrink to 0 uniformly as $n \to \infty$. So if $x, y \in S^2$ are distinct points and X^n and Y^n are tiles of level n with $x \in X^n$ and $y \in Y^n$, then $X^n \cap Y^n = \emptyset$ for sufficiently large n. This implies that the number

$$\begin{aligned} (1.5) \qquad m(x,y) \coloneqq \max\{n \in \mathbb{N}_0 : {} &\text{there exist non-disjoint } n\text{-tiles} \\ &X^n \text{ and } Y^n \text{ with } x \in X^n, y \in Y^n\} \end{aligned}$$

is finite. Similar to (1.3), it records the level at which x and y can be separated by tiles. In Figure 8.1 we have illustrated this separation by tiles in an example.

Generalizing (1.4), we consider metrics ϱ on S^2 satisfying

$$\varrho(x,y) \asymp \Lambda^{-m(x,y)},$$

for some $\Lambda > 1$. We call such a metric a *visual metric* for f, and Λ its *expansion factor*. We will start investigating visual metrics in earnest in Chapter 8.

Visual metrics for a given Thurston map f are not unique, but two different visual metrics with the same expansion factor are bi-Lipschitz equivalent. They are snowflake equivalent if they have different expansion factors (see Section 4.1 for this terminology). Whether a metric is visual does not depend on the choice of the Jordan curve $\mathcal{C}$ that was used to define the quantity (1.5) via the cell decompositions $\mathcal{D}^n(f,\mathcal{C})$. Moreover, if $F = f^k$ is an iterate of f (where $k \in \mathbb{N}$), then a metric is visual for f if and only if it is visual for F. These (and other) basic properties of visual metrics can be found in Proposition 8.3.

If σ is a tile or an edge in the cell decomposition $\mathcal{D}^n = \mathcal{D}^n(f,\mathcal{C})$, then $\operatorname{diam}_\varrho(\sigma) \asymp \Lambda^{-n}$. In addition, any two disjoint cells $\sigma, \tau \in \mathcal{D}^n$ satisfy $\operatorname{dist}_\varrho(\sigma,\tau) \gtrsim \Lambda^{-n}$. This notation means that there is a constant $C > 0$ such that $C \operatorname{dist}_\varrho(\sigma,\tau) \geq \Lambda^{-n}$. We refer to the constant C as $C(\gtrsim)$. Equivalently, we write $\Lambda^{-n} \lesssim \operatorname{dist}_\varrho(\sigma,\tau)$ and refer to the constant C as $C(\lesssim)$. Here the constants $C(\asymp)$ and $C(\gtrsim)$ do not depend on n or the cells involved. In fact, these two geometric properties characterize visual metrics (see Proposition 8.4).

For the map g from Section 1.1 the length metric induced by the Euclidean metric on the pillow P is a visual metric with expansion factor $\Lambda = 2$. Similarly, the particular metric ϱ defined in Section 1.3 is a visual metric for h with expansion factor $\Lambda = 2$ (here we identify $\mathcal{S}$ with S^2 by the homeomorphism φ). In this case, we obtain visual metrics with arbitrary expansion factor $1 < \Lambda \leq 2$ if we consider a "snowflaked" metric ϱ^α with suitable $\alpha \in (0,1]$, but there is no visual metric for h with $\Lambda > 2$. Indeed, if ϱ is a visual metric with expansion factor $\Lambda > 1$ and X^n is an n-tile, then $\operatorname{diam}_\varrho(X^n) \lesssim \Lambda^{-n}$. Now it is easy to see that one can form a connected chain of n-tiles with 2^n elements that joins two non-adjacent 0-edges (as Figure 1.3 suggests, one obtains such a chain by running along the bottom 0-edge). Then by the triangle inequality $2^n \cdot \Lambda^{-n} \gtrsim 1$ for all $n \in \mathbb{N}$, and so $\Lambda \leq 2$.

Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then the supremum of all $\Lambda > 1$ for which there exist visual metrics with expansion factor Λ agrees with the *combinatorial expansion factor* of f, denoted by $\Lambda_0(f)$. It is computed from data associated with the cell decompositions $\mathcal{D}^n(f,\mathcal{C})$ determined by the map f and a Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$. For this we consider the combinatorial quantity $D_n(f,\mathcal{C})$ defined to be the minimal number of tiles in $\mathcal{D}^n(f,\mathcal{C})$ that are needed to form a connected set joining opposite sides of $\mathcal{C}$, i.e., two non-adjacent 0-edges (the definition is slightly different in the case $\#\operatorname{post}(f) = 3$; see Section 5.7). For the examples discussed in Sections 1.1 and 1.3 we have $D_n(g,\mathcal{C}) = 2^n$ and $D_n(h,\mathcal{C}) = 2^n$ for $n \in \mathbb{N}_0$.

In general, the number $D_n(f,\mathcal{C})$ depends on $\mathcal{C}$. For an expanding Thurston map it grows at an exponential rate as $n \to \infty$. This growth rate is independent of $\mathcal{C}$, and determined only by f. Moreover, the limit

$$\Lambda_0(f) := \lim_{n\to\infty} D_n(f,\mathcal{C})^{1/n} \tag{1.6}$$

exists, satisfies $1 < \Lambda_0(f) < \infty$, and is defined to be the combinatorial expansion factor of f (see Proposition 16.1). It is invariant under topological conjugacy and well-behaved under iteration (see Proposition 16.2). For our two examples we have $\Lambda_0(g) = 2$ and $\Lambda_0(h) = 2$.

As already mentioned above, $\Lambda_0(f)$ gives the range of possible expansion factors of visual metrics for an expanding Thurston map f. This is made precise in the following theorem.

THEOREM 16.3 (Visual metrics and their expansion factors). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\Lambda_0(f) \in (1, \infty)$ be its combinatorial expansion factor. Then the following statements are true:*

(i) *If Λ is the expansion factor of a visual metric for f, then $1 < \Lambda \leq \Lambda_0(f)$.*

(ii) *Conversely, if $1 < \Lambda < \Lambda_0(f)$, then there exists a visual metric ϱ for f with expansion factor Λ. Moreover, the visual metric ϱ can be chosen to have the following additional property:*

For every $x \in S^2$ there exists a neighborhood U_x of x such that

$$\varrho(f(x), f(y)) = \Lambda \varrho(x, y) \text{ for all } y \in U_x.$$

(Note that in this introduction we label the results as they appear in later chapters.)

In general, one cannot guarantee the existence of a visual metric with expansion factor $\Lambda = \Lambda_0(f)$ (see Example 16.8).

The combinatorial expansion factor always satisfies the inequality $\Lambda_0(f) \leq \deg(f)^{1/2}$, where $\deg(f)$ is the (topological) degree of f (see Proposition 20.1). For our examples g and h from the previous sections we have $\Lambda_0(g) = 2 = \deg(g)^{1/2}$ and $\Lambda_0(h) = 2 < \deg(h)^{1/2} = \sqrt{5}$. The equality for the Lattès map g is not a coincidence. Closely related results will be discussed in Section 1.7.

According to Theorem 16.3 (ii), for each expanding Thurston map f we can find a visual metric ϱ so that f scales the metric ϱ by a constant factor at each point. The Lattès map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ discussed in Section 1.1 illustrates this statement: if we equip $\widehat{\mathbb{C}}$ with a suitable visual metric for g (the path metric on the pillow in Figure 1.1), then g behaves like a piecewise similarity map, where distances are scaled by the factor $\Lambda = 2$.

The space $\mathcal{S}$ from Section 1.3 equipped with the visual metric ϱ in (1.2) is a fractal sphere. It is self-similar in the sense that the part of the surface that is "built on top" of some n-tile $\mathcal{X}^n$ is similar (i.e., is isometric up to scaling by the factor 2^n) to the part of the surface that is "built on top" of the white or the black 0-tile. Similarly, we can find visual metrics for any Thurston map f such that f scales tiles by a constant factor. Then the metric behavior of the dynamics on tiles becomes very simple, while the space on which f acts is a fractal sphere and geometrically more complicated.

Our choice of the term "visual metric" is motivated by the close relation of this concept to the notion of a visual metric on the boundary of a Gromov hyperbolic space (see Section 4.2 for general background; very similar ideas can be found in [**HP09**]). Namely, if $f\colon S^2 \to S^2$ is an expanding Thurston map and $\mathcal{C} \subset S^2$ a Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$, then one can define an associated *tile graph* $\mathcal{G}(f, \mathcal{C})$ as follows. Its vertices are given by the tiles in the cell decompositions $\mathcal{D}^n(f, \mathcal{C})$ on all levels $n \in \mathbb{N}_0$. We consider $X^{-1} := S^2$ as a tile of level -1 and add it as a vertex. One joins two vertices by an edge if the corresponding tiles intersect and have levels differing by at most 1 (see Chapter 10). The graph $\mathcal{G}(f, \mathcal{C})$ depends on the choice of $\mathcal{C}$, but if $\mathcal{C}' \subset S^2$ is another Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}'$, then $\mathcal{G}(f, \mathcal{C})$ and

$\mathcal{G}(f,\mathcal{C}')$ are rough-isometric (Theorem 10.4); note that this is much stronger than being quasi-isometric (see Section 4.2 for the terminology).

The graph $\mathcal{G}(f,\mathcal{C})$ is Gromov hyperbolic (Theorem 10.1). Its boundary at infinity $\partial_\infty\mathcal{G}(f,\mathcal{C})$ can be identified with S^2. Under this identification the class of visual metrics in the sense of Gromov hyperbolic spaces coincides with the class of visual metrics for f in our sense (Theorem 10.2). The number $m(x,y)$ defined in (1.5) is the Gromov product of the points $x,y\in S^2\cong\partial_\infty\mathcal{G}(f,\mathcal{C})$ up to a uniformly bounded additive constant (Lemma 10.3).

If $f\colon S^2\to S^2$ is an expanding Thurston map and ϱ a visual metric for f, then we call the metric space (S^2,ϱ) the *visual sphere* of f. For fixed f different visual metrics ϱ_1 and ϱ_2 give snowflake equivalent spaces (S^2,ϱ_1) and (S^2,ϱ_1). So an expanding Thurston map determines its visual sphere uniquely up to snowflake equivalence.

Many dynamical properties of f are encoded in the geometry of its visual sphere. The following statement is one of the main results of this work.

THEOREM 18.1 (Properties of f and its associated visual sphere). *Suppose $f\colon S^2\to S^2$ is an expanding Thurston map and ϱ is a visual metric for f. Then the following statements are true:*

(i) *(S^2,ϱ) is doubling if and only if f has no periodic critical points.*

(ii) *(S^2,ϱ) is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$ if and only if f is topologically conjugate to a rational map.*

(iii) *(S^2,ϱ) is snowflake equivalent to $\widehat{\mathbb{C}}$ if and only if f is topologically conjugate to a Lattès map.*

Here it is understood that $\widehat{\mathbb{C}}$ is equipped with the chordal metric. For the terminology used in the statements see Section 4.1.

As we already discussed, part (ii) of the previous theorem provides an analog of Cannon's conjecture in geometric group theory (see Section 4.3 for a more detailed discussion). According to this conjecture every Gromov hyperbolic group G whose boundary at infinity $\partial_\infty G$ is a 2-sphere should arise from some standard situation in hyperbolic geometry. The conjecture is equivalent to showing that $\partial_\infty G$ equipped with a visual metric (in the sense of Gromov hyperbolic spaces) is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$. One of the reasons why Cannon's conjecture is still open may be the lack of non-trivial examples that guide the intuition (see the paper [**BK11**] though, which in a sense addresses this issue). All examples come from fundamental groups G of compact hyperbolic manifolds where one already has a natural identification of $\partial_\infty G$ with $\widehat{\mathbb{C}}$; according to Cannon's conjecture there are no other examples. In contrast, the visual spheres of expanding Thurston maps provide a rich supply of metric 2-spheres that sometimes are and sometimes are not quasisymmetrically equivalent to $\widehat{\mathbb{C}}$ (see Section 4.4).

The proof of one of the implications in Theorem 18.1 (ii) (the "only if" part) uses some well-known ingredients. Namely, if (S^2,ϱ) is quasisymmetrically equivalent to the standard sphere $\widehat{\mathbb{C}}$, then one can conjugate f to a map g on $\widehat{\mathbb{C}}$. Since the map f dilates distances with respect to a suitable visual metric by a fixed factor (see Theorem 16.3 (ii) mentioned above), the map g is uniformly quasiregular (see Section 4.1 for the terminology). Hence g, and therefore also f, are conjugate to a rational map by a standard theorem.

The converse direction (the "if" part) is harder to establish. If f is conjugate to a rational map, then we may assume without loss of generality that f is a rational expanding Thurston map on $\widehat{\mathbb{C}}$ to begin with. If ϱ is a visual metric for f, then one shows that the identity map from $(\widehat{\mathbb{C}}, \varrho)$ to $(\widehat{\mathbb{C}}, \sigma)$ is a quasisymmetry, where σ is the chordal metric. This follows from a careful analysis of the geometry of the tiles in the cell decompositions $\mathcal{D}^n(f, \mathcal{C})$ with respect to the metric σ (see Proposition 18.8). For example, while it is fairly obvious from the definitions that adjacent tiles in $\mathcal{D}^n(f, \mathcal{C})$ have comparable diameter with respect to a visual metric ϱ (with uniform constants independent of the level n), the same assertion is also true for the chordal metric σ. Our proof of this and related statements is based on Koebe's distortion theorem and the fact that if f has no periodic critical points, then in the cell decompositions $\mathcal{D}^n(f, \mathcal{C})$ we see locally only finitely many different combinatorial types.

Thurston studied the question when a given Thurston map is represented by a conformal dynamical system from a point of view different from the one suggested by Theorem 18.1 (ii) (see Section 2.6 for a short overview). He asked when a Thurston map $f\colon S^2 \to S^2$ is in a suitable sense *(Thurston) equivalent* (see Definition 2.4) to a rational map and obtained a necessary and sufficient condition (see [**DH93**]). For expanding Thurston maps his notion of equivalence actually means the same as topological conjugacy of the maps (Theorem 11.1).

The proof of part (ii) of Theorem 18.1 does not use Thurston's theorem. Indeed, none of our statements relies on this, and so our methods possibly provide a different approach for its proof.

It is not clear how useful Theorem 18.1 (ii) is for deciding whether an explicitly given expanding Thurston map is topologically conjugate to a rational map. It is likely that our techniques can be used to formulate a more efficient criterion, but we will not pursue this further here.

1.5. Invariant curves

The Jordan curve $\mathcal{C}$ chosen in Section 1.1 is invariant for the map g in the sense that $g(\mathcal{C}) \subset \mathcal{C}$. In this case, the cell decomposition $\mathcal{D}^{n+1}(g, \mathcal{C})$ is a refinement of $\mathcal{D}^n(g, \mathcal{C})$ for each $n \in \mathbb{N}_0$. We have a similar situation for the Jordan curve $\mathcal{C}$ and the map h in Section 1.3.

Some of our main results are about the existence and uniqueness of such invariant Jordan curves $\mathcal{C}$. In particular, we will show that they exist for sufficiently high iterates of *every* expanding Thurston map.

