Introduction

Part I

Arrangements. (Chapter 1.) A hyperplane arrangement $\mathcal{A}$ is a set of hyperplanes (codimension-one subspaces) in a fixed real vector space. We assume that the number of hyperplanes is finite and all of them pass through the origin. The intersection of all hyperplanes is the central face. The rank of an arrangement is the dimension of the ambient vector space minus the dimension of the central face. An arrangement has rank 0 if it has no hyperplanes, rank 1 if it has one hyperplane, and rank 2 if it has at least two hyperplanes and all of them pass through a codimension-two subspace.

Flats and faces. (Chapter 1.) Subspaces obtained by intersecting hyperplanes are called the *flats* of the arrangement. We let $\Pi[\mathcal{A}]$ denote the set of flats. It is a graded lattice with partial order given by inclusion. The minimum element is the central face and the maximum element is the ambient space. The codimension-one flats are the hyperplanes. Each hyperplane divides the ambient space into two half-spaces. Their intersection is the given hyperplane. Subsets obtained by intersecting half-spaces, with at least one half-space chosen for each hyperplane, are called the *faces* of the arrangement. We let $\Sigma[\mathcal{A}]$ denote the set of faces. It is a graded poset under inclusion. The central face is the minimum element. However, there is no unique maximum face, so $\Sigma[\mathcal{A}]$ is *not* a lattice. A maximal face is called a *chamber*. We let $\Gamma[\mathcal{A}]$ denote the set of chambers. The linear span of any face is a flat. This defines a surjective map

$$\mathrm{s} : \Sigma[\mathcal{A}] \twoheadrightarrow \Pi[\mathcal{A}].$$

We call this the *support map*. It is order-preserving.

Birkhoff monoid and Tits monoid. (Chapter 1.) We view the lattice of flats $\Pi[\mathcal{A}]$ as a (commutative) monoid with product given by the join operation. We call this the *Birkhoff monoid*. For flats X and Y, their Birkhoff product is $\mathrm{X} \vee \mathrm{Y}$. The poset of faces $\Sigma[\mathcal{A}]$ is not a lattice. Nonetheless, it carries a (noncommutative) monoid structure. We call this the *Tits monoid*. It is an example of a left regular band (since it satisfies the axiom $xyx = xy$). For faces F and G, we denote their Tits product by FG. The set of chambers $\Gamma[\mathcal{A}]$ is a left $\Sigma[\mathcal{A}]$-set, that is, for F a face and C a chamber, FC is a chamber. The support map is a monoid homomorphism.

Janus monoid. (Chapter 1.) A *bi-face* is a pair (F, F') of faces such that F and F' have the same support. Let $\mathrm{J}[\mathcal{A}]$ denote the set of bi-faces. The operation

$$(F, F')(G, G') := (FG, G'F')$$

turns $J[\mathcal{A}]$ into a monoid. We call it the *Janus monoid.* [1] It is the fiber product of the Tits monoid $\Sigma[\mathcal{A}]$ and its opposite $\Sigma[\mathcal{A}]^{\mathrm{op}}$ over the Birkhoff monoid $\Pi[\mathcal{A}]$. This can be pictured as follows.

$$\begin{array}{ccc} & \text{Janus monoid} & \\ \swarrow & & \searrow \\ \text{Tits monoid} & & \text{opposite Tits monoid} \\ \searrow & & \swarrow \\ & \text{Birkhoff monoid} & \end{array}$$

The Janus monoid is a band (since every element is idempotent) which is neither left regular nor right regular in general.

Arrangements under and over a flat. (Chapter 1.) From a flat X of an arrangement $\mathcal{A}$, one may construct two new arrangements: $\mathcal{A}^{\mathrm{X}}$, the arrangement *under* X, and $\mathcal{A}_{\mathrm{X}}$, the arrangement *over* X. The former is the arrangement obtained by intersecting the hyperplanes in $\mathcal{A}$ with X, while the latter is the subarrangement consisting of those hyperplanes which contain X. For flats $\mathrm{X} \leq \mathrm{Y}$, the arrangement under Y in $\mathcal{A}_{\mathrm{X}}$ is the same as the arrangement over X in $\mathcal{A}^{\mathrm{Y}}$. We denote this arrangement by $\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}$.

Cones. (Chapter 2.) Subsets obtained by intersecting half-spaces (with no restriction) are called the *cones* of the arrangement. In particular, faces and flats are cones. (A hyperplane is the intersection of the two half-spaces it bounds.) Let $\Omega[\mathcal{A}]$ denote the set of all cones. It is a lattice under inclusion. The support map extends to an order-preserving map

$$\mathrm{c} : \Omega[\mathcal{A}] \to \Pi[\mathcal{A}].$$

We call this the *case map.* It sends a cone to the smallest flat containing that cone. The case map is the left adjoint of the inclusion map $\Pi[\mathcal{A}] \to \Omega[\mathcal{A}]$. There is another order-preserving map

$$\mathrm{b} : \Omega[\mathcal{A}] \to \Pi[\mathcal{A}]$$

which we call the *base map.* It sends a cone to the largest flat which is contained in that cone. The base map is the right adjoint of the inclusion map. Note that the base and case of a flat is the flat itself.

