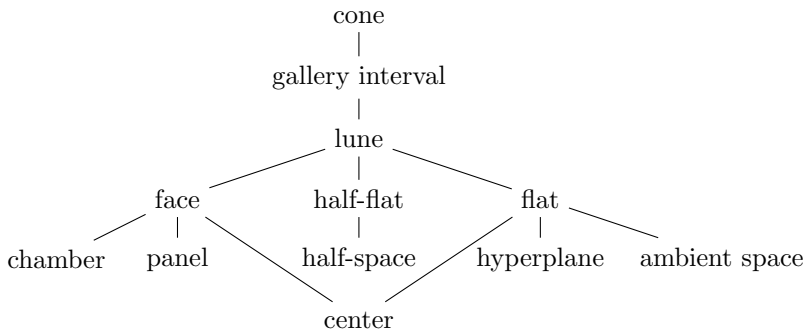


Hyperplane arrangements

Some basic geometric objects associated to a hyperplane arrangement are summarized below.



The diagram is to be read as follows. The most general object is a cone. Every gallery interval is a cone, and every lune is a gallery interval. Faces, half-flats and flats are lunes. Chambers and panels are faces, a half-space is a half-flat, hyperplanes and the ambient space are flats, and the center is both a face and a flat. Some other important objects which are not seen in the diagram are bi-faces, nested faces, charts, dicharts and partial-flats.

In this chapter, we discuss

- faces, chambers, flats and bi-faces,
- Tits monoid, Birkhoff monoid and Janus monoid,
- minimal galleries and gate property,
- arrangements under and over a flat of an arrangement,
- other operations on arrangements such as adjoint and cartesian product,
- enumerative aspects such as Möbius functions and characteristic polynomial.

Cones, gallery intervals, lunes, charts, dicharts and partial-flats are discussed in Chapters 2 and 3.

Many of the sets that we consider, such as the set of faces, carry the structure of a poset. All our posets are finite. For posets, we will employ the terminology given in Section B.1. Note in particular that ‘smaller than’ means \leq and ‘greater than’ means \geq . Graded posets are reviewed in Section B.2.

Convention 1.1. An arrangement is usually denoted by \mathcal{A} . To show the dependence of an object on \mathcal{A} , we use $[\mathcal{A}]$. For instance, $\Sigma[\mathcal{A}]$ denotes the set of faces of \mathcal{A} and $\Pi[\mathcal{A}]$ denotes the set of flats of \mathcal{A} . When \mathcal{A} is understood from the context, we may simply write Σ , Π , and so on.

CHAPTER 2

Cones

Cones form a nice general class of objects which encompass many other objects associated to hyperplane arrangements. For instance, faces, flats and half-spaces are examples of cones. Cones are related to the geometric notion of convexity. Each cone has a convexity dimension. Cones of convexity dimension either 1 or 2 are called gallery intervals. The support map on faces extends to cones. We call it the case map. Top-cones are cones whose case is the maximum flat. There is another map from cones to flats which we call the base map. The poset of top-cones is join-distributive, and in particular, upper semimodular and graded. We also discuss charts and dicharts and relate them to flats and cones, respectively. Finally, we introduce the notion of a partial-flat as an interpolating object between faces and flats. Partial-flats are also cones.

Adjunctions between posets play an important role in this chapter. Our examples include adjunctions between faces and top-cones, between cones and flats, between flats and charts, between charts and dicharts and between cones and dicharts. Background information on adjunctions is given in Section B.5.

2.1. Cones and convexity

We begin with cones. Cones bear to half-spaces the same relation that flats bear to hyperplanes. They can be characterized by using a combinatorial notion of convexity involving minimal galleries.

2.1.1. Cones. A *cone* of an arrangement \mathcal{A} is a subset of the ambient space which can be obtained by intersecting some subset of half-spaces in the arrangement.

Let $\Omega[\mathcal{A}]$ denote the set of all cones. It is a poset under inclusion. The center of \mathcal{A} is the minimum element. Since the intersection of two cones is a cone, meets exist in this poset. Further, it has a maximum element, namely, the ambient space, so joins exist as well, and $\Omega[\mathcal{A}]$ is a lattice. Explicitly, the join of two cones is the intersection of those half-spaces which contain both of them. We will usually denote cones by V and W ; we will denote their meet by $V \wedge W$ and join by $V \vee W$.