THEOREM 15.1 (High iterates have invariant curves). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with* $\operatorname{post}(f) \subset \mathcal{C}$*. Then for each sufficiently large $n \in \mathbb{N}$ there exists a Jordan curve $\widetilde{\mathcal{C}} \subset S^2$ that is invariant for f^n and isotopic to $\mathcal{C}$ rel.* $\operatorname{post}(f)$*.*

A discussion of isotopies and related terminology can be found in Section 2.4. Since $\widetilde{\mathcal{C}}$ is isotopic to $\mathcal{C}$ rel. $\operatorname{post}(f)$, it will also contain the set $\operatorname{post}(f)$.

In Example 15.11 we exhibit an expanding Thurston map $f\colon S^2 \to S^2$ that has no f-invariant Jordan curve $\widetilde{\mathcal{C}} \subset S^2$ with $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$. This shows that in general it is necessary to pass to an iterate in Theorem 15.1.

If a curve $\widetilde{\mathcal{C}}$ is invariant for some iterate f^n, then one cannot expect it to be invariant for some other iterate f^k unless k is a multiple of n (see Remark 15.16). So typically the curve $\widetilde{\mathcal{C}}$ in the previous theorem will depend on n.

The proof of Theorem 15.1 is based on a necessary and sufficient criterion for the existence of f-invariant curves given in Theorem 15.4. An outline of the proof of this latter theorem is presented in Example 15.6 (see Figure 15.1 for an illustration).

One can actually formulate a related criterion for the existence of an invariant curve in a given isotopy class rel. $\operatorname{post}(f)$ or rel. $f^{-1}(\operatorname{post}(f))$. Moreover, if an f-invariant Jordan curve $\widetilde{\mathcal{C}}$ exists, then it is the Hausdorff limit of a sequence of Jordan curves $\mathcal{C}^n$ that can be obtained from a simple iterative procedure (see Remark 15.13 (iii) and Proposition 15.20).

Our existence results are complemented by the following uniqueness statement for invariant Jordan curves.

THEOREM 15.5 (Uniqueness of invariant curves). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C}, \mathcal{C}' \subset S^2$ be f-invariant Jordan curves that both contain the set* $\operatorname{post}(f)$. *Then $\mathcal{C} = \mathcal{C}'$ if and only if $\mathcal{C}$ and $\mathcal{C}'$ are isotopic rel.* $f^{-1}(\operatorname{post}(f))$.

As a consequence one can prove that if $\#\operatorname{post}(f) = 3$, then there are at most finitely many f-invariant Jordan curves $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$ (Corollary 15.8). This is also true if f is rational and has a hyperbolic orbifold (see Theorem 15.10). In general, a Thurston map f can have infinitely many such invariant curves (Example 15.9), but there are at most finitely many in a given isotopy class rel. $\operatorname{post}(f)$ (Corollary 15.7).

Let $f\colon S^2 \to S^2$ be a Thurston map and suppose $\mathcal{C} \subset S^2$ is an f-invariant Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$. We consider the cell decompositions $\mathcal{D}^n = \mathcal{D}^n(f, \mathcal{C})$ as discussed in Section 1.2. The f-invariance of $\mathcal{C}$ implies that $f^{n+1}(f^{-n}(\mathcal{C})) \subset \mathcal{C}$, or equivalently that $f^{-n}(\mathcal{C}) \subset f^{-(n+1)}(\mathcal{C})$ for all $n \in \mathbb{N}_0$. Since n-tiles and $(n+1)$-tiles were defined to be the closures of the complementary components of $f^{-n}(\mathcal{C})$ and $f^{-(n+1)}(\mathcal{C})$, respectively, each $(n+1)$-tile is contained in an n-tile. Similarly, every cell in $\mathcal{D}^{n+1}$ is contained in a cell in $\mathcal{D}^n$. More precisely, $\mathcal{D}^{n+1}$ is a refinement of $\mathcal{D}^n$ (see Definition 5.6 and Proposition 12.5). In particular, $\mathcal{D}^1$ refines $\mathcal{D}^0$.

One can essentially recover the Thurston map f from the cell decomposition $\mathcal{D}^0$ and its refinement $\mathcal{D}^1$ if one specifies some additional data. In the example in Figure 1.1 we labeled the vertices in domain and range to indicate their correspondence under the map. Similarly, in the general case this additional information is provided by a *labeling*, which is a map $L\colon \mathcal{D}^1 \to \mathcal{D}^0$ (see Section 5.4).

The triple $(\mathcal{D}^1, \mathcal{D}^0, L)$ records how 0-cells are subdivided by 1-cells and how 1-cells are mapped to 0-cells. We call such triples $(\mathcal{D}^1, \mathcal{D}^0, L)$ *two-tile subdivision rules* (see Definition 12.1), because $\mathcal{D}^0$ contains two tiles (namely the two closed Jordan regions bounded by $\mathcal{C}$). Every Thurston map f with an f-invariant Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$ gives rise to a two-tile subdivision rule (see Proposition 12.2).

Note that L is a map between finite sets. This means that the information encoded in $(\mathcal{D}^1, \mathcal{D}^0, L)$ is given in terms of finite data. So a two-tile subdivision rule can be considered as a combinatorial object.

Conversely, every two-tile subdivision rule can be *realized* by a Thurston map (see Proposition 12.3). It is unique up to Thurston equivalence. In this way, two-tile subdivision rules give simple combinatorial models for Thurston maps. There is no obvious difference between the two-tile subdivision rules realized by rational maps and the ones that are not. This is another motivation for investigating general Thurston maps.

The map g in Section 1.1 and the map h in Section 1.3 were constructed from Figure 1.1 and Figure 1.2, respectively. This really means that the pictures represent two-tile subdivision rules, and the maps realize these subdivision rules according to Proposition 12.3. This is our preferred way to define Thurston maps. To be able to discuss examples, we will use this way of constructing Thurston maps in an informal way even before we provide the theoretical foundations in Chapter 12.

Our concept of a two-tile subdivision rule is closely related to the general *subdivision rules* that have been studied extensively by Cannon, Floyd, and Parry (see for example [**CFP01**]).

The main consequence of Theorem 15.1 is that we have a combinatorial description by a two-tile subdivision rule for sufficiently high iterates $F = f^n$ of every expanding Thurston map f.

Corollary 15.2 (Thurston maps and subdivision rules). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then for each sufficiently large $n \in \mathbb{N}$ there exists a two-tile subdivision rule that is realized by $F = f^n$.*

In particular, we obtain a cellular Markov partition for F. There are several other approaches to providing combinatorial models for certain classes of maps. For example, a postcritically-finite polynomial can be described by its Hubbard tree (see [**DH84**]) or a rational map with three critical values by a dessin d'enfant (see [**Gro97**] and [**LZ04**]). A very general setting that allows one to address similar questions is the recently developed theory of self-similar group actions. In this context one investigates algebraic objects such as the iterated monodromy group and the biset (or bimodule) defined for a Thurston map (see [**Ne05**], in particular Chapter 6).

Our approach is more geometric and adapted to Thurston maps. One of its main features is that we have good geometric control for the cells in the decompositions $\mathcal{D}^n = \mathcal{D}^n(f, \mathcal{C})$ if $\mathcal{C}$ is f-invariant. In particular, with respect to any visual metric the curve $\mathcal{C}$ is actually a quasicircle (Theorem 15.3) and the boundaries of the tiles in $\mathcal{D}^n$ are quasicircles with uniform parameters independent of the level n (Proposition 15.26). For a rational expanding Thurston map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the tiles in $\mathcal{D}^n$ are in fact uniform quasidisks with respect to the chordal metric σ on $\widehat{\mathbb{C}}$ (Theorem 18.4 (iii)).

1.6. Miscellaneous results

In this section we collect various noteworthy results that may be useful for the orientation of the reader.

The concept of *Thurston equivalence* for Thurston maps was already mentioned before. We record its slightly technical definition in Section 2.4. At first sight the concept does not seem to be adapted to the dynamics under iteration. However, in Theorem 11.1 we will prove the important fact that two expanding Thurston maps are Thurston equivalent if and only if they are topologically conjugate.

In Chapter 9 we make a brief excursion to symbolic dynamics. The properties of visual metrics are essential for proving the following statement (see Chapter 9 for the relevant definitions).

THEOREM 9.1. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then f is a factor of the left-shift $\Sigma\colon J^\omega \to J^\omega$ on the space J^ω of all sequences in a finite set J of cardinality $\#J = \deg(f)$.*

The proof of this theorem does not use invariant Jordan curves for f or its iterates; so in a sense it is independent of Theorem 15.1 mentioned above. It can be used to obtain another Markov partition for f, but we have very little control for the geometric shape of the "tiles".

An immediate consequence of this theorem and its proof is the fact that the periodic points of an expanding Thurston map $f\colon S^2 \to S^2$ form a dense subset of S^2 (Corollary 9.2).

In Chapter 13 we investigate equivalence relations $\sim$ on the sphere S^2, and the question when a Thurston map $f\colon S^2 \to S^2$ descends to a Thurston map on the quotient space $S^2/{\sim}$. Here we assume that $\sim$ is of *Moore-type* (see Definition 13.7), which implies that $S^2/{\sim}$ is a 2-sphere. Under this assumption the relevant condition is that $\sim$ is *strongly invariant* for f in the sense that f maps each equivalence class onto another equivalence class (see Definition 13.1 and Lemma 13.19). We will prove that for a given equivalence relation $\sim$ of Moore-type on S^2 a Thurston map $f\colon S^2 \to S^2$ descends to a Thurston map if and only if $\sim$ is strongly invariant for f (see Theorem 13.2 and Corollary 13.3).

Often it is desirable to promote a given Thurston map f that is not expanding to an expanding one. More precisely, we want to find an expanding Thurston map $\widetilde{f}$ that is Thurston equivalent to f. In general, the existence of $\widetilde{f}$ is not guaranteed. However, if f is *combinatorially expanding* (see Definition 12.4) such a map $\widetilde{f}$ does exist. Roughly speaking, it is constructed by defining an equivalence relation that collapses the sets where f fails to be expanding to points (see Chapter 14).

We will also investigate some measure-theoretic aspects of expanding Thurston maps. Each such map has a natural measure adapted to its dynamics.

THEOREM 17.1. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then there exists a unique measure ν_f of maximal entropy for f. The map f is mixing for ν_f.*

This theorem follows from results due to Haïssinsky-Pilgrim [**HP09**, Theorem 3.4.1]. We will present a different proof and give an explicit description of ν_f in terms of the cell decompositions $\mathcal{D}^n(F, \mathcal{C})$, where $F = f^n$ is a suitable iterate and $\mathcal{C}$ is an invariant curve as in Theorem 15.1. In particular, $\nu_f = \nu_F$ assigns equal mass to all tiles in the cell decompositions $\mathcal{D}^n(F, \mathcal{C})$ of a given "color" (see Proposition 17.12 and Theorem 17.13).

The measure ν_f can be used to study the topological and measure-theoretic dynamics of f under iteration. For example, we will see that $h_{top}(f) = \log(\deg(f))$, where $h_{top}(f)$ is the topological entropy and $\deg(f)$ the topological degree of f (Corollary 17.2).

If μ is a Borel measure on a metric space (X, d), then we call the metric measure space (X, d, μ) *Ahlfors Q-regular* for $Q > 0$ if

$$\mu(\overline{B}_d(x, r)) \asymp r^Q$$

for each closed ball $\overline{B}_d(x, r)$ in X whose radius r does not exceed the diameter of the space. If an expanding Thurston map has no periodic critical points, then its visual sphere together with its measure of maximal entropy has this property.

PROPOSITION 18.2. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map without periodic critical points, ϱ be a visual metric for f with expansion factor $\Lambda > 1$, and ν_f be the measure of maximal entropy of f. Then the metric measure space (S^2, ϱ, ν_f) is Ahlfors Q-regular with*

$$Q := \frac{\log(\deg(f))}{\log(\Lambda)}.$$

In particular, (S^2, ϱ) has Hausdorff dimension Q and

$$0 < \mathcal{H}^Q_\varrho(S^2) < \infty.$$

Here $\mathcal{H}^Q_\varrho$ denotes Hausdorff Q-measure on the metric space (S^2, ϱ).

In Chapter 19 we delve into a deeper analysis of measure-theoretic properties of rational expanding Thurston maps. We denote by $\mathcal{L}_{\widehat{\mathbb{C}}}$ Lebesgue measure on $\widehat{\mathbb{C}}$, normalized such that $\mathcal{L}_{\widehat{\mathbb{C}}}(\widehat{\mathbb{C}}) = 1$. Then the following (well-known) statement is true.

THEOREM 19.1. *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational expanding Thurston map. Then Lebesgue measure $\mathcal{L}_{\widehat{\mathbb{C}}}$ is ergodic for f.*

Note that $\mathcal{L}_{\widehat{\mathbb{C}}}$ is essentially never f-invariant, but ergodicity is interpreted as for f-invariant measures (see the discussion in Section 17.1): if $A \subset \widehat{\mathbb{C}}$ is a Borel set with $f^{-1}(A) = A$, then $\mathcal{L}_{\widehat{\mathbb{C}}}(A) = 0$ or $\mathcal{L}_{\widehat{\mathbb{C}}}(A) = 1$.

In our context one can actually find an f-invariant measure that is absolutely continuous with respect to Lebesgue measure.

THEOREM 19.2. *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational expanding Thurston map. Then there exists a unique f-invariant (Borel) probability measure λ_f on $\widehat{\mathbb{C}}$ that is absolutely continuous with respect to Lebesgue measure $\mathcal{L}_{\widehat{\mathbb{C}}}$. This measure has the form $d\lambda_f = \rho\, d\mathcal{L}_{\widehat{\mathbb{C}}}$, where ρ is a positive continuous function on $\widehat{\mathbb{C}} \setminus \operatorname{post}(f)$. Moreover, the measure λ_f is ergodic for f.*

Again this statement is essentially well known. We will prove it by interpreting the existence of λ_f as a fixed point problem for a suitable *Ruelle operator*. This is a standard technique in ergodic theory reviewed in Chapter 19.

1.7. Characterizations of Lattès maps

Lattès maps form another major theme in this book. One may define such a map as a rational expanding Thurston map with a parabolic orbifold (see Section 2.5 for the terminology). Equivalently, they are characterized as quotients of holomorphic torus endomorphisms, or as quotients of holomorphic automorphisms on the complex plane $\mathbb{C}$ by a crystallographic group (see Theorem 3.1). While these latter descriptions are more technical to state, they contain more information and allow us to construct all Lattès maps explicitly.

In Section 1.1 the Lattès map g was constructed from maps $A\colon \mathbb{C} \to \mathbb{C}$ and $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$. For these maps we have $g \circ \Theta = \Theta \circ A$, and so we obtain a commutative diagram as in (1.1). The push-forward of the Euclidean metric on $\mathbb{C}$ by Θ is the *canonical orbifold metric* ω of g. In this example, it is the path metric on the pillow in Figure 1.1. Moreover, ω is a visual metric for g. This is characteristic for Lattès maps.