Cones whose case is the maximum flat are called top-cones. The poset of top-cones is a join-semilattice which is join-distributive, and in particular, graded and upper semimodular (Theorems 2.56, 2.58 and 2.60).

Lunes. (Chapters 3 and 4.) A cone is a *lune* if it has the property that for any hyperplane containing its base, the entire cone lies on one side of that hyperplane. Faces and flats are lunes. In general, any cone can be optimally cut up into lunes by using hyperplanes containing the base of the cone (Theorem 3.27). Finer decompositions can be obtained by using hyperplanes containing a fixed flat lying inside the base (Proposition 3.22). For instance, it is possible to cut a lune itself into smaller lunes. The optimal decomposition of a flat X is X itself (since it is a lune). An instance of a finer decomposition is to write X as a union of faces having support X.

[1] Janus Bifrons is a Roman god with two faces.

Lunes which are top-cones are called top-lunes. The poset of top-lunes under inclusion is graded (Theorem 4.9). We consider two partial orders on lunes. The first partial order is the inclusion of lune closures (and is the restriction of the partial order on cones), while the second is the inclusion of lune interiors. Both extend the partial order on top-lunes and are graded (Theorems 4.12 and 4.26).

Lunes can be composed when the case of the first lune equals the base of the second lune. This yields a category whose objects are flats and morphisms are lunes. We call it the *category of lunes*. It is internal to posets under the second partial order on lunes (Proposition 4.31). It also admits a nice presentation (Proposition 4.42). A lune with base X and case Y is the same as a chamber in the arrangement $\mathcal{A}^{\mathrm{Y}}_{\mathrm{X}}$. Using this, composition of lunes can be recast as follows. For any flat X, there is a map

$$\Gamma[\mathcal{A}^{\mathrm{X}}] \otimes \Gamma[\mathcal{A}_{\mathrm{X}}] \to \Gamma[\mathcal{A}].$$

We call this the substitution product of chambers, see (4.18).

Braid arrangement. (Chapters 5 and 6.) The braid arrangement is the motivating example for many of our considerations. The key observation is that for this arrangement, geometric notions of faces, flats, top-cones, and so on can be encoded by combinatorial notions of set compositions, set partitions, partial orders and so on. This correspondence between geometry and combinatorics is summarized in Table 6.2. The braid arrangement is an example of a reflection arrangement whose associated Coxeter group is the group of permutations. In the Coxeter case, one can define face-types and flat-types. Face-types are orbits of the set of faces under the Coxeter group action. Similarly, flat-types are orbits of the set of flats. For the braid arrangement, face-types and flat-types correspond to integer compositions and integer partitions.

Descent equation and lune equation. (Chapter 7.) Fix chambers C and D. The *descent equation* is $HC = D$. In other words, we need to solve for faces H such that the Tits product of H and C equals D. (This is related to descents of permutations in the case of the braid arrangement which motivates our terminology.) More generally, we can fix faces F and G, and consider the equation $HF = G$. In fact, one can do the following. For any left $\Sigma[\mathcal{A}]$-set h, the descent equation is $H \cdot x = y$, where x and y are fixed elements of h, the variable is H, and $\cdot$ denotes the action of $\Sigma[\mathcal{A}]$ on h. Apart from finding the solutions, there is also interest in computing the sum $\sum (-1)^{\mathrm{rk}(H)}$ as H ranges over the solution set, with $\mathrm{rk}(H)$ denoting the rank of H. For this, we attach to the solution set a relative pair (X, A) of cell complexes whose Euler characteristic is the given sum, see (7.32). By construction X is either a ball or sphere, but the topology of A is complicated in general. In our starting examples h is either $\Gamma[\mathcal{A}]$ or $\Sigma[\mathcal{A}]$. In these cases, A also has the topology of a ball or sphere. This leads to explicit identities, see (7.10) and (7.11a).

Fix a face H and a chamber D. The *lune equation* is $HC = D$. The difference is that now we need to solve for C. For a solution to exist H must be smaller than D. Assuming this condition, the solution set is precisely the set of chambers contained in some top-lune (which explains our terminology). More generally, an arbitrary lune can be obtained as the solution set of the equation $HF = G$ for some fixed H and G. Since lunes have the topology of a ball or sphere, we can again compute $\sum (-1)^{\mathrm{rk}(F)}$ explicitly, see (7.12a). An analysis with relative pairs, similar to the one for the descent equation, can be carried out for right $\Sigma[\mathcal{A}]$-sets h, see (7.41). The lune equation in this case is $x \cdot F = y$, with $x, y \in$ h.