Recall the poset of faces $\Sigma[\mathcal{A}]$. Every face is a cone. Further, if a cone V is smaller than a face G , that is, if $V \leq G$, then V is necessarily a face. It follows that $\Sigma[\mathcal{A}]$ is a convex subposet of $\Omega[\mathcal{A}]$. Hence the inclusion $\Sigma[\mathcal{A}] \hookrightarrow \Omega[\mathcal{A}]$ always preserves meets. It preserves joins whenever they exist in $\Sigma[\mathcal{A}]$. This makes the notations $F \vee G$ and $F \wedge G$ unambiguous. Note very carefully that $F \vee G$ is in general only a cone (and not a face).

Recall the lattice of flats $\Pi[\mathcal{A}]$. Every flat is a cone. This yields a map $\Pi[\mathcal{A}] \hookrightarrow \Omega[\mathcal{A}]$. This is a lattice homomorphism. This makes the notations $X \vee Y$ and $X \wedge Y$ unambiguous. We point out that $\Pi[\mathcal{A}]$ is *not* a convex subposet of $\Omega[\mathcal{A}]$ in

CHAPTER 3

Lunes

We now study an important family of cones known as lunes. Faces, flats and half-flats are examples of lunes. Geometrically, lunes are cones which cannot be cut along their base. They can be identified with faces of arrangements over their base. A nested face is a pair of faces one contained in the other. There is a support map from nested faces to lunes which parallels the support map from faces to flats. This allows us to study lunes using the Tits product. Lunes are gallery intervals; in fact, they can be realized as gallery intervals in multiple ways. Lunes also serve as building blocks of cones. More precisely, any cone can be decomposed into lunes by cutting it along a fixed flat contained in its base. In particular, cutting along its base yields the optimal decomposition.

We also continue the discussion on cones. Cones offer the flexibility for interesting local operations, which we call restriction and extension. There is also a notion of conjugate top-cones; an important example of conjugate pairs is provided by top-stars and top-lunes. We also introduce top-star-lunes which are top-cones constructed inductively by using the extension operation. Both top-stars and top-lunes are examples of top-star-lunes. We discuss in detail the compatibility of cones, gallery intervals and lunes with the cartesian product of arrangements.

3.1. Lunes

We begin by defining lunes geometrically as cones which cannot be cut along their base, and then relate them to faces of arrangements over their base.

3.1.1. Lunes. Recall that a cone is a subset of the ambient space obtained by intersecting some half-spaces. The base of a cone V , denoted $b(V)$, is the largest flat contained in that cone. For a hyperplane H , let H^+ and H^- denote its two associated half-spaces. A *lune* is a cone V with the following property.

(3.1) If a hyperplane H contains $b(V)$,

then either H^+ contains V or H^- contains V .

In other words, for a cone to be a lune, a hyperplane containing the base of the cone is not allowed to cut the cone. In (3.1), it is possible that both H^+ and H^- contain V in which case $H = H^+ \cap H^-$ contains V .

Since lunes are cones, they have a *base* and a *case*. A *top-lune* is a lune whose top-dimensional faces are chambers, or equivalently, whose case is the maximum flat. In other words, a top-lune is a lune which is a top-cone.

Just as with cones, one can take a combinatorial approach to lunes. A *combinatorial lune* is a subset of the set of faces consisting of precisely the top-dimensional faces of some lune.

Category of lunes

We study two interesting partial orders on lunes. The first one is the restriction of the partial order on cones, and thus is defined by inclusion of lune closures. The second partial order is defined by inclusion of lune interiors. Both partial orders are graded. In the first case, the rank of a lune is the sum of the ranks of its base and its case, while in the second, it is just the rank of its base. The poset of top-lunes is an upper set in either partial order.

Lunes can be composed when the case of the first lune equals the base of the second lune. This yields the category of lunes whose objects are flats and morphisms are lunes. Further, this category is internal to posets under the second partial order on lunes. Also it has a nice presentation with generators being lunes of slack 1 (half-flats) subject to quadratic relations involving lunes of slack 2. In addition, the Birkhoff monoid acts on the category of lunes.