To make this precise, we will introduce some terminology in an informal way. Each rational Thurston map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has an associated orbifold $\mathcal{O}_f$ (see Section 2.5), for which there is in turn a universal orbifold covering map $\Theta\colon X \to \widehat{\mathbb{C}}$ (see Section A.9). Here $X = \mathbb{C}$ or $X = \mathbb{D}$ depending on whether f has a parabolic or hyperbolic orbifold. The map Θ is holomorphic. The *canonical orbifold metric* ω of f is the push-forward of the Euclidean metric (if $X = \mathbb{C}$) or hyperbolic metric (if $X = \mathbb{D}$) by Θ (see Section A.10). The metric ω is a conformal metric on $\widehat{\mathbb{C}}$ (see Section A.1 for the terminology), and closely related to the chordal metric σ on $\widehat{\mathbb{C}}$. In fact, if f does not have periodic critical points, the metric space $(\widehat{\mathbb{C}}, \omega)$ is bi-Lipschitz equivalent to $(\widehat{\mathbb{C}}, \sigma)$ (see Lemma A.34). Note that all this is tied to a holomorphic setting, and so we cannot define such a canonical metric ω unless the Thurston map f is rational.

PROPOSITION 8.5 (Canonical orbifold metric as visual metric). *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational Thurston map without periodic critical points, and ω be the canonical orbifold metric for f. Then ω is a visual metric for f if and only if f is a Lattès map.*

In Theorem 18.1 (iii) we have already encountered another (much deeper) characterization of Lattès maps in terms of visual metrics.

It is not hard to see that for a Lattès map f the space $(\widehat{\mathbb{C}}, \omega)$ is Ahlfors 2-regular. The existence of a visual metric with this property again characterizes Lattès maps.

THEOREM 20.4. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then f is topologically conjugate to a Lattès map if and only if there is a visual metric ϱ for f such that (S^2, ϱ) is Ahlfors 2-regular.*

Together with Proposition 18.2 (mentioned above) the previous theorem implies that an expanding Thurston map without periodic critical points is topologically conjugate to a Lattès map if and only if there is a visual metric with expansion factor $\Lambda = \deg(f)^{1/2}$ (see Corollary 20.5).

Recall from Theorem 16.3 that for an expanding Thurston map f the supremum of all expansion factors of visual metrics is given by the combinatorial expansion factor $\Lambda_0(f)$. This number was defined in (1.6) as the limit of $D_n(f, \mathcal{C})^{1/n}$ as $n \to \infty$. Here $D_n(f, \mathcal{C})$ is the minimal number of n-tiles that are needed to form a connected set joining opposite sides of $\mathcal{C}$. We will show in Proposition 20.1 that $D_n(f, \mathcal{C}) \lesssim \deg(f)^{n/2}$. Lattès maps are precisely those expanding Thurston maps for which this maximal growth rate for $D_n(f, \mathcal{C})$ is attained.

THEOREM 20.2. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then f is topologically conjugate to a Lattès map if and only if the following conditions are true:*

(i) *f has no periodic critical points.*

(ii) *There exists $c > 0$, and a Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$ such that for all $n \in \mathbb{N}_0$ we have*

$$D_n(f, \mathcal{C}) \geq c \deg(f)^{n/2}.$$

This theorem is due to Qian Yin [**Yi15**]. It is remarkable that one can characterize the conformal dynamical systems given by iteration of Lattès maps in terms of essentially combinatorial data.

An immediate consequence of the maximal growth rate of $D_n(f, \mathcal{C})$ is the inequality $\Lambda_0(f) \leq \deg(f)^{1/2}$ for each expanding Thurston map. Here equality is attained for Lattès maps, and so one might expect that this property again characterizes Lattès maps similar to Theorem 20.2. However, there are Thurston maps f satisfying $\Lambda_0(f) = \deg(f)^{1/2}$ that are not topologically conjugate to a Lattès map (see Example 16.8).

The proof of Theorem 20.2 relies on Theorem 20.4, which in turn depends on yet another characterization of Lattès maps. For this we compare the Lebesgue measure $\mathcal{L}_{\widehat{\mathbb{C}}}$ on $\widehat{\mathbb{C}}$ with the measure of maximal entropy ν_f for a given rational expanding Thurston map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. As the following result shows, ν_f and $\mathcal{L}_{\widehat{\mathbb{C}}}$ lie in different measure classes unless f is a Lattès map.

THEOREM 19.4. *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational expanding Thurston map. Then its measure of maximal entropy ν_f is absolutely continuous with respect to Lebesgue measure $\mathcal{L}_{\widehat{\mathbb{C}}}$ if and only if f is a Lattès map.*

This is a special case of a more general theorem due to Zdunik [**Zd90**] which gives a similar characterization of Lattès maps among all, not only postcritically-finite, rational maps. Crucial in the proof of Theorem 19.4 is the existence of the f-invariant measure λ_f on $\widehat{\mathbb{C}}$ that is absolutely continuous with respect to Lebesgue measure $\mathcal{L}_{\widehat{\mathbb{C}}}$ (see Theorem 19.2 mentioned above). In fact, for a Lattès map f the measures λ_f, ν_f, and the *canonical orbifold measure* Ω_f (see Section A.10) all agree (Theorem 19.3). In the example g from Section 1.1 these measures are given by the Euclidean area measure on the pillow in Figure 1.1 (normalized to be a probability measure).

1.8. Outline of the presentation

Our work is an introduction to the subject. We hope that it will stimulate more research in the area and will serve as a foundation for future investigations. Therefore, we kept our presentation elementary, as self-contained as possible, and rather detailed.

For the most part, the prerequisites for the reader are modest and include some basic knowledge of complex analysis and topology, in particular plane topology and the topology of surfaces. A background in complex dynamics is helpful, but not absolutely necessary. In later chapters our demands on the reader are more substantial. In particular, in Chapter 17 and Chapter 19 we require some concepts and results from topological and measure-theoretic dynamics, but we will state and review the necessary facts.

When writing this book, we were faced with conflicting objectives. On the one hand, it was desirable to present the material in linear order, meaning that we should only use results in an argument that have been discussed or established before. On the other hand, a too rigid implementation of this idea would have resulted in long detours that might have distracted the reader from the subject at hand. Moreover, we often had to invoke results that are "well known", but difficult to find in the required form in the literature. For this reason we have included an appendix, where such results are collected. We use and refer to the appendix throughout the text.

We will now give an outline of how of this book is organized and will briefly discuss some main concepts and ideas.

In the last section of this introduction the reader can find a list of all examples of Thurston maps that we consider in this work (Section 1.9).

In Chapter 2 we turn to Thurston maps, the main object of our investigation. We first review branched covering maps in Section 2.1 and then define Thurston maps in Section 2.2. Our notion of *expansion* is introduced in Section 2.3 (see Definition 2.2). In this section we also give a characterization when a rational Thurston map is expanding. While the concept of expansion is discussed more systematically later (in Chapter 6), readers familiar with complex dynamics may find it helpful to get some perspective early on. The notion of *(Thurston) equivalence* for Thurston maps is discussed in Section 2.4.

Every Thurston map f has an associated *orbifold* $\mathcal{O}_f$, defined in terms of the *ramification function* of f. This orbifold can be parabolic in some exceptional cases (including Lattès maps) and is hyperbolic otherwise (see Section 2.5). For rational Thurston maps the *universal orbifold covering map* induces a natural metric (the *canonical orbifold metric*) and a natural measure (the *canonical orbifold measure*). As we did not want to overburden the reader with technicalities at this early stage, we delegated a detailed discussion of these topics to the appendix (see Sections A.9 and A.10).

As already mentioned, Thurston gave a characterization when a Thurston map is equivalent to a rational map. We will review this without proofs in Section 2.6. This material is not really essential for the rest of the book, but we included this discussion for general background.

In Chapter 3 we discuss a large class of Thurston maps, namely Lattès maps and a related class that we call Lattès-type maps. Lattès-type maps are quotients of torus endomorphisms, but in contrast to the Lattès case we do not require the endomorphism to be holomorphic. We will classify Lattès maps, which is surprisingly involved, and will discuss many examples.

In Chapter 4 we collect facts from quasiconformal geometry (Section 4.1) and from the theory of Gromov hyperbolic spaces (Section 4.2) that will be relevant later on. We then turn to Cannon's conjecture in geometric group theory (Section 4.3). As we already remarked earlier, this conjecture gives an intriguing analog to some of the main themes of our study of expanding Thurston maps. We illustrate this with an explicit description of some examples of fractal 2-spheres that arise as visual spheres of expanding Thurston maps (Section 4.4). We included Sections 4.3 and 4.4 mainly to give some motivating background for our investigation.

This starts in earnest in Chapter 5, the technical core of our combinatorial approach. Here we discuss cell decompositions and their relation to Thurston maps. In Section 5.1 we collect some general (well-known) facts about cell decompositions, including the definition of a cell decomposition and related concepts such as refinements and cellular maps. In Section 5.2 we specialize to cell decompositions on 2-spheres.

In Section 5.3 we consider cell decompositions induced by a Thurston map f. Here we define cell decompositions $\mathcal{D}^n = \mathcal{D}^n(f, \mathcal{C})$ for each level $n \in \mathbb{N}_0$ from a Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$ as outlined in Section 1.2. These cell decompositions are our most important technical tool for studying Thurston maps. Their properties are summarized in Proposition 5.16.

Given such a sequence $\mathcal{D}^n$ of cell decompositions induced by (the iterates of) a Thurston map f, we may *label* the cells in $\mathcal{D}^n$ to record to which cells in $\mathcal{D}^0$

they are mapped by f^n. This is explained in Section 5.4. By using two cell decompositions $\mathcal{D}^0$ and $\mathcal{D}^1$ and a labeling (satisfying some additional assumptions), it is possible to construct a Thurston map f that *realizes* this data in a suitable way (see Proposition 5.26). Roughly speaking, this means that we may construct Thurston maps in a geometric fashion, very similar to the example indicated in Figure 1.1.

In Section 5.6 we introduce the concept of an *n-flower* $W^n(p)$ of a vertex p in the cell decomposition $\mathcal{D}^n$. The set $W^n(p)$ is formed by the interiors of all cells in $\mathcal{D}^n$ that meet p (see Definition 5.27 and Lemma 5.28). An important fact is that while in general a component of the preimage $f^{-n}(K)$ of a small connected set K will not be contained in an n-tile (i.e., a 2-dimensional cell in $\mathcal{D}^n$), it is always contained in an n-flower (Lemma 5.34).

In Section 5.7 we give a precise definition for a connected set to *join opposite sides* of $\mathcal{C}$. In addition, we define the quantity $D_n = D_n(f, \mathcal{C})$ that measures the combinatorial expansion rate of a Thurston map. It is given as the minimal number of n-tiles needed to form a connected set joining opposite sides of $\mathcal{C}$ (see (5.15)).

In Chapter 6 we revisit our notion of expansion. The main result here is Proposition 6.4, which gives several equivalent conditions for a Thurston map to be expanding. In particular, it follows that expansion is a topological condition, and does not depend on the choice of the metric on S^2 used in the definition. Section 6.2 collects some additional results about expansion, and Section 6.3 provides a simple criterion when a Lattès-type map is expanding.

In Chapter 7 we consider Thurston maps with two or three postcritical points. Every such map is equivalent to a rational map. In fact, every Thurston map f with $\#\operatorname{post}(f) = 2$ is equivalent to $z \mapsto z^n$ for $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$ (see Theorem 7.2 and Proposition 7.1). In Section 7.2 we consider Thurston maps with an associated parabolic orbifold of signature (∞, ∞) or $(2, 2, \infty)$. Such a map is equivalent to $z \mapsto z^n$ in the case (∞, ∞), or to a *Chebyshev polynomial* in the case $(2, 2, \infty)$ (up to a sign). This completes the classification of Thurston maps with parabolic orbifold begun in Chapter 3.

In Chapter 8 we introduce *visual metrics* for expanding Thurston maps, one of our central concepts, as outlined in Section 1.4. Basic properties of visual metrics are listed in Proposition 8.3. A characterization (already mentioned in Section 1.4) is given in Proposition 8.4.

If an expanding Thurston map f is a rational map on the Riemann sphere $\widehat{\mathbb{C}}$, then one would like to know whether some natural metrics on $\widehat{\mathbb{C}}$ are visual metrics for f. The chordal metric on $\widehat{\mathbb{C}}$ is never a visual metric. The canonical orbifold metric of f is visual if and only if f is a Lattès map. This is discussed in Section 8.3.

In Chapter 9 we briefly turn to symbolic dynamics and show that every expanding Thurston map is a factor of a shift operator (see Theorem 9.1).

In Chapter 10 we connect our development of expanding Thurston maps to the theory of *Gromov hyperbolic spaces*. We define the *tile graph* $\mathcal{G} = \mathcal{G}(f, \mathcal{C})$ associated with an expanding Thurston map $f \colon S^2 \to S^2$ and a Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset S^2$. The graph $\mathcal{G}(f, \mathcal{C})$ only depends on $\mathcal{C}$ up to rough-isometry (Theorem 10.4). We show that it is Gromov hyperbolic (Theorem 10.1) and that its boundary at infinity $\partial_\infty \mathcal{G}$ can be identified with S^2 (Theorem 10.2). Under this identification a metric ϱ on $S^2 \cong \partial_\infty \mathcal{G}$ is visual in the sense of Gromov hyperbolic spaces if and only if it is visual for f as defined in Chapter 8 (see

Theorem 10.2). This is the reason why we chose the term "visual" for the metrics associated with a Thurston map.

In Chapter 11 we consider isotopies on S^2 and their lifts by Thurston maps. We show that if two expanding Thurston maps are Thurston equivalent, then they are in fact topologically conjugate (Theorem 11.1). We also prove some results on isotopies of Jordan curves (Section 11.2). The subsequent Section 11.3 contains some auxiliary statements on graphs. The main result is the important, but rather technical Lemma 11.17, which gives a sufficient criterion when a Jordan curve can be isotoped into the 1-skeleton of a given cell decomposition of a 2-sphere.

In Chapter 12 we study the cell decompositions $\mathcal{D}^n = \mathcal{D}^n(f, \mathcal{C})$ under the additional assumption that $\mathcal{C}$ is f-invariant. Then $\mathcal{D}^{n+k}$ is a refinement of $\mathcal{D}^n$ for all $n, k \in \mathbb{N}_0$; so each cell in any of the cell decompositions $\mathcal{D}^n$ is "subdivided" by cells of higher levels. Moreover, the pair $(\mathcal{D}^{n+k}, \mathcal{D}^n)$ is a cellular Markov partition for f^k (Proposition 12.5). In this case the Thurston map f can be described by a *two-tile subdivision rule* (Definition 12.1) as discussed in Section 12.2. Conversely, we may construct a Thurston map from a two-tile subdivision rule by Proposition 12.3. This is the main result in this chapter. This way to construct Thurston maps from a combinatorial viewpoint is illustrated in Section 12.3, where we consider many examples of Thurston maps given in this form.