Distance function and Varchenko matrix. (Chapter 8.) A hyperplane separates two chambers if they lie on its opposite sides. The distance between two chambers is defined to be the number of hyperplanes which separate them. Fix a scalar q, and define a bilinear form on the set of chambers $\Gamma[\mathcal{A}]$ by

$$\langle C, D \rangle := q^{\mathrm{dist}(C,D)}.$$

Here C and D are chambers and $\mathrm{dist}(C, D)$ denotes the distance between them. The determinant of this matrix factorizes with factors of the form $1 - q^i$, see (8.41). In particular, the bilinear form is nondegenerate if q is not a root of unity.

More generally, assign a weight to each half-space, and define $\langle C, D \rangle$ to be the product of the weights of all half-spaces which contain C but do not contain D. Setting each weight to be q recovers the previous case. A factorization of the determinant of this matrix was obtained by Varchenko (Theorem 8.11). (He worked in the special case when the two opposite half-spaces bound by each hyperplane carry the same weight.) Lunes play a key role in the proof. The Varchenko matrix can be formally inverted using non-stuttering paths, see (8.30).

It is fruitful to consider a more general situation where we start with an arbitrary top-cone, and restrict the Varchenko matrix to chambers of this top-cone. The determinant of this matrix also factorizes. This more general result is given in Theorem 8.12. Specializing the top-cone to the ambient space recovers the previous situation. The special case of weights on hyperplanes is given in Theorem 8.22. This latter result has been obtained recently by Gente independent of our work.

Part II

Birkhoff algebra and Tits algebra. (Chapter 9.) The linearization of a monoid over a field $\Bbbk$ yields an algebra. Let $\mathsf{\Pi}[\mathcal{A}]$ denote the linearization of $\Pi[\mathcal{A}]$, and $\mathsf{\Sigma}[\mathcal{A}]$ denote the linearization of $\Sigma[\mathcal{A}]$ over $\Bbbk$. We call these the *Birkhoff algebra* and the *Tits algebra*, respectively. These are finite-dimensional $\Bbbk$-algebras (since the original monoids are finite). The linearization of $\Gamma[\mathcal{A}]$, denoted $\mathsf{\Gamma}[\mathcal{A}]$, is a left module over $\mathsf{\Sigma}[\mathcal{A}]$. One can linearize the support map as well to obtain an algebra homomorphism $\mathrm{s} : \mathsf{\Sigma}[\mathcal{A}] \twoheadrightarrow \mathsf{\Pi}[\mathcal{A}]$.

The Birkhoff algebra $\mathsf{\Pi}[\mathcal{A}]$ is isomorphic to $\Bbbk^n$, where n is the number of flats. In other words, $\mathsf{\Pi}[\mathcal{A}]$ is a split-semisimple commutative algebra (Theorem 9.2). (By a result of Solomon, this holds for any algebra obtained by linearizing a lattice.) The coordinate vectors of $\Bbbk^n$ yield a unique complete system of primitive orthogonal idempotents of $\mathsf{\Pi}[\mathcal{A}]$. We denote them by Q_{X}, as X varies over flats. The simple modules over $\mathsf{\Pi}[\mathcal{A}]$ are all one-dimensional, and given by $\mathsf{Q}_{\mathrm{X}} \cdot \mathsf{\Pi}[\mathcal{A}]$. Further, any module h is a direct sum of simple modules. More precisely, we have the Peirce decomposition [2]

$$\mathsf{h} = \bigoplus_{\mathrm{X}} \mathsf{Q}_{\mathrm{X}} \cdot \mathsf{h},$$

[2] A decomposition of a module using an orthogonal family of idempotents is called a Peirce decomposition.

and the simple module $\mathtt{Q}_{\mathrm{X}} \cdot \Pi[\mathcal{A}]$ occurs in the summand $\mathtt{Q}_{\mathrm{X}} \cdot \mathsf{h}$ with multiplicity equal to its dimension (Theorems 9.7 and 9.8). As a consequence, the action of any element of $\Pi[\mathcal{A}]$ on any module h is diagonalizable (Theorem 9.9).