Recall that a lune is the same as a chamber in the arrangement over its base and under its case. Thus, composition of lunes is equivalent to an operation on chambers in arrangements over and under flats. We call this the substitution product of chambers. Using the same idea, one can also multiply chambers and faces, and top-lunes and chambers.

We consider the categories associated to the poset of faces and to the poset of flats. Since these posets are strongly connected, both categories have nice presentations. We also relate them to the category of lunes by functors which are internal to posets.

These ideas are further developed in Chapter 15.

4.1. Poset of top-lunes

Let us begin with the set of top-lunes $\widehat{\Lambda}[\mathcal{A}]$. It is a poset under inclusion, that is, $L \leq M$ iff M contains L (as subsets of the ambient space). The poset of top-lunes has a maximum element, namely, the ambient space. Each chamber is a minimal element. In combinatorial terms,

$$(4.1) \quad L \leq M \iff \text{Cl}(L) \subseteq \text{Cl}(M).$$

By definition, $\widehat{\Lambda}[\mathcal{A}]$ is a subposet of the poset of top-cones $\widehat{\Omega}[\mathcal{A}]$. Recall from Section 2.7 that the poset of top-cones is graded. We now proceed to show that the same is true for the poset of top-lunes. The strategy remains the same, namely, to find an order-preserving map to a graded poset which preserves cover relations. In the case at hand, this will be accomplished by the base map

$$(4.2) \quad \text{b} : \widehat{\Lambda}[\mathcal{A}] \rightarrow \Pi[\mathcal{A}]$$

obtained by restricting (2.9).

Reflection arrangements

We review reflection arrangements. Roughly speaking, these are hyperplane arrangements equipped with reflection symmetries. The group generated by these symmetries is the Coxeter group of the arrangement. In addition to everything that comes with an arrangement, these symmetries allow us to define concepts such as face-types, flat-types, nested face-types and lune-types. These are orbits of faces, flats, nested faces and lunes under the Coxeter group action and display similar inter-relationships. Another interesting concept is that of the cycle-type function. One can also construct new objects like the Coxeter-Tits monoid by taking the semidirect product of the Coxeter group and Tits monoid.

Among reflection arrangements, there is a further subclass of good reflection arrangements which is closed under passage to arrangements over and under a flat. We recall the classification of reflection arrangements, and then list out those which are good.

5.1. Coxeter groups and reflection arrangements

5.1.1. Reflections. Let V be a finite-dimensional vector space over \mathbb{R} equipped with an inner product. For any hyperplane H passing through the origin, reflection in H defines an orthogonal transformation of V . It fixes H pointwise, and sends any point on the line through the origin orthogonal to H to its negative. Let us denote this transformation by s_H .

A *Coxeter group* W on V is a finite group of orthogonal transformations of V generated by reflections in some finite set of hyperplanes through the origin. (The condition that the group generated by reflections be finite is very nontrivial. For instance, the group generated by reflections in two lines in \mathbb{R}^2 passing through the origin is finite iff the angle between them is a rational multiple of π .) For a Coxeter group W , the set of hyperplanes H such that $s_H \in W$ is the *reflection arrangement* associated with W . This arrangement is central but not necessarily essential. Its ambient space is V .

Above we started with the group and constructed the arrangement from it. This procedure can also be reversed: We say that a hyperplane arrangement $\mathcal{A} = \{H_i\}_{i \in I}$ with ambient space V is a *reflection arrangement* if for each i , the reflection s_{H_i} preserves \mathcal{A} . For a reflection arrangement \mathcal{A} , the group generated by the reflections s_{H_i} is called the *Coxeter group* of \mathcal{A} . (One can show that the Coxeter group does not have any more reflections than what we started with.)