In Chapter 13 we study the question when a Thurston map $f\colon S^2 \to S^2$ descends to a Thurston map on the quotient space $S^2/{\sim}$ obtained from an equivalence relation $\sim$ on S^2. For this, we first review closed equivalence relations and Moore's theorem in Section 13.1. We also require some auxiliary results on the mapping behavior of branched covering maps discussed in Section 13.2. We prove the main result of this chapter in Section 13.3: a Thurston map $f\colon S^2 \to S^2$ descends to a Thurston map on the quotient $S^2/{\sim}$ obtained from an equivalence relation $\sim$ on S^2 of Moore-type (see Definition 13.7) if and only if $\sim$ is strongly f-invariant (see Definition 13.1, Theorem 13.2, and Corollary 13.3).

Can a given two-tile subdivision rule be realized by an *expanding* Thurston map? This question is addressed in Chapter 14. A necessary condition is that the subdivision rule is *combinatorially expanding* (see Definition 12.18 and Definition 12.4). We show that every combinatorially expanding Thurston map is equivalent to an expanding Thurston map (Proposition 14.3 and Theorem 14.2).

To prove this statement, we "correct" a Thurston map $f\colon S^2 \to S^2$ that realizes a combinatorially expanding two-tile subdivision rule so that the map becomes expanding. On an intuitive level, it is very plausible that this should be possible (see the discussion at the beginning of Chapter 14), but a rigorous implementation is somewhat cumbersome. For this we define an equivalence relation $\sim$ on S^2 that essentially collapses components where the map fails to be expanding to single points. We show that $\sim$ is of Moore-type and will obtain a suitable Thurston map on the quotient $S^2/{\sim}$.

Existence and uniqueness results for *invariant Jordan curves* are proved in Chapter 15. It is one of the central chapters of the present work. Here we establish Theorems 15.1, 15.4, and 15.5, and Corollary 15.2 about existence and uniqueness of invariant curves mentioned in Section 1.5. One can obtain invariant curves from an iterative procedure discussed in detail in Section 15.2.

In Section 15.3 we prove that a Jordan curve $\mathcal{C}$ is a *quasicircle* if it is invariant for an expanding Thurston map f (Theorem 15.3). If the cell decompositions

$\mathcal{D}^n(f, \mathcal{C})$, $n \in \mathbb{N}_0$, are obtained from such an f-invariant Jordan curve $\mathcal{C}$, then the edges in these cell decompositions are uniform quasiarcs and the boundaries of tiles are uniform quasicircles (Proposition 15.26). The underlying metric in all these statements is any visual metric for f.

In Chapter 16 we revisit visual metrics. We introduce the combinatorial expansion factor $\Lambda_0(f)$ and prove Theorem 16.3 (see the outline in Section 1.4). In its proof we use the invariant curves constructed in Chapter 15 to obtain particularly nice visual metrics for a given expanding Thurston map f. They have the property that the map f expands distances locally by the constant factor Λ (see (16.1)).

Chapter 17 is devoted to the measure-theoretic dynamics of an expanding Thurston map f. The main result is Theorem 17.1 about existence and uniqueness of a measure of maximal entropy ν_f for f. For the convenience of the reader we review (mostly standard) material from measure-theoretic dynamics in Section 17.1.

The geometry of the visual sphere (S^2, ϱ) of an expanding Thurston map $f\colon S^2 \to S^2$ is explored in Chapter 18, another central part of our work. Here we show the important fact—already discussed in Section 1.4—that f is conjugate to a rational map if and only if (S^2, ϱ) is quasisymmetrically equivalent to the standard 2-sphere (see Theorem 18.1 (ii)). In addition, we prove linear local connectivity of the visual sphere (Proposition 18.5), as well as its Ahlfors regularity in the absence of periodic critical points of the map (Proposition 18.2).

In Chapter 19 we study measure-theoretic properties of rational expanding Thurston maps $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. We first construct a measure λ_f on $\widehat{\mathbb{C}}$ that is f-invariant and absolutely continuous with respect to Lebesgue measure on $\widehat{\mathbb{C}}$ (Theorem 19.2). This allows us to apply methods from ergodic theory. As a consequence we recover (a weak form of) Zdunik's result that the measure of maximal entropy of f is absolutely continuous with respect to Lebesgue measure if and only if f is a Lattès map (Theorem 19.4).

This in turn allows us to finish the discussion of the visual sphere of an expanding Thurston map f begun in Chapter 18. Specifically, in Section 19.4 we prove that the visual sphere of such a map f is snowflake equivalent to the standard 2-sphere if and only if f is topologically conjugate to a Lattès map (Theorem 18.1 (iii)).

Chapter 20 gives another application of (Zdunik's) Theorem 19.4. By using this theorem we show that Lattès maps can be characterized in terms of their combinatorial expansion behavior (see Theorem 20.2 mentioned in Section 1.6).

Some further developments and future perspectives are presented in Chapter 21. Here we discuss some recent related work and open problems.

The appendix is devoted to several subjects whose inclusion in the main text would have been too distracting. For example, in one of its sections we establish a useful variant of Janiszewski's theorem in plane topology, whose proof is a bit technical. We also review some fairly standard material about conformal metrics, Koebe's distortion theorem, orientation on surfaces, covering maps, lattices and tori, and quotient spaces. We discuss these topics so that we can refer to them in the main text and to make this work as self-contained as possible.

The appendix also contains fairly lengthy sections on branched covering maps, orbifolds, and the canonical orbifold metric. Though the expert will find no surprises here, it is hard to track down this material in an accessible form in the literature.

1.9. List of examples for Thurston maps

Throughout the book we consider many examples of Thurston maps in order to illustrate various phenomena. We list them here with a short description for the reader's convenience. The relevant terms used in these descriptions are defined in later chapters. Often the maps in our examples are Lattès maps. While Lattès maps sometimes have special properties compared to general Thurston maps, they often provide convenient examples with generic behavior.

A Lattès map is discussed in Section 1.1. In the terminology of Chapter 3 it is a flexible Lattès map.

In Section 1.3 we consider an expanding Thurston map h that "generates" a fractal sphere $\mathcal{S}$. The map h is not topologically conjugate (or Thurston equivalent) to a rational map. This is examined in Example 2.19. Closely related is the fact that the fractal sphere $\mathcal{S}$ is not quasisymmetrically equivalent to the standard sphere (see Example 4.7).

The examples $f(z) = 1 - 2/z^2$ and $g(z) = \frac{\boldsymbol{i}}{2}(z + 1/z)$ are used in Section 2.2 to explain the concept of a ramification portrait. Both are in fact Lattès maps.

In Example 2.6 our main purpose is to familiarize the reader with the concept of Thurston equivalence. To this end we consider two Thurston maps f and g and show that they are Thurston equivalent. While the map g is given in a combinatorial fashion and realizes a certain two-tile subdivision rule, the map f is rational and given by an explicit formula.

A general construction for Lattès-type maps with signature $(2, 2, 2, 2)$ is presented in Example 3.20.

In Section 3.6 we look at several Lattès maps. In Example 3.23 the map is $f(z) = 1 - 2/z^2$, which is a Lattès map with orbifold signature $(2, 4, 4)$. Similarly, in Example 3.24 and Example 3.25 we have Lattès maps with orbifold signatures $(3, 3, 3)$ and $(2, 3, 6)$, respectively.

Certain types of Lattès maps are called *flexible*; see Definition 3.26 and (3.38) for an example. They have orbifold signature $(2, 2, 2, 2)$, but not all Lattès maps with this signature are flexible. Such a non-flexible Lattès map with signature $(2, 2, 2, 2)$ is given in Example 3.27.

The Thurston map in Example 6.11 has a Levy cycle.

In Example 6.15 we present a Thurston map that is eventually onto, but not expanding.

In Section 7.2 we consider Thurston maps with signatures (∞, ∞) and $(2, 2, \infty)$. They are equivalent to $z \mapsto z^n$ (where $n \in \mathbb{Z} \setminus \{-1, 0, 1\}$) and to Chebyshev polynomials (up to sign), respectively.

The example $f(z) = \boldsymbol{i}(z^4 - \boldsymbol{i})/(z^4 + \boldsymbol{i})$ is used in Figure 8.1 to illustrate the definition of a visual metric.

In Example 12.6 we show some cell decompositions generated by an f-invariant curve for the map $f(z) = 1 - 2/z^4$.

In Example 12.11 we consider two maps that realize two-tile subdivision rules that only differ by the labeling.

Several examples of two-tile subdivision rules and the Thurston maps realizing them are discussed in Section 12.3. The map in Example 12.20 is $f_1(z) = z^2 - 1$; it realizes a two-tile subdivision rule that is not combinatorially expanding.

The two maps f_2 and $\widetilde{f}_2$ in Example 12.21 both realize the barycentric subdivision rule. The map f_2 is a rational map, but it is not expanding (i.e., its Julia set is not the whole Riemann sphere $\widehat{\mathbb{C}}$). However, the map $\widetilde{f}_2$ is expanding. It is an example of an expanding Thurston map with periodic critical points.

The map f_3 in Example 12.22 is the map h considered in Section 1.3. It realizes a certain two-tile subdivision rule and is an obstructed map. This means that f_3 is not Thurston equivalent to a rational map.

The map f_4 in Example 12.23 is a Lattès-type map that is again not Thurston equivalent to a rational map. While it is somewhat easier to define than the map f_3 in Example 12.22, it is less generic, since f_4 has a parabolic orbifold, while f_3 has a hyperbolic orbifold. The map f_4 realizes the 2-by-3 subdivision rule. If we equip the underlying sphere S^2 with a suitable visual metric for f_4, then S^2 consists of two copies of Rickman's rug.

Example 12.24 provides a whole class of Thurston maps. One of them is $f_5(z) = 1 - 2/z^2$ which realizes a simple two-tile subdivision rule. By "adding flaps" we obtain the other maps. All these maps are rational; in fact, they are given by an explicit formula, which makes them easy to understand and visualize.

We discuss a general method for constructing Thurston maps from tilings of the Euclidean or the hyperbolic plane in Example 12.25. One can use this to find Thurston maps with arbitrarily large sets of postcritical points.

In Example 13.17 we consider an equivalence relation of Moore-type on $\widehat{\mathbb{C}}$ such that the map $z \mapsto z^2$ does not descend to a branched covering map. In contrast, the equivalence relation on $\widehat{\mathbb{C}}$ in Example 13.18 is not of Moore-type, but $z \mapsto z^2$ descends to a Thurston map on the quotient.

Example 14.22 shows why the additional condition $\operatorname{post}(f) = \mathbf{V}^0$ in Theorem 14.1 is necessary.

The Thurston map f in Example 14.23 is not combinatorially expanding, yet Thurston equivalent to an expanding Thurston map g. This shows that the sufficient condition in Proposition 14.3 is not necessary.

The map f in Example 15.6 is the same as in Example 2.6. We use it to outline the main ideas of Chapter 15. In particular, we show how to construct an f-invariant curve $\widetilde{\mathcal{C}}$ with $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$ (see Figure 15.1).

In Example 15.9 we return to the Lattès map g from Section 1.1 and prove that it has infinitely many distinct g-invariant curves $\mathcal{C}$ with $\operatorname{post}(g) \subset \mathcal{C}$.

In Example 15.11 we exhibit an expanding Thurston map f for which no f-invariant Jordan curve $\mathcal{C}$ with $\operatorname{post}(f) \subset \mathcal{C}$ exists.

Remark 15.16 justifies why the f^n-invariant curve $\widetilde{\mathcal{C}}$ given by Theorem 15.1 will in general depend on n.

In Example 15.17 we use another Lattès map to illustrate an iterative construction of invariant curves (see Figure 15.4). The invariant curve obtained is quite "fractal" (its Hausdorff dimension is > 1).

Example 15.23 shows what can happen if one of the necessary conditions in the iterative procedure for producing invariant curve is violated. Namely, the limiting object $\widetilde{\mathcal{C}}$ is not a Jordan curve anymore. The map here is again a Lattès map.

The map in Example 15.24 (yet another Lattès map) has a non-trivial (in particular non-smooth) invariant curve that is rectifiable.

In Example 16.8 we exhibit an expanding Thurston map f that has no visual metric with an expansion factor Λ equal to its combinatorial expansion factor $\Lambda_0(f)$. Therefore, statement (ii) in Theorem 16.3 cannot be improved in general.

In Example 18.11 we revisit the two maps from Example 12.21 that realize the barycentric subdivision rule; these maps show that in Theorem 18.1 (ii) we cannot replace "topologically conjugate to a rational map" with "Thurston equivalent to a rational map".

CHAPTER 3

Lattès maps

A *Lattès map* is a rational Thurston map that is expanding and has a parabolic orbifold. There are other equivalent ways to characterize these maps. For example, a Lattès map is a quotient of a holomorphic automorphism of the complex plane by the action of a crystallographic group or a quotient of a holomorphic endomorphism on a complex torus. We will explain this more precisely below.

These maps play a special role in the theory. On the one hand, they are very easy to construct and visualize, and provide a convenient class of examples (one was given in Section 1.1). On the other hand, they often show exceptional behavior compared to generic rational Thurston maps (that are expanding). This is already apparent in Thurston's characterization of rational maps (Theorem 2.18). In general, Lattès maps are distinguished among typical rational Thurston maps in terms of metric geometry (Theorem 18.1 (iii)), by their measure-theoretic properties (Theorem 19.4), or by their "combinatorial expansion rate" (Theorem 20.2). These statements are among the main results of this work and so we will take a closer look at these maps. We will also define the related class of *Lattès-type* maps. These are Thurston maps with a parabolic orbifold and no periodic critical points, but they are not necessarily (equivalent to) rational maps.

Some aspects of a thorough treatment of the underlying theory are rather technical. As we do not want to overburden the reader with details at this point, we will rely on various results that are more fully developed in the appendix.

To motivate our definition of Lattès maps in terms of three equivalent conditions, we will now consider a specific example. For precise definitions of the terminology in the ensuing discussion we refer to the beginning of Section 3.1.

Let f be the map from Section 1.1 (there denoted by g). Then f is a rational Thurston map that is expanding, or equivalently, has no periodic critical points. Its orbifold has signature $(2,2,2,2)$, and is hence parabolic.

The map f is a quotient of the automorphism $A\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto A(z) = 2z$, on $\mathbb{C}$ by a holomorphic map $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$ in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{A} & \mathbb{C} \\ \Theta \downarrow & & \downarrow \Theta \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}}. \end{array} \tag{3.1}$$

Here Θ is essentially a Weierstrass $\wp$-function for the lattice $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\boldsymbol{i}$ (see Section 3.5 for the definition of $\wp$ and a related discussion). Note that $\Theta(z) = \Theta(w)$ for $z, w \in \mathbb{C}$ if and only if $w = \pm z + m + n\boldsymbol{i}$ with $m, n \in \mathbb{Z}$. This last condition can most conveniently be expressed in terms of an action of a *crystallographic group* of orientation-preserving isometries on $\mathbb{C}$ (see Section 3.1).