The largest nilpotent ideal of an algebra A is called its radical, denoted $\mathrm{rad}(A)$. The Birkhoff algebra has no nonzero nilpotent elements, so $\mathrm{rad}(\Pi[\mathcal{A}]) = 0$. In contrast, the Tits algebra has many nilpotent elements. In fact, $\mathrm{rad}(\Sigma[\mathcal{A}])$ is precisely the kernel of the (linearized) support map s, hence

$$\Sigma[\mathcal{A}]/\,\mathrm{rad}(\Sigma[\mathcal{A}]) \cong \Pi[\mathcal{A}].$$

This was proved by Bidigare. We say that $\Sigma[\mathcal{A}]$ is an elementary algebra since the quotient by its radical is a split-semisimple commutative algebra. The simple modules over $\Sigma[\mathcal{A}]$ coincide with those over $\Pi[\mathcal{A}]$ (since $\mathrm{rad}(\Sigma[\mathcal{A}])$ is forced to act by zero on such modules). However, a module of $\Sigma[\mathcal{A}]$ does not split as a direct sum of simple modules in general. (An example is provided by the module of chambers $\Gamma[\mathcal{A}]$.) Similarly, the action of an element of $\Sigma[\mathcal{A}]$ on a module h is not diagonalizable in general. Nonetheless, by taking a filtration of h, one can gain detailed information about the eigenvalues and multiplicities of the action (Theorem 9.44). This result for $\mathsf{h} := \Gamma[\mathcal{A}]$ was first obtained by Bidigare, Hanlon and Rockmore (Theorem 9.46); their motivation for considering this problem came from random walks. The above line of argument was given by Brown.

Any left module h over the Tits algebra has a *primitive part* which we denote by $\mathcal{P}(\mathsf{h})$. It consists of those elements of h which are annihilated by all faces except the central face (which acts by the identity). Dually, any right module h has a *decomposable part* which we denote by $\mathcal{D}(\mathsf{h})$. The duality is made precise in Proposition 9.61.

Janus algebra. (Chapter 9.) Let $\mathsf{J}[\mathcal{A}]$ denote the linearization of $\mathrm{J}[\mathcal{A}]$. We call this the *Janus algebra*. Just like the Tits algebra, the Janus algebra is elementary, and its split-semisimple quotient is the Birkhoff algebra. Interestingly, the Janus algebra admits a deformation by a scalar q. When q is not a root of unity, the q-Janus algebra is in fact split-semisimple, that is, isomorphic to a product of matrix algebras over $\Bbbk$. There is one matrix algebra for each flat X, with the size of the matrix being the number of faces with support X (Theorem 9.75). As a consequence, the q-Janus algebra, for q not a root of unity, is Morita equivalent to the Birkhoff algebra (Theorem 9.76). This is completely different from what happens for $q = 1$.

Eulerian idempotents. (Chapter 11.) Let us go back to the Tits algebra $\Sigma[\mathcal{A}]$. An *Eulerian family* $\mathtt{E}$ is a complete system of primitive orthogonal idempotents of $\Sigma[\mathcal{A}]$. Eulerian families are in correspondence with algebra sections $\Pi[\mathcal{A}] \hookrightarrow \Sigma[\mathcal{A}]$ of the support map s. The construction of such sections is the idempotent lifting problem in ring theory. For elementary algebras, lifts always exist and any two lifts are conjugate by an invertible element in the algebra. For each X, we let $\mathtt{E}_{\mathrm{X}}$ denote the image of $\mathtt{Q}_{\mathrm{X}}$ under an algebra section, thus, $\mathrm{s}(\mathtt{E}_{\mathrm{X}}) = \mathtt{Q}_{\mathrm{X}}$. The $\mathtt{E}_{\mathrm{X}}$ are called *Eulerian idempotents* and constitute the Eulerian family $\mathtt{E}$. Apart from being elementary, the Tits algebra is also the linearization of a left regular band. This allows for many interesting characterizations of Eulerian families (Theorems 11.20, 11.40 and 15.44). A highlight here is a construction of Saliola which produces an Eulerian family starting with a homogeneous section of the support map. (A homogeneous section is equivalent to an assignment of a scalar $\mathtt{u}^F$ to each face F such that for any flat X, the sum of $\mathtt{u}^F$ over all F with support X is 1.) This construction employs the *Saliola lemma* (Lemma 11.12), which is an important property of any Eulerian family. For a good reflection arrangement, we give cancelation-free formulas for the Eulerian idempotents arising from the uniform homogeneous section (Theorem 11.53).

Diagonalizability. (Chapter 12.) An element of an algebra is diagonalizable if it can be expressed as a linear combination of orthogonal idempotents. All elements of the Birkhoff algebra are diagonalizable. However, that is not true for the Tits algebra. For instance, no nonzero element of the radical of $\Sigma[\mathcal{A}]$ is diagonalizable. Following another method of Saliola, one can characterize diagonalizable elements using existence of eigensections (Corollary 12.15). Examples include nonnegative elements (Theorem 12.20) and separating elements (Theorem 12.17). The separating condition was introduced by Brown. For separating elements, there is a formula for the eigensection (arising from the Brown-Diaconis stationary distribution formula (12.6)), and a formula for the Eulerian idempotents due to Brown, see (12.12) and (12.13). Apart from these families, we also consider diagonalizability of specific elements such as the Takeuchi element (12.23) and the Fulman elements (12.38). For the braid arrangement, these include the Adams elements; their diagonalization is given in (12.49).

