Preface

Synopsis

The goal of this monograph is to study the interplay between various algebraic, geometric and combinatorial aspects of real hyperplane arrangements. The text contains many new ideas and results. It also gathers and organizes material from various sources in the literature, sometimes highlighting previously unnoticed connections. We briefly outline the contents below. They are explained in more detail in the main introduction.

We provide a detailed discussion on faces, flats, chambers, cones, gallery intervals, lunes, the support map, the case and base maps, and other geometric notions associated to real hyperplane arrangements. We show that any cone can be optimally decomposed into lunes. We introduce the category of lunes. This beautiful structure is intimately related to the substitution product of chambers a generalization of the classical associative operad). The classical case is obtained by specializing to the braid arrangement. We give several generalizations of the classical identity of Witt from Coxeter theory under the broad umbrella of descent and lune identities. The topological invariant involved here is the Euler characteristic of a relative pair of cell complexes. We generalize a well-known factorization theorem of Varchenko to cones, and also initiate an abstract approach to distance functions on chambers.

The main algebraic objects are the Birkhoff monoid and the Tits monoid, and their linearized algebras. The former is commutative and its elements are the flats of the arrangement, while the latter is not commutative and its elements are the faces. A module whose elements are chambers also plays a central role. Both monoids carry natural partial orders. The Birkhoff monoid is a lattice and its product is the join operation in the lattice. One may think of the Tits monoid as a noncommutative lattice. Its abelianization is the Birkhoff monoid, via the map that sends a face to the flat which supports it. We introduce the Janus monoid which is built out of the Tits and Birkhoff monoids.

We initiate a noncommutative Möbius theory of the Tits monoid and relate it to the representation theory of its linerization which is the Tits algebra. The central object is the lune-incidence algebra, which is a certain reduced incidence algebra of the poset of faces. It contains noncommutative zeta functions characterized by lune-additivity, and noncommutative Möbius functions characterized by the noncommutative Weisner formula. This theory lifts the usual Möbius theory for lattices, where the central object is the incidence algebra of the lattice of flats.

We introduce Lie and Zie elements. The latter belong to the Tits algebra, and the former to the module of chambers. The space of Zie elements is a right ideal of the Tits algebra. Any special Zie element defines an idempotent operator on chambers whose image is the space of Lie elements. To any generic half-space, we associate a special Zie element called the Dynkin element. Its action on chambers generalizes the left bracketing operator in classical Lie theory. We define a substitution product and establish a presentation of Lie. This generalizes the familiar presentation of the classical Lie operad. Antisymmetry is encoded in the notion of orientation of the rank-one arrangement and the Jacobi identity in the form of a linear relation among chambers obtained by "unbracketing" lines of the rank-two arrangements. This is same as saying that the space of Lie elements is isomorphic, up to orientation, to the top cohomology of the lattice of flats. This generalizes a celebrated theorem due to the combined work of Joyal, Klyachko and Stanley. We introduce the Lie-incidence algebra and show that it is isomorphic to the Tits algebra. This is intimately connected to the two-sided Peirce decomposition of the Tits algebra. The latter can be understood in terms of left and right Peirce decompositions of chambers and Zie elements respectively.

The Birkhoff algebra is split-semisimple. For the Tits algebra, complete systems of primitive orthogonal idempotents are in correspondence with algebra sections of the support map. We obtain many interesting characterizations of such sections. This aspect of the theory generalizes the classical theory of Eulerian idempotents. Noncommutative zeta and Möbius functions, and special Zie families are among the various concepts in correspondence. For reflection arrangements, there is a similar theory for the subalgebra of the Tits algebra invariant under the action of the Coxeter group. (The opposite of this algebra is the Solomon descent algebra.)