Indeed, let G be the group of all maps $g\colon \mathbb{C} \to \mathbb{C}$ of the form

$$g(z) = \pm z + m + n\boldsymbol{i},$$

where $m, n \in \mathbb{Z}$. Then G is a crystallographic group and the map Θ is *induced* by G in the sense that $\Theta(z) = \Theta(w)$ for $z, w \in \mathbb{C}$ if and only if there exists $g \in G$ such that $w = g(z)$ (related concepts and facts are discussed in more detail in Section A.7). This implies that the quotient $\mathbb{C}/G$ can be identified with $\widehat{\mathbb{C}}$, and the map Θ with the quotient map $\mathbb{C} \to \mathbb{C}/G$ (see Corollary A.23).

The property that allows us to pass to a quotient f in (3.1) is that the map $z \mapsto A(z) = 2z$ is *G-equivariant* (see Lemma A.24). This means that A maps points that are in the same G-orbit to points that are also in the same G-orbit, or equivalently that

$$A \circ g \circ A^{-1} \in G \text{ for all } g \in G. \tag{3.2}$$

The translations in G form a subgroup G_{tr} isomorphic (as a group) to $\mathbb{Z}^2 \cong \mathbb{Z} \oplus \mathbb{Z}\boldsymbol{i}$. The quotient $\mathbb{C}/G_{\mathrm{tr}}$ is naturally a *complex torus* $\mathbb{T}$, i.e., a Riemann surface whose underlying 2-manifold is a 2-dimensional torus (for more on tori see Section A.8). The maps $A\colon \mathbb{C} \to \mathbb{C}$ and $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$ descend to $\mathbb{T}$, and we obtain holomorphic maps $\overline{A}\colon \mathbb{T} \to \mathbb{T}$ and $\overline{\Theta}\colon \mathbb{T} \to \widehat{\mathbb{C}}$ such that $f \circ \overline{\Theta} = \overline{\Theta} \circ \overline{A}$. So we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{T} & \xrightarrow{\overline{A}} & \mathbb{T} \\ {\scriptstyle \overline{\Theta}}\downarrow & & \downarrow{\scriptstyle \overline{\Theta}} \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}}. \end{array} \tag{3.3}$$

We call a non-constant holomorphic map $\overline{A}\colon \mathbb{T} \to \mathbb{T}$ on a complex torus $\mathbb{T}$ a *holomorphic torus endomorphism*. The Riemann-Hurwitz formula (2.3) implies that such a map $\overline{A}$ has no critical points and is hence a covering map (in the usual topological sense; see Section A.5).

The relations of our example f to crystallographic groups or to holomorphic torus endomorphisms hold for a more general class of rational maps, called *Lattès maps*, as the following statement shows.

THEOREM 3.1 (Characterization of Lattès maps). *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a map. Then the following conditions are equivalent:*

(i) *f is a rational Thurston map that has a parabolic orbifold and no periodic critical points.*

(ii) *There exists a crystallographic group G, a G-equivariant holomorphic map $A\colon \mathbb{C} \to \mathbb{C}$ of the form $A(z) = \alpha z + \beta$, where $\alpha, \beta \in \mathbb{C}$, $|\alpha| > 1$, and a holomorphic map $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$ induced by G such that $f \circ \Theta = \Theta \circ A$.*

(iii) *There exists a complex torus $\mathbb{T}$, a holomorphic torus endomorphism $\overline{A}\colon \mathbb{T} \to \mathbb{T}$ with $\deg(\overline{A}) > 1$, and a non-constant holomorphic map $\overline{\Theta}\colon \mathbb{T} \to \widehat{\mathbb{C}}$ such that $f \circ \overline{\Theta} = \overline{\Theta} \circ \overline{A}$.*

So in (ii) the map f is given as in (3.1), and in (iii) as in (3.3). We will see that we have $\deg(f) = \deg(\overline{A}) = |\alpha|^2 > 1$ (Lemma 3.16).

As we already indicated, the previous theorem motivates the following definition.

Definition 3.2 (Lattès maps). A map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is called a *Lattès map* if it satisfies one of the conditions (and hence every condition) in Theorem 3.1.

The terminology is not uniform in the literature and some authors use the term "Lattès map" with a slightly different meaning (see the discussion in Section 3.6). Lattès maps became more widely known through Lattès paper [**La18**], but they had been studied about half a century earlier by Schroeder, for example. See [**Mi06a**] for more on the history of these maps.

Theorem 3.1 is well known (see, for example, [**Mi06a**]). We will prove it in Sections 3.1 and 3.2. The map A in (ii) is subject to strong further restrictions. See Proposition 3.14 (or [**Mi06a**] and [**DH84**, Appendix]) for more details.

By Proposition 2.3, condition (i) can equivalently be expressed as:

(i') *f is a rational Thurston map that has a parabolic orbifold and is expanding.*

This is how we introduced Lattès maps in the beginning of the chapter.

The most convenient way to construct Lattès maps is based on condition (ii) in Theorem 3.1. One starts with a crystallographic group G not isomorphic to $\mathbb{Z}^2$ and an G-equivariant map A as in this statement. Then there exists a holomorphic branched covering map $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$ induced by G; it is unique up to postcomposition with a Möbius transformation (see Proposition 3.9). The existence of a Lattès map f as in (3.1) then follows from the G-equivariance of A (see Lemma A.24).

If $\alpha_f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{N}}$ is the ramification function of f (see Definition 2.7), then $\Theta\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is a holomorphic branched covering map such that $\deg(\Theta, z) = \alpha_f(\Theta(z))$ for all $z \in \mathbb{C}$ (see Corollary 3.17). Therefore, Θ is in fact the *universal orbifold covering map* of $\mathcal{O}_f = (\widehat{\mathbb{C}}, \alpha_f)$ (see Theorem 3.10 and Section A.9).

It is quite natural to consider more general maps f as in (3.1) or as in (3.3), where the maps involved are branched covering maps, but not necessarily holomorphic. To state this more precisely, we first recall some terminology.

As usual, we call a map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ *affine*, if it has the form

$$A(u) = L_A(u) + u_0, \quad u \in \mathbb{R}^2, \tag{3.4}$$

where $L_A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-linear and $u_0 \in \mathbb{R}^2$. We call L_A the *linear part* of A.

Let G be a crystallographic group not isomorphic to $\mathbb{Z}^2$. Then one can show that the quotient $\mathbb{R}^2/G$ is homeomorphic to a 2-sphere S^2 and the quotient map $\Theta\colon \mathbb{R}^2 \to S^2 \cong \mathbb{R}^2/G$ is a branched covering map induced by G. If, in addition, $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ is an affine map that is G-equivariant and whose linear part L_A satisfies $\det(L_A) > 1$, then there is a Thurston map $f\colon S^2 \to S^2$ such that the diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ {\scriptstyle\Theta}\downarrow & & \downarrow{\scriptstyle\Theta} \\ S^2 & \xrightarrow{f} & S^2 \end{array} \tag{3.5}$$

commutes (see the beginning of Section 3.4). This is the basis of the following definition.

Definition 3.3 (Lattès-type maps). Let $f\colon S^2 \to S^2$ be a map such that there exists a crystallographic group G, an affine map $A\colon \mathbb{R}^2 \to \mathbb{R}^2$ with $\det(L_A) > 1$ that is G-equivariant, and a branched covering map $\Theta\colon \mathbb{R}^2 \to S^2$ induced by G such that $f \circ \Theta = \Theta \circ A$. Then f is called a *Lattès-type map*.

So Lattès-type maps are given as in (3.5), where A is affine. It is clear that every Lattès map belongs to this class. For the degree of a Lattès-type map f we have $\deg(f) = \det(L_A)$ (see Lemma 3.16). The requirement $\det(L_A) > 1$ guarantees the condition $\deg(f) \geq 2$ which is part of our definition of a Thurston map. One can also consider maps as in Definition 3.3 with $\det(L_A) = 1$ or $\det(L_A) < 0$. This gives homeomorphisms and orientation-reversing maps, respectively. According to our definition, no such map is a Thurston map.

Before we discuss some other properties of Lattès-type maps, we will first define another generalization of Lattès maps based on (3.3). We denote by T^2 a 2-dimensional torus, now considered as a purely topological object with no conformal structure. If $\overline{A}\colon T^2 \to T^2$ is a branched covering map, then again by the Riemann-Hurwitz formula (2.3) the map $\overline{A}$ cannot have any critical points and must be an orientation-preserving covering map. We call such a map $\overline{A}$ a *(topological) torus endomorphism*. So a continuous map $\overline{A}\colon T^2 \to T^2$ is a torus endomorphism precisely if it is an orientation-preserving local homeomorphism.

Definition 3.4 (Quotients of torus endomorphisms). Let $f\colon S^2 \to S^2$ be a map on a 2-sphere S^2 such that there exists a torus endomorphism $\overline{A}\colon T^2 \to T^2$ with $\deg(\overline{A}) \geq 2$, and a branched covering map $\overline{\Theta}\colon T^2 \to S^2$ such that $f \circ \overline{\Theta} = \overline{\Theta} \circ \overline{A}$. Then f is called a *quotient of a torus endomorphism*.

In this case, we have a commutative diagram of the form

$$\begin{array}{ccc} T^2 & \xrightarrow{\overline{A}} & T^2 \\ {\scriptstyle\overline{\Theta}}\downarrow & & \downarrow{\scriptstyle\overline{\Theta}} \\ S^2 & \xrightarrow{f} & S^2. \end{array} \tag{3.6}$$

Properties of quotients of torus endomorphisms are recorded in Lemma 3.12. In particular, every such map is a Thurston map without periodic critical points. Lattès-type maps are in this class.

Proposition 3.5. *Every Lattès-type map $f\colon S^2 \to S^2$ is a quotient of a torus endomorphism and hence a Thurston map. It has a parabolic orbifold and no periodic critical points.*

This implies that the orbifold of every Lattès-type map has one of the signatures $(2,2,2,2)$, $(2,4,4)$, $(3,3,3)$, or $(2,3,6)$. The last three signatures do not lead to genuinely new maps, as each Lattès-type map whose orbifold has such a signature is topologically conjugate to a Lattès map (Proposition 3.18). The most interesting case is signature $(2,2,2,2)$. More details on these maps can be found in Example 3.20 (see also Proposition 3.21 and Theorem 3.22). These maps include *flexible Lattès maps* (see Definition 3.26 and the discussion that follows there). The last statement in Proposition 3.5 essentially characterizes Lattès-type maps among Thurston maps.

Proposition 3.6. *Let $f\colon S^2 \to S^2$ be a Thurston map. Then f is Thurston equivalent to a Lattès-type map if and only if f has a parabolic orbifold and no periodic critical points.*

If f has a parabolic orbifold $\mathcal{O}_f$, but also periodic critical points, then the signature of $\mathcal{O}_f$ is (∞,∞) or $(2,\infty,\infty)$. It is easy to classify these maps up to Thurston equivalence as well (see Theorem 7.3).

Each Lattès map is expanding, but this is not always true for a Lattès-type map (see Example 6.15). One can state a simple criterion though when this is the case. Namely, a Lattès-type map is expanding if and only if the two (possibly complex) eigenvalues λ_1 and λ_2 of the linear part L_A of the affine map A in (3.5) satisfy $|\lambda_1|, |\lambda_2| > 1$ (Proposition 6.12).

Every Lattès map is a Lattès-type map, and every Lattès-type map is a quotient of a torus endomorphism. On the other hand, not every Lattès-type map is (conjugate to) a Lattès map. It is a natural question whether every quotient of a torus endomorphism f is Thurston equivalent to a Lattès-type map. One can show that this is true if f is expanding (in this case, f is even conjugate to a Lattès-type map), but we have been unable to answer this question in full generality.

Our presentation in this chapter is as follows. In Section 3.1 we review crystallographic groups. We also formulate two important existence and uniqueness statements for maps related to crystallographic groups or parabolic orbifolds (Proposition 3.9 and Theorem 3.10), but we postpone the proofs of these facts to Section 3.5. We then prove the implications (ii) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (ii) in Theorem 3.1. The final implication (iii) $\Rightarrow$ (i) is established in Section 3.2 after we discussed some relevant facts about quotients of torus endomorphisms.

In Section 3.3 we analyze the restrictions on α and β for the map $A(z) = \alpha z + \beta$ in Theorem 3.1 in detail. This is mostly for potential future reference and can be omitted at first reading. Section 3.4 is devoted to Lattès-type maps and their properties. Here we justify Proposition 3.5 and Proposition 3.6. The proof of this last statement is rather involved and uses some facts about mapping class groups that we will only cite from the literature, but not discuss in detail. As we will not use Proposition 3.6 later, its proof can safely be skipped.

We revisit crystallographic groups and parabolic orbifolds in Section 3.5. Here we give proofs of Proposition 3.9 and Theorem 3.10. We will emphasize a geometric point of view. This will help us in the discussion of some explicit Lattès maps in Section 3.6.

3.1. Crystallographic groups and Lattès maps

In this section we focus on maps as in statement (ii) of Theorem 3.1. We first review some facts related to crystallographic groups. For a more detailed discussion related to group actions and quotient spaces see Section A.7.

We use the notation

$$\mathrm{Aut}(\mathbb{C}) = \{z \in \mathbb{C} \mapsto \alpha z + \beta : \alpha, \beta \in \mathbb{C},\, \alpha \neq 0\}$$

for the group of all holomorphic automorphisms of $\mathbb{C}$ and

$$\mathrm{Isom}(\mathbb{C}) = \{z \in \mathbb{C} \mapsto \alpha z + \beta : \alpha, \beta \in \mathbb{C},\, |\alpha| = 1\} \subset \mathrm{Aut}(\mathbb{C})$$

for the group of all orientation-preserving isometries of $\mathbb{C}$ (equipped with the Euclidean metric).

Let G be a group of homeomorphisms acting on $\mathbb{C}$. If $z \in \mathbb{C}$, then we denote by $G_z := \{g \in G : g(z) = z\}$ its *stabilizer subgroup* and by $Gz := \{g(z) : g \in G\}$ its *orbit under G* or *G-orbit*. The group G induces a natural equivalence relation on $\mathbb{C}$ whose equivalence classes are given by the G-orbits. The corresponding quotient space is denoted by $\mathbb{C}/G$.

The group G acts *properly discontinuously* on $\mathbb{C}$ if for each compact set $K \subset \mathbb{C}$ there are only finitely many maps $g \in G$ with $g(K) \cap K \neq \emptyset$. Then the stabilizer

CHAPTER 8

Visual Metrics

In this chapter we construct a natural class of metrics for an expanding Thurston map that we call *visual metrics*. We haven chosen this name, because there is a close relation between these metrics and visual metrics on the boundary at infinity of a Gromov hyperbolic space. Indeed, for an expanding Thurston map $f\colon S^2 \to S^2$ one can define a Gromov hyperbolic *tile graph* whose boundary at infinity can naturally be identified with S^2. By this identification, a metric ϱ on S^2 is visual in the sense of Gromov hyperbolic spaces if and only if it is visual as it will be defined in this chapter (see Chapter 10 and in particular Theorem 10.2). In general, a visual metric ϱ is not a length metric on S^2. In Chapter 18 we will investigate the resulting metric space (S^2, ϱ) in more detail.