Lie elements and JKS. (Chapters 10 and 14.) Recall that the Tits algebra $\Sigma[\mathcal{A}]$ acts on the space of chambers $\mathsf{\Gamma}[\mathcal{A}]$. We put

$$\mathsf{Lie}[\mathcal{A}] := \mathcal{P}(\mathsf{\Gamma}[\mathcal{A}]),$$

the primitive part of $\mathsf{\Gamma}[\mathcal{A}]$. This is the space of *Lie elements*. We refer to this description of $\mathsf{Lie}[\mathcal{A}]$ as the Friedrichs criterion. There are other characterizations of $\mathsf{Lie}[\mathcal{A}]$ such as the top-lune criterion and the descent criterion. In the case of the braid arrangement, $\mathsf{Lie}[\mathcal{A}]$ is the space of classical Lie elements (the multilinear part of the free Lie algebra). The top-lune criterion extends a classical result of Ree for the free Lie algebra, while the descent criterion extends a result of Garsia. The top-lune criterion says the following: A Lie element is an assignment of a scalar x^C to each chamber C such that the sum of these scalars in any top-lune (containing more than one chamber) is zero. In fact, by cutting a top-lune into smaller top-lunes, it suffices to restrict to top-lunes whose base is of rank 1. The dimension of $\mathsf{Lie}[\mathcal{A}]$ equals the absolute value of the Möbius number of $\mathcal{A}$. There are many ways to deduce this, see for instance (10.24) or (11.63). There are also many interesting bases for $\mathsf{Lie}[\mathcal{A}]$. We discuss the Dynkin basis (which depends on a generic half-space) and the Lyndon basis (which depends on a choice function).

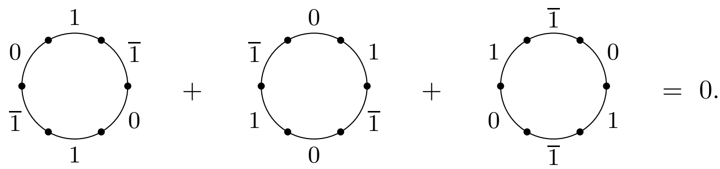
For any flat X, there is a map

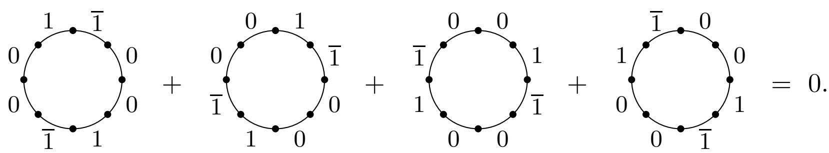
$$\mathsf{Lie}[\mathcal{A}^{\mathrm{X}}] \otimes \mathsf{Lie}[\mathcal{A}_{\mathrm{X}}] \to \mathsf{Lie}[\mathcal{A}].$$

We call this the substitution product of Lie, see (10.28). It is obtained by restricting the substitution product of chambers. All Lie elements of $\mathcal{A}$ can be generated by repeated substitutions starting with Lie elements of rank-one arrangements (which incorporate antisymmetry), subject to the Jacobi identities in rank-two arrangements (Theorem 14.41). Antisymmetry can be visualized as follows.

$$(\overset{1}{\bullet} \qquad \overset{\bar{1}}{\bullet}) + (\overset{\bar{1}}{\bullet} \qquad \overset{1}{\bullet}) = 0.$$

(By convention, $\overline{1}$ denotes -1.) The two vertices are the two chambers of a rank-one arrangement. The Jacobi identity for the hexagon and octagon (which are the spherical models of rank-two arrangements of 3 and 4 lines, respectively) are shown below. The figures show the coefficients of each chamber in a Lie element.





The Ree criterion says that the sum of the scalars in any semicircle is 0.