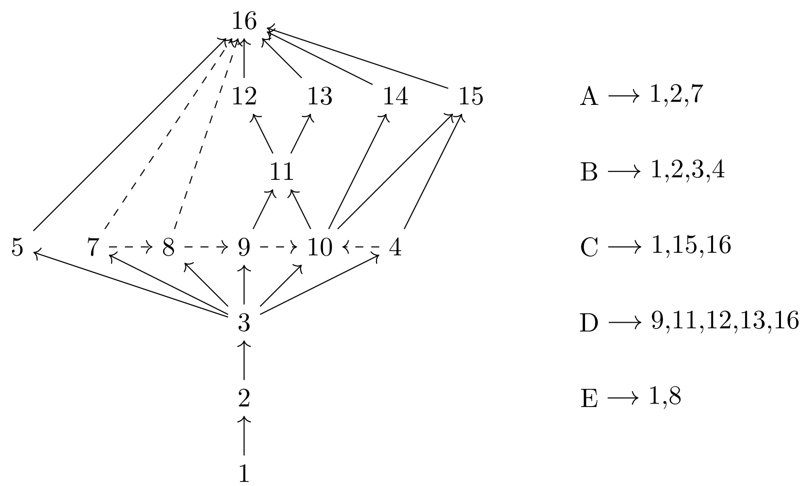
Precedents

This work benefits from and builds on some important recent developments. For the representation theory of the Tits algebra, we mention work of Brown, Diaconis and Saliola propelled by a landmark paper of Bidigare, Hanlon and Rockmore. (Older work of Solomon on the descent algebra has also been influential.) Some of these results are given in the generality of left regular bands and even bands. Further generalizations of this kind appear in work of Steinberg. For Lie theory, we mention work of Barcelo, Bergeron, Björner, Garsia, Patras, Reutenauer and Wachs. Saliola's work also implicitly contains elements of Lie theory. Explicit references to Lie are made only for the braid arrangement and the reflection arrangement of type B. The work of Joyal, Klyachko and Stanley relating Lie to order homology is for the braid arrangement. On the other hand, related results on order homology in the literature are usually given in the generality of arrangements or beyond. There have been several other contributors; most of them are mentioned in the main introduction. Two new entrants are the mathematicians Janus and Zie.

Organization

The text is organized in two parts. In Part I, the emphasis is on set-theoretic objects associated to hyperplane arrangements such as posets, monoids and the action of monoids on sets. In Part II, the emphasis is on linear objects such as algebras and their modules. There is a Notes section at the end of each chapter where detailed references to the literature, including discussions on alternative terminology and notation, are provided. Background information on topics such as Möbius functions, incidence algebras, representation theory of algebras and bands is provided in Appendices at the end of the main text. A notation index and a subject index are provided at the end of the book. Pictures and diagrams form an important component of our exposition which has a distinct geometric flavor. Numerous exercises are interspersed throughout the book.

The text is not meant to be read linearly from start to finish. We encourage readers to take up a particular chapter or section of their interest and backtrack as necessary. As an aid, the diagram of interdependence of chapters and appendices is displayed below.



A directed path from i to j indicates that some basic familiarity with Chapter i is necessary before proceeding to Chapter j. A dashed arrow from i to j means that the dependence of Chapter j on Chapter i is minimal, that is, restricted to some section or example.

Chapter 6 is not shown in the above diagram. It discusses the braid arrangement, the reflection arrangement of type B and other examples. They are employed frequently in later chapters for illustration.

Readership

We have strived to keep the text self-contained and with minimum prerequisites with the objective of making it accessible to advanced undergraduate and beginning graduate students. We hope it also serves as a useful reference on hyperplane arrangements to experts. The book touches upon several fields of mathematics such as representation theory of monoids and associative algebras, posets and their incidence algebras, lattice theory, random walks, invariant theory, discrete geometry, algebraic and geometric combinatorics, and algebraic Lie theory.

Scope

The theory of hyperplane arrangements has grown enormously in several different directions in the past two decades. The text is not meant to be a comprehensive survey of the entire theory. For instance, topics such as singularities, integral systems, hypergeometric functions and resonance varieties find no mention in the book. For these, one may look at [**15, 121, 138, 157, 177, 329, 413**] and references therein.

Future directions

Our constructions are all based on the choice of a real hyperplane arrangement. It is apparent, moreover, that a central role is played by the Tits monoid of faces of the arrangement. It is tempting to try to extend the theory to more general classes of monoids, particularly bands and left regular bands. We have kept our focus on arrangements, although such generalizations offer a promising line of research. We also mention the Janus monoid, the category of lunes and noncommutative Möbius functions as important objects worthy of further study. Our choice of topics has mainly been guided by applications to the theory of species, operads and Hopf algebras which we plan to develop in future work.

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