We will first give a quick overview of the definition and the basic properties of visual metrics. In Sections 8.1 and 8.2 we will then provide the technical details. We conclude this chapter with Section 8.3 where we consider rational expanding Thurston maps. In particular, we show that the canonical orbifold metric (see Section A.10) for such a map f is a visual metric precisely if f is a Lattès map (see Proposition 8.5).

Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$. We consider the cell decompositions of S^2 for $(f, \mathcal{C})$ as defined in Section 5.3. One can think of the set of n-tiles as a discrete approximation of the sphere S^2 and measure distances of points by a quantity related to combinatorics of n-tiles.

Indeed, let $x, y \in S^2$ be two distinct points, and X and Y be n-tiles with $x \in X$, $y \in Y$. Since f is expanding, X and Y must be disjoint if n is sufficiently large (see (6.1) and Lemma 6.2). This leads to the following definition.

DEFINITION 8.1. Let $f\colon S^2 \to S^2$ be an expanding Thurston map, $\mathcal{C} \subset S^2$ be a Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$, and $x, y \in S^2$. For $x \neq y$ we define

$$m_{f,\mathcal{C}}(x,y) := \max\{n \in \mathbb{N}_0 : \text{there exist non-disjoint } n\text{-tiles } X \text{ and } Y \text{ for } (f,\mathcal{C}) \text{ with } x \in X,\, y \in Y\}.$$

If $x = y$ we define $m_{f,\mathcal{C}}(x,x) := \infty$.

Note that $m_{f,\mathcal{C}}(x,y) \in \mathbb{N}_0$ if $x \neq y$. We usually drop both subscripts in $m_{f,\mathcal{C}}(x,y)$ if f and $\mathcal{C}$ are clear from the context. A similar combinatorial quantity that is essentially equivalent to $m_{f,\mathcal{C}}(x,y)$ (see Lemma 8.7 (v)) is

$$m'_{f,\mathcal{C}}(x,y) := \min\{n \in \mathbb{N}_0 : \text{there exist disjoint } n\text{-tiles } X \text{ and } Y \text{ for } (f,\mathcal{C}) \text{ with } x \in X,\, y \in Y\} \tag{8.1}$$

for $x \neq y$, and $m'_{f,\mathcal{C}}(x,x) := \infty$.

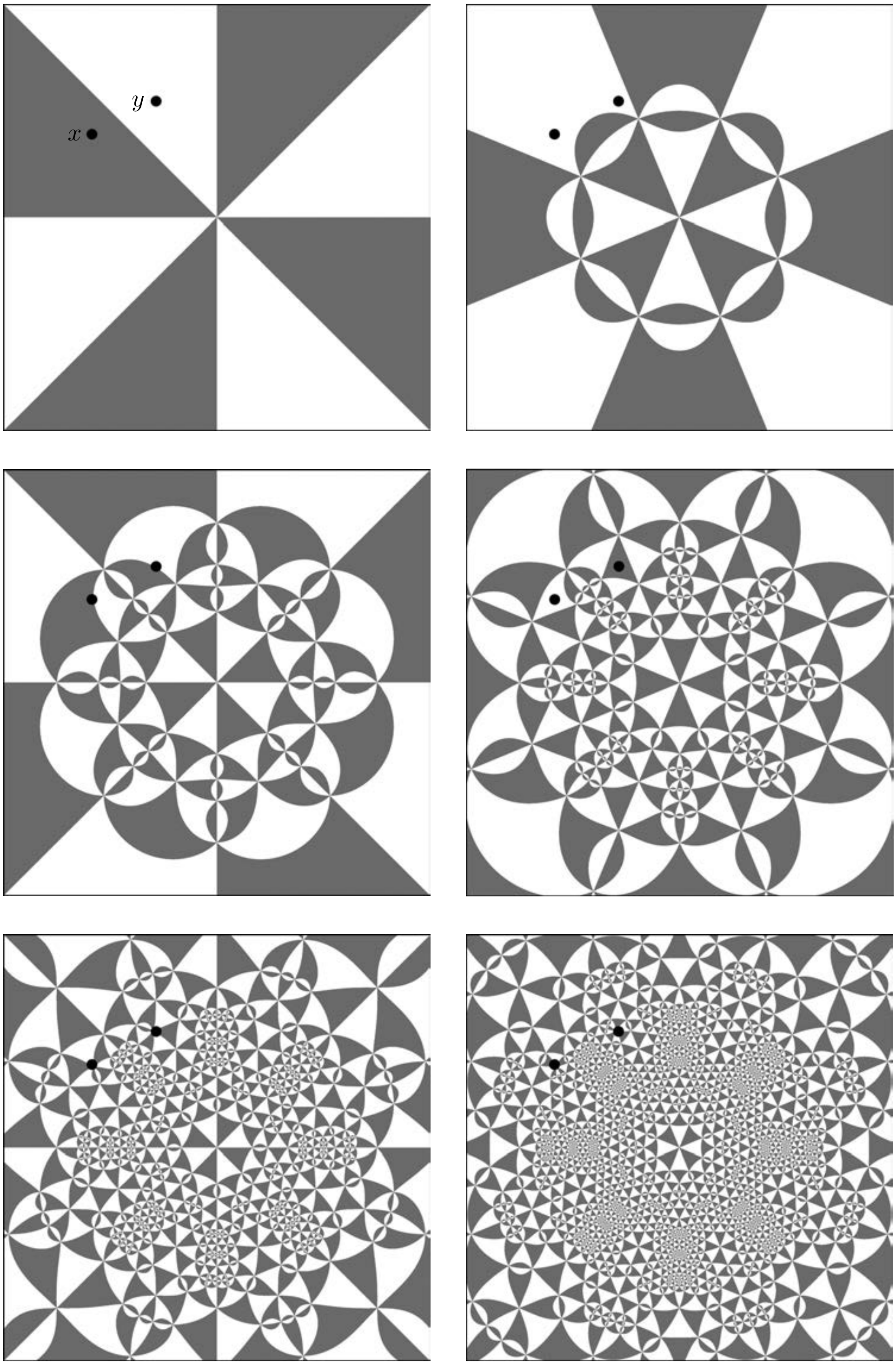

FIGURE 8.1. Separating points by tiles.

These quantities are illustrated in Figure 8.1. Here we use the map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ given by

$$f(z) = \boldsymbol{i}\frac{z^4 - \boldsymbol{i}}{z^4 + \boldsymbol{i}}$$

for $z \in \widehat{\mathbb{C}}$ (we will consider this map again in Example 15.11). We have $\operatorname{post}(f) = \{1, \boldsymbol{i}, -\boldsymbol{i}\}$, and so we can choose the unit circle $\mathcal{C} = \partial\mathbb{D}$ as a Jordan curve containing the postcritical set of f. In Figure 8.1 the n-tiles for $(f, \mathcal{C})$ are shown for $n = 1, \dots, 6$. For the points x and y as indicated in the figure, we have $m_{f,\mathcal{C}}(x,y) = 3$ and $m'_{f,\mathcal{C}}(x,y) = 4$.

The number $m_{f,\mathcal{C}}(x,y)$ is large if x and y are close together, i.e., if n-tiles of high level n are needed to separate the points. This is the basis of the following definition.

Definition 8.2 (Visual metrics). Let $f\colon S^2 \to S^2$ be an expanding Thurston map. A metric ϱ on S^2 is called a *visual metric* (for f) if there exists a Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$, and a constant $\Lambda > 1$ such that

$$\varrho(x,y) \asymp \Lambda^{-m(x,y)} \tag{8.2}$$

for all $x, y \in S^2$, where $m(x,y) = m_{f,\mathcal{C}}(x,y)$ and where the constant $C(\asymp)$ is independent of x and y.

Here we use the convention $\Lambda^{-\infty} = 0$. The number Λ is called the *expansion factor* of the metric ϱ. It is easy to see that the expansion factor of each visual metric is uniquely determined. Different visual metrics may have different expansion factors.

As mentioned above, it is possible to identify the sphere S^2 with the boundary at infinity of a certain Gromov hyperbolic graph constructed from tiles. Under this identification, the numbers $m_{f,\mathcal{C}}(x,y)$ and $m'_{f,\mathcal{C}}(x,y)$ are the Gromov product of x and y, up to some additive constants (see Section 4.2, Chapter 10, and Lemma 10.3).

Obvious questions are whether visual metrics exist, and how they depend on the chosen Jordan curve $\mathcal{C}$ and the expansion factor Λ. This is answered by the following proposition.

Proposition 8.3. *For an expanding Thurston map $f\colon S^2 \to S^2$ the following statements are true:*

(i) *There exist visual metrics for f.*
(ii) *Every visual metric induces the given topology on S^2.*
(iii) *Let ϱ be a visual metric for f with expansion factor Λ, $\widetilde{\mathcal{C}} \subset S^2$ be a Jordan curve with* $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$, *and $m = m_{f,\widetilde{\mathcal{C}}}$ be defined as in Definition* 8.1. *Then a relation as in* (8.2) *is true with the same expansion factor Λ, where the constant $C(\asymp)$ depends on $\widetilde{\mathcal{C}}$.*
(iv) *Any two visual metrics are snowflake equivalent, and bi-Lipschitz equivalent if they have the same expansion factor Λ.*
(v) *A metric ϱ is a visual metric for some iterate $F = f^n$ with $n \in \mathbb{N}$ if and only if it is a visual metric for f. If $\Lambda > 1$ is the expansion factor of ϱ for f, then $\Lambda_F = \Lambda^n$ is the expansion factor of ϱ for $F = f^n$.*
(vi) *If ϱ is a visual metric for f, then $f\colon (S^2, \varrho) \to (S^2, \varrho)$ is a Lipschitz map.*

The notions of snowflake and bi-Lipschitz equivalence were defined in Section 4.1. The proof of Proposition 8.3 will be provided in Section 8.2. Theorem 16.3 gives a stronger result on the existence of visual metrics.

In Section 1.3 and Section 4.4 we introduced visual metrics on a more intuitive level, where we considered certain self-similar fractal spheres constructed as limits of polyhedral surfaces $\mathcal{S}^n$. Each surface $\mathcal{S}^n$ was built from tiles whose size was about Λ^{-n} for some constant $\Lambda > 1$. A similar statement is true in general for visual metrics and gives in fact a characterization of these metrics.

Proposition 8.4 (Characterization of visual metric). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map and ϱ be a metric on S^2. Then ϱ is a visual metric for f with expansion factor $\Lambda > 1$ if and only if the following two conditions hold for all $n \in \mathbb{N}_0$:*

(i) $\operatorname{dist}_\varrho(\sigma, \tau) \gtrsim \Lambda^{-n}$*, whenever σ and τ are disjoint n-cells.*

(ii) $\operatorname{diam}_\varrho(\tau) \asymp \Lambda^{-n}$ *for all n-edges and all n-tiles τ.*

Here cells are defined in terms of some Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$, and the constants $C(\gtrsim)$ and $C(\asymp)$ are independent of the cells and their level n.

We will prove this proposition in Section 8.2.

For a rational expanding Thurston map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ there are some other natural metrics on the Riemann sphere $\widehat{\mathbb{C}}$ besides the visual metrics for f, in particular the chordal metric σ and the canonical orbifold metric ω of f as introduced in Section 2.5. The chordal metric σ is never visual for f (see Lemma 8.12). For the canonical orbifold metric we will prove the following statement in Section 8.3.

Proposition 8.5 (Canonical orbifold metric as visual metric). *Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a rational Thurston map without periodic critical points, and ω be the canonical orbifold metric for f. Then ω is a visual metric for f if and only if f is a Lattès map.*

8.1. The number $m(x, y)$

We now turn to a more detailed exposition of the basic properties of the quantity $m(x, y) = m_{f,\mathcal{C}}(x, y)$ as in Definition 8.1. In the following, $f\colon S^2 \to S^2$ will be an expanding Thurston map and $\mathcal{C} \subset S^2$ be a Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$. Note that $\#\operatorname{post}(f) \geq 3$, because f is expanding (see Lemma 6.1).

Recall the quantity $D_n = D_n(f, \mathcal{C})$ as defined in (5.15) that measures distances in terms of lengths of tile chains. We consider a slight variant here.

We define $\widetilde{D}_n = \widetilde{D}_n(f, \mathcal{C})$ as the minimal number of tiles of levels $k \geq n$ for $(f, \mathcal{C})$ required to join opposite sides of $\mathcal{C}$, i.e., the smallest number $N \in \mathbb{N}$ for which there are tiles $X_i \in \bigcup_{k \geq n} \mathbf{X}^k$, $i = 1, \dots, N$, such that $K = \bigcup_{i=1}^N X_i$ is connected and joins opposite sides of $\mathcal{C}$ (see Definition 5.32).

While the sets K used to define D_n are unions of tiles of level n, the sets K in the definition of $\widetilde{D}_n$ are unions of tiles of levels $k \geq n$; in particular, $D_k \geq \widetilde{D}_n$ for $k \geq n$.

Lemma 8.6. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$. Let $D_n = D_n(f, \mathcal{C})$ and $\widetilde{D}_n = \widetilde{D}_n(f, \mathcal{C})$ for $n \in \mathbb{N}_0$.*

Then $D_n \to \infty$ and $\widetilde{D}_n \to \infty$ as $n \to \infty$.

CHAPTER 15

Invariant curves

This chapter is central for this work. We will prove existence and uniqueness results for invariant curves $\mathcal{C}$ of an expanding Thurston map f. We will also show that if an invariant curve exists, then it can be obtained from an iterative procedure, and that it is a quasicircle. We always require that $\mathcal{C}$ is a Jordan curve and that $\operatorname{post}(f) \subset \mathcal{C}$, but in the following discussion we will often refer to such a curve $\mathcal{C}$ simply as an invariant curve for brevity.

One of our main results can be formulated as follows.

Theorem 15.1 (High iterates have invariant curves). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$. Then for each sufficiently large $n \in \mathbb{N}$ there exists a Jordan curve $\widetilde{\mathcal{C}} \subset S^2$ that is invariant for f^n and isotopic to $\mathcal{C}$ rel.* $\operatorname{post}(f)$.

This existence result has the following important implication.

Corollary 15.2 (Thurston maps and subdivision rules). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then for each sufficiently large $n \in \mathbb{N}$ there exists a two-tile subdivision rule that is realized by $F = f^n$.*

This justifies our approach of studying expanding Thurston maps from a combinatorial perspective based on cellular Markov partitions.

Invariant curves are quasicircles if the underlying metric is visual.

Theorem 15.3 (Invariant curves are quasicircles). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C} \subset S^2$ be a Jordan curve with* $\operatorname{post}(f) \subset \mathcal{C}$. *If $\mathcal{C}$ is f-invariant, then $\mathcal{C}$ equipped with (the restriction of) a visual metric for f is a quasicircle.*

This also applies to invariant curves of iterates, because if $f\colon S^2 \to S^2$ is an expanding Thurston map, then the same is true for each iterate $F = f^n$, $n \in \mathbb{N}$.

If one studies rational Thurston maps f that are expanding, then the underlying 2-sphere is the Riemann sphere $\widehat{\mathbb{C}}$, and it is natural to equip it with the chordal metric σ. Then an invariant $\mathcal{C}$ curve as in the previous theorem is also a quasicircle with respect to σ. This can be deduced from Theorem 15.3 once we know that for such maps the chordal metric is quasisymmetrically equivalent to each visual metric. This will be proved in Chapter 18; see in particular Lemma 18.10.