A closely related object to $\mathsf{Lie}[\mathcal{A}]$ is the order homology of the lattice of flats $\Pi[\mathcal{A}]$. The latter is a well-studied object. The order homology is nonzero only in top rank and has dimension equal to the absolute value of the Möbius number of $\mathcal{A}$. Again, there are many bases for this space. We discuss the Björner-Wachs basis and the Björner basis. One of our main results, the *Joyal-Klyachko-Stanley theorem*, or JKS for short, states that up to the one-dimensional orientation space of $\mathcal{A}$, there is a natural isomorphism between $\mathsf{Lie}[\mathcal{A}]$ and the top-cohomology of $\Pi[\mathcal{A}]$ (Theorem 14.38). We write this as

$$\mathcal{H}^{\mathrm{top}}(\Pi[\mathcal{A}]) \otimes \mathsf{E}^{\mathsf{o}}[\mathcal{A}] \cong \mathsf{Lie}[\mathcal{A}].$$

The special case of the braid arrangement is a classical result due to separate work by Joyal, Klyachko and Stanley. Under the JKS isomorphism and the duality between homology and cohomology, the Dynkin basis corresponds to the Björner-Wachs basis (Corollary 14.39) while the Lyndon basis corresponds to the Björner basis (Propositions 14.51 and 14.52). The latter correspondence was used by Barcelo to give the first combinatorial proof of the classical JKS.

Zie elements. (Chapter 10, 11 and 14.) Consider the left action of the Tits algebra $\Sigma[\mathcal{A}]$ on itself, and put

$$\mathsf{Zie}[\mathcal{A}] := \mathcal{P}(\Sigma[\mathcal{A}]),$$

the primitive part of $\Sigma[\mathcal{A}]$. This is the space of *Zie elements* (defined using the Friedrichs criterion). In analogy with $\mathsf{Lie}[\mathcal{A}]$, we also have other criteria such as the lune and descent criteria. A Zie element is a particular element of the Tits algebra. It is called special if its coefficient of the central face is 1. The space $\mathsf{Zie}[\mathcal{A}]$ is a right ideal of $\Sigma[\mathcal{A}]$ generated by any special Zie element (Lemma 10.21). Any special Zie element is an idempotent. In fact, an element of the Tits algebra is a special Zie element iff it is an idempotent whose support is $\mathtt{Q}_\perp$ (Lemma 10.24). The first Eulerian idempotent $\mathtt{E}_\perp$ of any Eulerian family is a special Zie element, and conversely every special Zie element arises in this manner (Lemma 11.42). More generally, the higher Eulerian idempotent $\mathtt{E}_{\mathrm{X}}$ (of any Eulerian family) yields a special Zie element of the arrangement $\mathcal{A}_{\mathrm{X}}$ over X. This leads to a characterization of Eulerian families in terms of families of special Zie elements (Theorem 11.40).

For any left $\Sigma[\mathcal{A}]$-module h, a special Zie element projects h onto its primitive part $\mathcal{P}(\mathsf{h})$ (Proposition 10.35).

Given any generic half-space of $\mathcal{A}$, the alternating sum of faces contained in that half-space yields a special Zie element. We call this the *Dynkin element* (Proposition 14.1). It projects the module of chambers $\Gamma[\mathcal{A}]$ onto its primitive part which is $\mathsf{Lie}[\mathcal{A}]$. This generalizes the classical Dynkin operator (left nested bracketing) in the case of the braid arrangement. Under this projection, the images of chambers in the half-space opposite to the given generic half-space yields a basis of $\mathsf{Lie}[\mathcal{A}]$. This is precisely the Dynkin basis mentioned earlier (Proposition 14.16).

Loewy series and Peirce decompositions. (Chapter 13.) The *primitive series* of a left $\Sigma[\mathcal{A}]$-module h is a specific filtration of h with the primitive part $\mathcal{P}(\mathsf{h})$ as the first nontrivial term from the bottom. Dually, the *decomposable series* of a right $\Sigma[\mathcal{A}]$-module h is a specific filtration of h with the decomposable part $\mathcal{D}(\mathsf{h})$ as the first nontrivial term from the top. The primitive series and decomposable series are both examples of Loewy series (Propositions 13.4 and 13.6). By general theory, they are trapped between the radical and socle series; see Lemmas 13.8 and 13.18. The left module of chambers is rigid, that is, its radical, primitive and socle series coincide (Theorem 13.68). The right module of Zie elements is also rigid (Theorem 13.83).

For any left $\Sigma[\mathcal{A}]$-module h, we have the left Peirce decomposition

$$\mathsf{h} = \bigoplus_{\mathrm{X}} \mathtt{E}_{\mathrm{X}} \cdot \mathsf{h}.$$

This depends on the choice of the Eulerian family $\mathtt{E}$. However, the summand indexed by the minimum flat $\perp$ is independent of this choice. More precisely,

$$\mathtt{E}_{\perp} \cdot \mathsf{h} = \mathcal{P}(\mathsf{h}),$$

see Proposition 13.21. This is consistent with the earlier statement that a special Zie element projects h onto $\mathcal{P}(\mathsf{h})$. In general, the components $\mathtt{E}_{\mathrm{X}} \cdot \mathsf{h}$ are related to the primitive series of h (Proposition 13.22). Similarly, one can relate the components of the right Peirce decomposition of a right $\Sigma[\mathcal{A}]$-module to its decomposable series (Proposition 13.24).

The components of the left Peirce decompositions of $\Gamma[\mathcal{A}]$ and $\Sigma[\mathcal{A}]$ relate to Lie and Zie elements as follows.

$$\mathtt{E}_{\mathrm{X}} \cdot \Gamma[\mathcal{A}] \cong \mathsf{Lie}[\mathcal{A}_{\mathrm{X}}] \quad \text{and} \quad \mathtt{E}_{\mathrm{X}} \cdot \Sigma[\mathcal{A}] \cong \mathsf{Zie}[\mathcal{A}_{\mathrm{X}}].$$

See Lemmas 13.26 and 13.30. The former yields an algebraic form of the Zaslavsky formula, see (13.8). Similarly, the components of the right Peirce decompositions of $\mathsf{Zie}[\mathcal{A}]$ and $\Sigma[\mathcal{A}]$ relate to Lie and chamber elements as follows.

$$\mathsf{Zie}[\mathcal{A}] \cdot \mathtt{E}_{\mathrm{Y}} \cong \mathsf{Lie}[\mathcal{A}^{\mathrm{Y}}] \quad \text{and} \quad \Sigma[\mathcal{A}] \cdot \mathtt{E}_{\mathrm{Y}} \cong \Gamma[\mathcal{A}^{\mathrm{Y}}].$$

See Lemmas 13.43 and 13.40. The latter is present in work of Saliola. Combining these decompositions yields a vector space isomorphism

$$\mathtt{E}_{\mathrm{X}} \cdot \Sigma[\mathcal{A}] \cdot \mathtt{E}_{\mathrm{Y}} \cong \mathsf{Lie}[\mathcal{A}_{\mathrm{X}}^{\mathrm{Y}}].$$

See Proposition 13.50 and Table 13.1. These are components of the two-sided Peirce decomposition of $\Sigma[\mathcal{A}]$. By taking direct sum over all $\mathrm{X} \leq \mathrm{Y}$, we obtain an algebra isomorphism

$$\Sigma[\mathcal{A}] \cong \bigoplus_{X \le Y} \mathsf{Lie}[\mathcal{A}_X^Y].$$

In the rhs, elements in the (X, Y)-summand are multiplied with elements in the (Y, Z)-summand by substitution; the remaining products are all zero. This isomorphism is given in Theorem 13.54. As an application, we obtain the quiver of the Tits algebra (Theorem 13.73). This is a result of Saliola, who proved it by linking the Tits algebra to the top-cohomology of the lattice of flats.

Lune-incidence algebra and noncommutative zeta and Möbius functions. (Chapter 15.) A *nested flat* is a pair of flats (X, Y) with $Y \ge X$. Let $I_{flat}[\mathcal{A}]$ denote the incidence algebra of the poset of flats $\Pi[\mathcal{A}]$. We call it the *flat-incidence algebra*. It consists of functions f on nested flats, with the product of f and g given by

$$(fg)(X, Z) = \sum_{Y:\, X \le Y \le Z} f(X, Y) g(Y, Z).$$

The zeta function $\zeta \in I_{flat}[\mathcal{A}]$ is defined to be identically 1. It is invertible and its inverse is the Möbius function $\mu \in I_{flat}[\mathcal{A}]$. The Möbius function satisfies the Weisner formula, and in fact is completely characterized by it. A standard way to prove this formula is to use the split-semisimplicity of the Birkhoff algebra.

We propose a noncommutative version of this theory with $\Pi[\mathcal{A}]$ replaced by $\Sigma[\mathcal{A}]$. A *nested face* is a pair of faces (A, F) with $F \ge A$. Let $I_{face}[\mathcal{A}]$ denote the incidence algebra of $\Sigma[\mathcal{A}]$. We call it the *face-incidence algebra*. It consists of functions f on nested faces, with the product of f and g given by

$$(fg)(F, H) = \sum_{G:\, F \le G \le H} f(F, G) g(G, H).$$

We say two nested faces (A, F) and (B, G) are equivalent if $AB = A$, $BA = B$, $AG = F$ and $BF = G$. Equivalence classes are indexed by lunes (Proposition 3.13). Let $I_{lune}[\mathcal{A}]$ denote the subalgebra of $I_{face}[\mathcal{A}]$ consisting of those f which take the same value on equivalent nested faces. In particular, $I_{lune}[\mathcal{A}]$ has a basis indexed by lunes. It is an example of a reduced incidence algebra. We call it the *lune-incidence algebra*. It can also be interpreted as the incidence algebra of the category of lunes (Proposition 15.7).