As we discussed in Section 12.1, if $\mathcal{C}$ an f-invariant Jordan curve with $\operatorname{post}(f) \subset \mathcal{C}$, then we get a sequence of cell decompositions $\mathcal{D}^n = \mathcal{D}^n(f, \mathcal{C})$, $n \in \mathbb{N}_0$, so that each cell decomposition is refined by the cell decompositions of higher levels. We will see that Theorem 15.3 implies that we have good control for the geometry of edges and tiles in these cell decompositions. Namely, the family of edges consists of *uniform quasiarcs* and the boundary of tiles are *uniform quasicircles*. See Section 15.3 for an explanation of this terminology and Proposition 15.26 for a precise statement.

In Theorem 15.1 it is necessary to pass to an iterate of the map to guarantee existence of an invariant curve, because there are examples of maps for which an invariant curve does not exist (see Example 15.11). One can formulate a necessary and sufficient criterion for the existence of invariant curves.

THEOREM 15.4 (Existence of invariant curves). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map. Then the following conditions are equivalent:*

(i) *There exists a Jordan curve $\widetilde{\mathcal{C}} \subset S^2$ with $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$ that is f-invariant.*

(ii) *There exist Jordan curves $\mathcal{C}, \mathcal{C}' \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}, \mathcal{C}'$ and $\mathcal{C}' \subset f^{-1}(\mathcal{C})$, and an isotopy $H\colon S^2 \times I \to S^2$ rel. $\operatorname{post}(f)$ with $H_0 = \mathrm{id}_{S^2}$ and $H_1(\mathcal{C}) = \mathcal{C}'$ such that the map*

$$\widehat{f} := H_1 \circ f \text{ is combinatorially expanding for } \mathcal{C}'.$$

Moreover, if (ii) *is true, then there exists an f-invariant Jordan curve $\widetilde{\mathcal{C}} \subset S^2$ with* $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$ *that is isotopic to $\mathcal{C}$ rel.* $\operatorname{post}(f)$ *and isotopic to $\mathcal{C}'$ rel.* $f^{-1}(\operatorname{post}(f))$.

The first condition in (ii) says that there exists a Jordan curve $\mathcal{C}$ with $\operatorname{post}(f) \subset \mathcal{C}$ that can be isotoped rel. $\operatorname{post}(f)$ into its preimage under f. This condition alone ensures that an associated f-invariant set $\widetilde{\mathcal{C}}$ with $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$ exists, but in general $\widetilde{\mathcal{C}}$ will not be a Jordan curve (see Lemma 15.18 (viii) and Example 15.23). If, in addition, the map $\widehat{f}$ is combinatorially expanding as stipulated in (ii), then one obtains a Jordan curve $\widetilde{\mathcal{C}}$.

Invariant curves can be constructed by an iterative procedure that will be described in Section 15.2. In the situation of Theorem 15.4, one lifts the isotopy H by the map f repeatedly to obtain a sequence of isotopies $H^n\colon S^2 \times I \to S^2$, $n \in \mathbb{N}_0$, with $H^0 := H$ such that $H^n_0 = H^n(\cdot, 0) = \mathrm{id}_{S^2}$. One sets $\mathcal{C}^0 := \mathcal{C}$ and defines inductively $\mathcal{C}^{n+1} := H^n_1(\mathcal{C}^n)$ for $n \in \mathbb{N}_0$. It can then be shown that the sequence $\{\mathcal{C}^n\}$ Hausdorff converges to the desired invariant curve $\widetilde{\mathcal{C}}$ (see Proposition 15.20). An explicit knowledge of the isotopies is not really necessary, because one can interpret this as an edge replacement procedure (see Remark 15.22). In Section 15.2 we will discuss this and several examples that illustrate various phenomena in this context.

Theorem 15.1, which is our basic existence result for invariant curves, is complemented by the following uniqueness statement.

THEOREM 15.5 (Uniqueness of invariant curves). *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and $\mathcal{C}, \mathcal{C}' \subset S^2$ be f-invariant Jordan curves that both contain the set* $\operatorname{post}(f)$. *Then $\mathcal{C} = \mathcal{C}'$ if and only if $\mathcal{C}$ and $\mathcal{C}'$ are isotopic rel.* $f^{-1}(\operatorname{post}(f))$.

This implies that in a given isotopy class rel. $\operatorname{post}(f)$ there are only finitely many invariant curves $\mathcal{C}$ (Corollary 15.7). It follows that an expanding Thurston map f with $\# \operatorname{post}(f) = 3$ can have only finitely many invariant curves $\mathcal{C}$ (Corollary 15.8).

The situation changes if one does not restrict the isotopy class of $\mathcal{C}$. An expanding Thurston map f may have infinitely many invariant curves $\mathcal{C}$ in general (see Example 15.9). If, in addition, the map is rational and has a hyperbolic orbifold, then this cannot happen and f can have only finitely many invariant curves $\mathcal{C}$ (see Theorem 15.10).

The chapter is organized as follows. Section 15.1 is devoted to existence and uniqueness results, where we provide proofs for the statements discussed above. The iterative procedure for the construction of invariant curves is explained in Section 15.2. In Section 15.3 we discuss the quasiconformal geometry of invariant curves. Here we prove Theorem 15.3 and related results.

Much of our discussion in this chapter is quite technical. Before we go into the details, we look at a specific example that will illustrate some of the main ideas.

EXAMPLE 15.6. Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the map defined by

$$f(z) = 1 + (\omega - 1)/z^3$$

for $z \in \widehat{\mathbb{C}}$, where $\omega = e^{4\pi i/3}$. This map was already considered in Example 2.6 and Example 12.24. It realizes the two-tile subdivision rule shown in Figure 2.1.

Note that $f(z) = \tau(z^3)$, where $\tau(w) = 1+(\omega-1)/w$ is a Möbius transformation that maps the upper half-plane to the half-plane above the line through the points ω and 1 (indeed, τ maps $0, 1, \infty$ to $\infty, \omega, 1$, respectively). We have $\operatorname{crit}(f) = \{0, \infty\}$ and $\operatorname{post}(f) = \{\omega, 1, \infty\}$.

One can obtain an f-invariant Jordan curve $\widetilde{\mathcal{C}} \subset \widehat{\mathbb{C}}$ with $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$ as follows. We first pick a Jordan curve $\mathcal{C}^0 \subset \widehat{\mathbb{C}}$ containing all postcritical points of f. More specifically, let $\mathcal{C}^0$ be the (extended) line through ω and 1 (i.e., the circle on $\widehat{\mathbb{C}}$ through $\omega, 1, \infty$).

Now consider $f^{-1}(\mathcal{C}^0) = \bigcup_{k=0,\dots,5} R_k$, where

$$R_k = \{re^{ik\pi/3} : 0 \le r \le \infty\}$$

is the ray from 0 through the sixth root of unity $e^{ik\pi/3}$; see the top right in Figure 15.1. We choose a Jordan curve $\mathcal{C}^1 \subset \widehat{\mathbb{C}}$ such that

$$\mathcal{C}^1 \subset f^{-1}(\mathcal{C}^0),\ \operatorname{post}(f) \subset \mathcal{C}^1,\ \text{and } \mathcal{C}^1 \text{ is isotopic to } \mathcal{C}^0 \text{ rel. } \operatorname{post}(f).$$

For general Thurston maps a similar choice is not always possible, but in our specific case there is a unique Jordan curve $\mathcal{C}^1 \subset f^{-1}(\mathcal{C}^0)$ with $\operatorname{post}(f) \subset \mathcal{C}^1$, namely $\mathcal{C}^1 = R_0 \cup R_4$, the union of the two rays through ω and through 1. Since $\#\operatorname{post}(f) = 3$, the requirement that $\mathcal{C}^1$ is isotopic to $\mathcal{C}^0$ rel. $\operatorname{post}(f)$ is automatic for our specific map f by Lemma 11.10. Let $H\colon \widehat{\mathbb{C}} \times I \to \widehat{\mathbb{C}}$ be an isotopy rel. $\operatorname{post}(f)$ that deforms $\mathcal{C}^0$ to $\mathcal{C}^1$, i.e., $H_0 = \operatorname{id}_{\widehat{\mathbb{C}}}$ and $H_1(\mathcal{C}^0) = \mathcal{C}^1$.

Given the data $\mathcal{C}^0$, $\mathcal{C}^1$, and H, there are two (essentially equivalent) ways to obtain an f-invariant Jordan curve isotopic to $\mathcal{C}^1$ rel. $f^{-1}(\operatorname{post}(f))$ and hence also isotopic to $\mathcal{C}^0$ rel. $\operatorname{post}(f)$.

For the first approach we consider the Thurston map $\widehat{f} := H_1 \circ f$. Since $\mathcal{C}^1 \subset f^{-1}(\mathcal{C}^0)$ we have $f(\mathcal{C}^1) \subset \mathcal{C}^0$, and so

$$\widehat{f}(\mathcal{C}^1) = (H_1 \circ f)(\mathcal{C}^1) \subset H_1(\mathcal{C}^0) = \mathcal{C}^1.$$

Thus $\mathcal{C}^1$ is $\widehat{f}$-invariant. The two-tile subdivision rule given by $\mathcal{D}^1 = \mathcal{D}^1(\widehat{f}, \mathcal{C}^1)$, $\mathcal{D}^0 = \mathcal{D}^0(\widehat{f}, \mathcal{C}^1)$, and the labeling induced by $\widehat{f}$ is as in Figure 2.1. The map $\widehat{f}$ is combinatorially expanding for $\mathcal{C}^1$; indeed, no 2-tile for $(\widehat{f}, \mathcal{C}^1)$ joins opposite sides of $\mathcal{C}^1$. Thus by Theorem 14.2 there is a homeomorphism $\phi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ isotopic to the identity on $\widehat{\mathbb{C}}$ rel. $\operatorname{post}(\widehat{f}) = \operatorname{post}(f)$ such that $\phi(\mathcal{C}^1) = \mathcal{C}^1$ and $g := \phi \circ \widehat{f}$ is an expanding Thurston map. Since f is also expanding (as follows from Proposition 2.3) and g is Thurston equivalent to f, there is a homeomorphism $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$

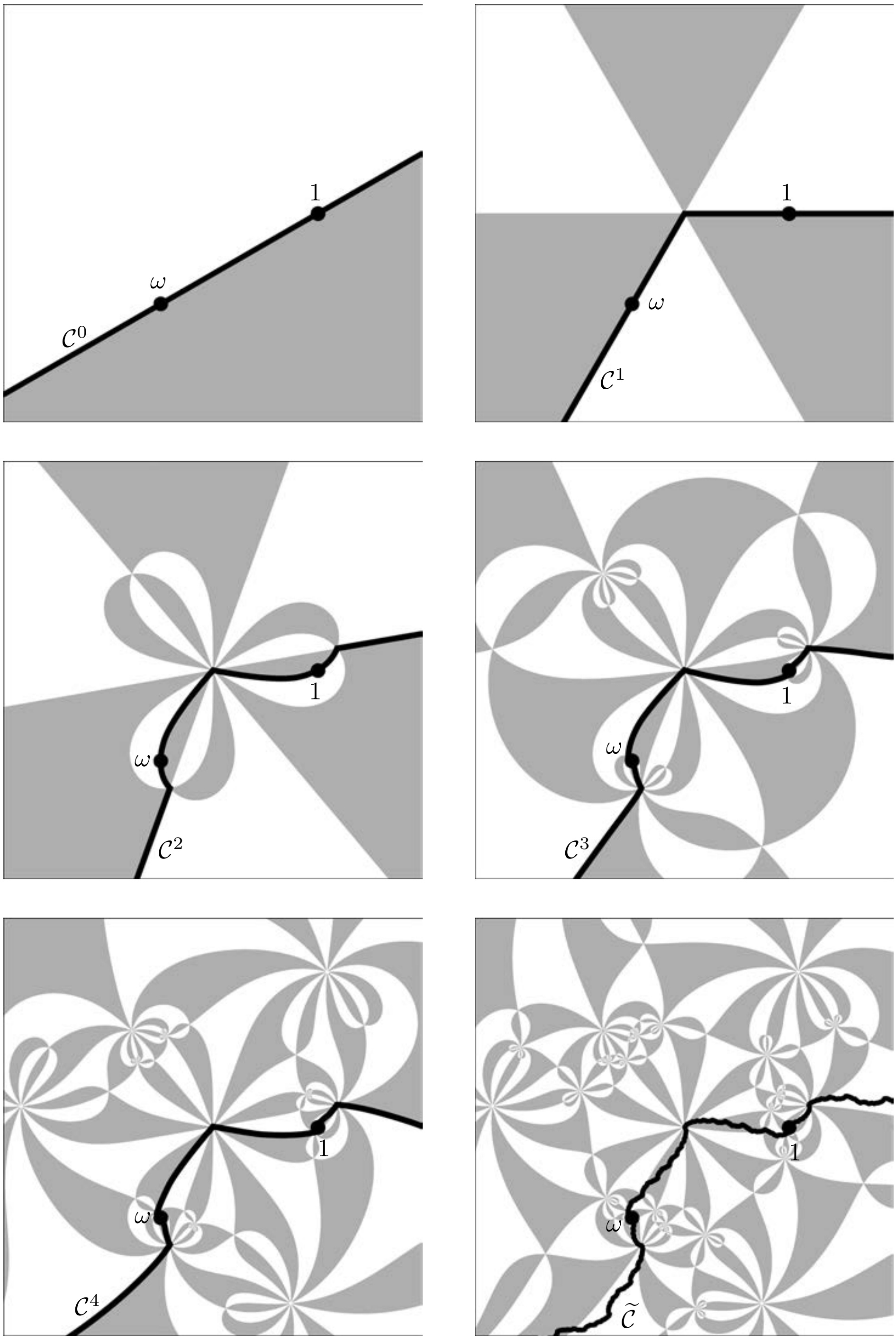

FIGURE 15.1. The invariant curve for Example 15.6.

such that $h \circ f = g \circ h$ (Theorem 11.1). Then $\widetilde{\mathcal{C}} := h^{-1}(\mathcal{C}^1)$ is an f-invariant Jordan curve containing $\operatorname{post}(f)$. The general existence result for invariant curves given by Theorem 15.4 is proved in the same way.

For the second approach, we use Proposition 11.3 to lift $H = H^0$ by the map f to an isotopy H^1 with $H^1_0 = \mathrm{id}_{\widehat{\mathbb{C}}}$. Then we lift H^1 to an isotopy H^2 with $H^2_0 = \mathrm{id}_{\widehat{\mathbb{C}}}$, etc. In this way, we find a sequence of isotopies H^n and inductively define $\mathcal{C}^{n+1} := H^n_1(\mathcal{C}^n)$. We will see in Proposition 15.20 that the sequence $\{\mathcal{C}^n\}$ of Jordan curves Hausdorff converges to an f-invariant Jordan curve $\widetilde{\mathcal{C}}$ containing all postcritical points of f as desired. This is illustrated in Figure 15.1; indeed, the invariant curve $\widetilde{\mathcal{C}}$ in this picture was obtained by approximating it by the curves $\mathcal{C}^n$ (as were the invariant curves in Figures 15.4, 15.6, and 15.7).