We define noncommutative zeta functions $\boldsymbol{\zeta}$ and noncommutative Möbius functions $\boldsymbol{\mu}$ as particular elements of the lune-incidence algebra. They are no longer unique; zeta functions are characterized by lune-additivity (15.23) and Möbius functions by the noncommutative Weisner formula (15.27). They correspond to each other under taking inverses in the lune-incidence algebra (Theorem 15.28). We relate this result to the representation theory of the Tits algebra. This circle of ideas is summarized in the important Theorem 15.44, which states in particular that noncommutative zeta and Möbius functions are in bijection with Eulerian families. Also see Table 15.1.

The flat-incidence algebra and lune-incidence algebra are both elementary and their quivers are acylic with vertices indexed by flats (Proposition 15.1 and Theorem 15.2, and Proposition 15.10 and Theorem 15.14).

Lie-incidence algebra. (Chapter 15.) We introduce the Lie-incidence algebra $\mathrm{I}_{\mathrm{Lie}}[\mathcal{A}]$. It is a subalgebra of the lune-incidence algebra (Proposition 15.51). It is isomorphic to the Tits algebra (Theorem 15.56). We also introduce additive and Weisner functions on lunes. These are linear subspaces of the lune-incidence algebra which respectively contain noncommutative zeta and Möbius functions as affine subspaces. Further, they are right and left modules respectively over $\mathrm{I}_{\mathrm{Lie}}[\mathcal{A}]$ with action induced from the product of $\mathrm{I}_{\mathrm{lune}}[\mathcal{A}]$ (Propositions 15.62 and 15.66). Moreover, they are isomorphic to the right and left regular representations of $\mathrm{I}_{\mathrm{Lie}}[\mathcal{A}]$ (Propositions 15.63 and 15.67).

Invariant objects. (Chapter 16.) In the discussion so far, the arrangement $\mathcal{A}$ and the field $\Bbbk$ have been arbitrary. Suppose now that $\mathcal{A}$ is a reflection arrangement with associated Coxeter group W, and the characteristic of $\Bbbk$ does not divide the order of W. Here W acts on both $\Sigma[\mathcal{A}]$ and $\Pi[\mathcal{A}]$ giving rise to the invariant subalgebras $\Sigma[\mathcal{A}]^W$ and $\Pi[\mathcal{A}]^W$. We call these the *invariant Tits algebra* and *invariant Birkhoff algebra*, respectively. The former is elementary, and the latter is its split-semisimple quotient. They have a basis indexed by face-types and flat-types. Complete systems of primitive orthogonal idempotents of $\Sigma[\mathcal{A}]^W$ (also called invariant Eulerian families) can be characterized in a manner similar to $\Sigma[\mathcal{A}]$ (Theorem 16.48). (The hypothesis on the characteristic of $\Bbbk$ is clarified by Lemma 16.42.) The Garsia-Reutenauer idempotents are the Eulerian idempotents which arise by specializing to the braid arrangement and taking the invariant homogeneous section to be uniform. By linking the invariant Tits algebra to invariant Lie elements, one can obtain information on the quiver of the invariant Tits algebra. The related result given in Proposition 16.55 is due to Saliola.

The Coxeter group acts on the lune-incidence algebra giving rise to the *invariant lune-incidence algebra*. This algebra can also be viewed as a reduced incidence algebra of the poset of face-types. It has a basis indexed by lune-types. For those noncommutative zeta and Möbius functions which belong to this algebra, lune-additivity and the noncommutative Weisner formula can be reformulated using face-types, see (16.41) and (16.42). The structure constants of the invariant Tits algebra intervene in this description.

There is an injective map from the Tits algebra to the space indexed by pairs of chambers. Taking invariants induces an injective map from $\Sigma[\mathcal{A}]^W$ to W (the group algebra of W). The image of this map is a subalgebra of W which is known as the *Solomon descent algebra*. This induces an isomorphism between the invariant Tits algebra and the opposite of the Solomon descent algebra (Theorem 16.8). This was proved by Bidigare. Invariant Eulerian families of the Solomon descent algebra appeared in work of Bergeron, Bergeron, Howlett and Taylor (Theorem 16.43).

Projective objects. Every arrangement carries a symmetry of order 2 given by the opposition map (which sends a point to its negative). The *projective Tits algebra* is the subalgebra of the Tits algebra which is invariant under the opposition map. It is elementary with the Birkhoff algebra as its split-semisimple quotient (Proposition 9.27). Its quiver is given in Theorem 13.75. A complete system of the projective Tits algebra is a projective Eulerian family. Its characterizations in terms of projective analogues of noncommutative zeta and Möbius functions, homogeneous sections and so on are summarized in Theorem 15.47. These results assume that the field characteristic is not 2.