In our example the f-invariant Jordan curve $\widetilde{\mathcal{C}} \subset \widehat{\mathbb{C}}$ with $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$ is in fact *unique*. To see this, note that since $\# \operatorname{post}(f) = 3$, every such curve $\widetilde{\mathcal{C}}$ is isotopic rel. $\operatorname{post}(f)$ to the curve $\mathcal{C}^0$ chosen above. Thus we can find an isotopy $K\colon \widehat{\mathbb{C}} \times I \to \widehat{\mathbb{C}}$ rel. $\operatorname{post}(f)$ with $K_0 = \mathrm{id}_{\widehat{\mathbb{C}}}$ and $K_1(\widetilde{\mathcal{C}}) = \mathcal{C}^0$. By Proposition 11.3 we can lift K to an isotopy $\widetilde{K}\colon \widehat{\mathbb{C}} \times I \to \widehat{\mathbb{C}}$ rel. $f^{-1}(\operatorname{post}(f))$ with $\widetilde{K}_0 = \mathrm{id}_{\widehat{\mathbb{C}}}$ and $K_t \circ f = f \circ \widetilde{K}_t$ for $t \in I$. Then by Lemma 11.2 we have

$$\mathcal{C}' := \widetilde{K}_1(\widetilde{\mathcal{C}}) \subset \widetilde{K}_1(f^{-1}(\widetilde{\mathcal{C}})) = f^{-1}(K_1(\widetilde{\mathcal{C}})) = f^{-1}(\mathcal{C}^0).$$

So $\mathcal{C}'$ is a Jordan curve in $\widehat{\mathbb{C}}$ with $\mathcal{C}' \subset f^{-1}(\mathcal{C}^0)$ and $\operatorname{post}(f) \subset \mathcal{C}'$. Since in this particular example $\mathcal{C}^1$ is the unique such curve, we conclude $\mathcal{C}' = \widetilde{K}_1(\widetilde{\mathcal{C}}) = \mathcal{C}^1$. In particular, $\widetilde{\mathcal{C}}$ is isotopic to $\mathcal{C}^1$ rel. $f^{-1}(\operatorname{post}(f))$ by the isotopy $\widetilde{K}$. So every f-invariant Jordan curve $\widetilde{\mathcal{C}}$ with $\operatorname{post}(f) \subset \widetilde{\mathcal{C}}$ lies in the same isotopy class rel. $f^{-1}(\operatorname{post}(f))$ as $\mathcal{C}^1$. Hence by Theorem 15.5 (which we will prove momentarily) there is at most one such Jordan curve $\widetilde{\mathcal{C}}$. The uniqueness of $\widetilde{\mathcal{C}}$ follows.

In Example 15.17 the reader can find another illustration for the construction of an invariant curve (see Figure 15.4).

15.1. Existence and uniqueness of invariant curves

We now turn to general expanding Thurston maps and establish existence and uniqueness results for invariant curves. We start with uniqueness results.

Proof of Theorem 15.5. Suppose $f\colon S^2 \to S^2$ is an expanding Thurston map, and $\mathcal{C}$ and $\mathcal{C}'$ are f-invariant Jordan curves in S^2 that both contain the set $\operatorname{post}(f)$ and are isotopic rel. $f^{-1}(\operatorname{post}(f))$. We have to show that $\mathcal{C} = \mathcal{C}'$.

Under the given assumptions, there exists an isotopy $H^0\colon S^2 \times I \to S^2$ rel. $f^{-1}(\operatorname{post}(f))$ with $H^0_0 = \mathrm{id}_{S^2}$ and $H^0_1(\mathcal{C}) = \mathcal{C}'$. Since $\operatorname{post}(f) \subset f^{-1}(\operatorname{post}(f))$, the map H^0 is also an isotopy rel. $\operatorname{post}(f)$. Hence by Proposition 11.3 we can find an isotopy $H^1\colon S^2 \times I \to S^2$ rel. $f^{-1}(\operatorname{post}(f))$ with $H^1_0 = \mathrm{id}_{S^2}$ and $f \circ H^1_t = H^0_t \circ f$ for $t \in I$. Repeating this argument, we obtain isotopies $H^n\colon S^2 \times I \to S^2$ rel. $f^{-1}(\operatorname{post}(f))$ with $H^n_0 = \mathrm{id}_{S^2}$ and $f \circ H^{n+1}_t = H^n_t \circ f$ for $t \in I$ and $n \in \mathbb{N}_0$.

Claim. $H^n_1(\mathcal{C}) = \mathcal{C}'$ for $n \in \mathbb{N}_0$.

To see this, we use induction on n. For $n = 0$ the claim is true by choice of H^0.

Suppose that $H^n_1(\mathcal{C}) = \mathcal{C}'$ for some $n \in \mathbb{N}_0$. Then Lemma 11.2 and the identity $f \circ H^{n+1}_1 = H^n_1 \circ f$ imply that

$$H^{n+1}_1(f^{-1}(\mathcal{C})) = f^{-1}(H^n_1(\mathcal{C})) = f^{-1}(\mathcal{C}').$$

CHAPTER 18

The geometry of the visual sphere

When $f\colon S^2 \to S^2$ is an expanding Thurston map and ϱ a visual metric for f, we call the metric space (S^2, ϱ) the *visual sphere* for f. Of course, (S^2, ϱ) depends on the choice of the visual metric ϱ, but by Proposition 8.3 (iv) any two such choices for a given Thurston map f produce snowflake equivalent metric spaces. Accordingly, we think of the visual sphere of a Thurston map as uniquely determined up to snowflake equivalence. In this chapter we investigate geometric features of the visual sphere that are invariant under such equivalences and relate them to the dynamics of f.

The following statement is one of the main results here.

THEOREM 18.1 (Properties of f and its associated visual sphere). *Suppose $f\colon S^2 \to S^2$ is an expanding Thurston map and ϱ is a visual metric for f. Then the following statements are true:*

(i) *(S^2, ϱ) is doubling if and only if f has no periodic critical points.*

(ii) *(S^2, ϱ) is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$ if and only if f is topologically conjugate to a rational map.*

(iii) *(S^2, ϱ) is snowflake equivalent to $\widehat{\mathbb{C}}$ if and only if f is topologically conjugate to a Lattès map.*

Here it is understood that $\widehat{\mathbb{C}}$ is equipped with the chordal metric. Statement (ii) characterizes the visual spheres that are quasispheres. As we have already discussed, it provides an interesting analog of Cannon's conjecture (see Section 4.3).

We know that two expanding Thurston maps are Thurston equivalent if and only if they are topologically conjugate (Theorem 11.1). Thus (ii) and (iii) can be reformulated as follows:

(ii') *(S^2, ϱ) is quasisymmetrically equivalent to $\widehat{\mathbb{C}}$ if and only if f is Thurston equivalent to a rational Thurston map with no periodic critical points.*

(iii') *(S^2, ϱ) is snowflake equivalent to $\widehat{\mathbb{C}}$ if and only if f is Thurston equivalent to a Lattès map.*

For the "if" implication in (ii') one has to assume that the rational map has no periodic critical points (or impose an equivalent condition); see Example 18.11.

Much more can be said in case (i) of the previous theorem.

PROPOSITION 18.2. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map without periodic critical points, ϱ be a visual metric for f with expansion factor $\Lambda > 1$, and ν_f be the measure of maximal entropy of f. Then the metric measure space (S^2, ϱ, ν_f) is Ahlfors Q-regular with*

$$Q := \frac{\log(\deg(f))}{\log(\Lambda)}.$$

In particular, (S^2, ϱ) *has Hausdorff dimension* Q *and*

$$0 < \mathcal{H}^Q_\varrho(S^2) < \infty.$$

Here $\mathcal{H}^Q_\varrho$ is Q-dimensional Hausdorff measure on (S^2, ϱ). For the definition of the measure of maximal entropy ν_f see Chapter 17. The statement implies that under the given assumptions we have $\mathcal{H}^Q_\varrho(M) \asymp \nu_f(M)$ for each Borel set $M \subset S^2$ with $C(\asymp)$ independent of M.

Since the Hausdorff dimension of (S^2, ϱ) must be ≥ 2, it also follows that $\Lambda \leq \deg(f)^{1/2}$. Combining this with Theorem 16.3 (ii), we obtain the upper bound $\Lambda_0(f) \leq \deg(f)^{1/2}$ for the combinatorial expansion factor of f. We will later see that this is true for every expanding Thurston map without the additional assumption that f has no periodic critical points (see Proposition 20.1).

Recall from Proposition 8.3 (v) that a metric ϱ on S^2 is a visual metric for an expanding Thurston map $f\colon S^2 \to S^2$ if and only if it is a visual metric for an iterate $F = f^n$ (which is also an expanding Thurston map by Lemma 6.5). Hence Theorem 18.1 immediately gives the following corollary.

COROLLARY 18.3. *Let* $f\colon S^2 \to S^2$ *be an expanding Thurston map, and* $F = f^n$ *with* $n \in \mathbb{N}$ *be an iterate of* f*. Then the following statements are true:*

(i) *The map* f *is topologically conjugate to a rational map if and only if* F *is topologically conjugate to a rational map.*
(ii) *The map* f *is topologically conjugate to a Lattès map if and only if* F *is topologically conjugate to a Lattès map.*

We now record some of the consequences of our results for rational Thurston maps explicitly.

THEOREM 18.4. *Let* $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ *be a rational Thurston map with no periodic critical points. Then the following statements are true:*

(i) *For each sufficiently large* $n \in \mathbb{N}$ *there exists a quasicircle* $\mathcal{C} \subset \widehat{\mathbb{C}}$ *with* $\operatorname{post}(f) \subset \mathcal{C}$ *that is* f^n*-invariant (i.e.,* $f^n(\mathcal{C}) \subset \mathcal{C}$*).*
(ii) *Each* f^n*-invariant Jordan curve* $\mathcal{C} \subset \widehat{\mathbb{C}}$ *with* $\operatorname{post}(f) \subset \mathcal{C}$ *is a quasicircle.*
(iii) *Let* $\mathcal{C}$ *be an* f*-invariant Jordan curve* $\mathcal{C} \subset \widehat{\mathbb{C}}$ *with* $\operatorname{post}(f) \subset \mathcal{C}$, $\mathbf{E}^n$ *be the set of* n*-edges, and* $\mathbf{X}^n$ *be the set of* n*-tiles for* $(f, \mathcal{C})$*. Then the family of all edges* $\{e : n \in \mathbb{N}_0,\, e \in \mathbf{E}^n\}$ *consists of uniform quasiarcs, and the family of all tiles* $\{X : n \in \mathbb{N}_0,\, X \in \mathbf{X}^n\}$ *of uniform quasidisks.*

Here the underlying metric is again the chordal metric on $\widehat{\mathbb{C}}$. For the concepts of uniformity used here see Section 15.3.

From (iii) it follows that the family $\{\partial X : n \in \mathbb{N}_0,\, X \in \mathbf{X}^n\}$ consists of uniform quasicircles. Note that this and the statement about the arcs in (iii) do not *a priori* follow from Proposition 15.26, because we use different underlying metrics. We will prove though that for a rational Thurston map f without periodic critical points the chordal metric is quasisymmetrically equivalent to each visual metric for f (see Lemma 18.10). Once we know this, Theorem 18.4 (iii) can easily be deduced from Proposition 15.26.

A consequence of Theorem 18.4 is that each sufficiently high iterate of a rational expanding Thurston map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has a particularly nice Markov partition, where the tiles are quasidisks.

Another important property of visual spheres from the viewpoint of quasiconformal geometry is that they are linearly locally connected. This and related properties will be discussed in Section 18.1.

Theorem 18.1 (i) and Proposition 18.2 will be proved in Section 18.2. In Section 18.3 we will establish Theorem 18.1 (ii). We postpone the proof of Theorem 18.1 (iii) to the end of Section 19.4 (this part was essentially proved in [**Me09a**]).

18.1. Linear local connectedness

Recall (see Section 4.1) that a metric space (X, d) is said to be *linearly locally connected* (often abbreviated as LLC) if there exists a constant $C \geq 1$ such that the following two conditions are satisfied:

(LLC1) If $p \in X$, $r > 0$, and $x, y \in B_d(p, r)$, then there exists a continuum $E \subset X$ with $x, y \in E$ and $E \subset B_d(p, Cr)$.

(LLC2) If $p \in X$, $r > 0$, and $x, y \in X \setminus B_d(p, r)$, then there exists a continuum $E \subset X$ with $x, y \in E$ and $E \subset X \setminus B_d(p, r/C)$.

It is easy to see that LLC1 is satisfied if and only if X is of bounded turning (as defined in (4.3)).

The space (X, d) is called *annularly linearly locally connected* (abbreviated as ALLC) if there exists a constant $C \geq 1$ with the following property: if $p \in X$, $r > 0$, and $x, y \in \overline{B}_d(p, 2r) \setminus B_d(p, r)$, then there exists a path γ in X joining x and y with

$$\gamma \subset \overline{B}_d(p, Cr) \setminus B_d(p, r/C).$$

The following proposition shows that the visual sphere of an expanding Thurston map is linearly locally connected and annularly linearly locally connected.

PROPOSITION 18.5. *Let $f\colon S^2 \to S^2$ be an expanding Thurston map, and ϱ be a visual metric for f. Then the following statements are true:*

(i) *(S^2, ϱ) is of bounded turning.*

(ii) *(S^2, ϱ) is annularly linearly locally connected.*

(iii) *(S^2, ϱ) is linearly locally connected.*

The statements (i)–(iii) are not logically independent, but one can show the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) for quite general spaces. The ensuing proof will not rely on this directly.

PROOF. Let $\Lambda > 1$ be the expansion factor of ϱ. Then for some Jordan curve $\mathcal{C} \subset S^2$ with $\operatorname{post}(f) \subset \mathcal{C}$ we have $\varrho(x, y) \asymp \Lambda^{-m(x,y)}$ for all $x, y \in S^2$, where $m(x, y) = m_{f,\mathcal{C}}(x, y)$ (see Definitions 8.1 and 8.2). In the following, all cells will be for $(f, \mathcal{C})$ and all metric concepts refer to ϱ.

(i) Let $x, y \in S^2$ be arbitrary. If $x = y$ there nothing to prove. So we may assume that $x \neq y$. Let $n = m(x, y) \in \mathbb{N}_0$. Then there exist n-tiles X and Y with $x \in X$, $y \in Y$, and $X \cap Y \neq \emptyset$. Since X and Y are Jordan regions, we can find a path α in $X \cup Y$ that joins x and y. Then by Proposition 8.4 (ii) we have

$$\operatorname{diam}(\alpha) \leq \operatorname{diam}(X) + \operatorname{diam}(Y) \lesssim \Lambda^{-n} \asymp \varrho(x, y).$$

In particular,

$$\operatorname{diam}(\alpha) \leq K \varrho(x, y) \tag{18.1}$$