5.1.2. Coxeter complex. A reflection arrangement \mathcal{A} is necessarily simplicial. We will refer to elements of its Coxeter group as the Coxeter symmetries of \mathcal{A} . For a Coxeter group W , the regular cell complex Σ associated to its reflection arrangement is the *Coxeter complex* of W . It is a pure simplicial complex. Moreover, it is

Braid arrangement and related examples

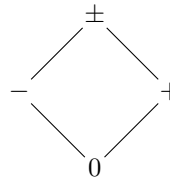
We discuss some important examples of arrangements. The coordinate arrangement is treated first. It is the simplest example. We then focus on the braid arrangement. This is the reflection arrangement of type A and is a main example. Subsequently we treat the reflection arrangements of types B and D , more briefly. Finally we discuss graphic arrangements. They are associated to simple graphs and are the subarrangements of the braid arrangement. In all these examples, there is a rich interplay between geometry and combinatorics.

We employ the notation $[n] := \{1, 2, \dots, n\}$.

6.1. Coordinate arrangement

The coordinate arrangement of rank n is the n -fold cartesian product of the arrangement of rank 1. We make explicit the notions of faces, flats, cones, lunes and so on for this arrangement, as summarized below.

Σ	$\{0, +, -\}^n$
Γ	$\{+, -\}^n$
$\Pi = G$	$\{0, \pm\}^n$
$\Omega = \Lambda = \vec{G}$	$\{0, +, -, \pm\}^n$
$\widehat{\Omega} = \widehat{\Lambda}$	$\{+, -, \pm\}^n$
W	\mathbb{Z}_2^n



In this section, \mathcal{A} denotes the coordinate arrangement of rank n , and W denotes its Coxeter group.

6.1.1. Coordinate arrangement. The *coordinate arrangement* of rank n consists of the n hyperplanes

$$x_i = 0$$

for $1 \leq i \leq n$. It is the smallest arrangement of rank n in terms of number of hyperplanes. It is the n -fold cartesian product of the arrangement of rank 1. It is a reflection arrangement. Its Coxeter group is \mathbb{Z}_2^n , the product of n copies of \mathbb{Z}_2 . The generator of the i -th copy of \mathbb{Z}_2 acts on \mathbb{R}^n by changing the i -th coordinate to its negative.

Consider the vector space \mathbb{R}^I consisting of functions from I to \mathbb{R} , with pointwise addition and scalar multiplication. It is useful to consider the coordinate arrangement in \mathbb{R}^I , where I is a finite set. It consists of the hyperplanes $x_a = 0$, with a varying over elements of I . It is isomorphic to the coordinate arrangement of rank n , where $n = |I|$.

Descent and lune equations

In any monoid, one may consider the equation $xy = a$, where a is fixed and either x or y or both are varying. We consider such equations for the Tits monoid. The main observation is that their solution sets (which consist of faces) are related to either topological balls or spheres. Hence the expression

$$\sum (-1)^{\text{rk}(F)}$$

with F ranging over the solution set (which is an Euler characteristic), can be computed. This leads to numerous identities which are called descent identities, lune identities and the Witt identities.

More generally, one may consider such equations in the context of (left or right) modules over the Tits monoid. We attach a relative pair (X, A) of cell complexes to the solution set. The notation is meant to remind us of relative pairs in algebraic topology. By construction X is either a ball or sphere, but the topology of A can be complicated. We give examples where A is a ball or sphere or more generally a wedge of spheres. Since the Euler characteristic is known in these cases, we again obtain explicit identities. Left actions give descent identities while right actions give lune identities.

We also apply these considerations to the Birkhoff monoid. Since it is commutative, there is no distinction between left and right modules. So in this case we obtain descent-lune identities. We illustrate this for the example of charts.

Background information on cell complexes and Euler characteristics is given in Section A.1.

7.1. Descent equation

We introduce the descent equation. We study it first for chambers and then for faces. In the simplicial case, its solution set involves the notion of descent between chambers, and more generally, between faces. We discuss identities involving the Euler characteristic of the solution set.

7.1.1. Descent equation. Consider the equation $HC = D$, where C and D are given chambers and we want to find H . We call this the *descent equation for chambers*. For faces, there are two situations one may look at, namely, $HF = G$ and $HF \leq G$, where F and G are arbitrary fixed faces. We refer to either of these as the *descent equation for faces*.

Simplicial case. Let us first assume that \mathcal{A} is a simplicial arrangement.

Proposition 7.1. *Let C and D be any chambers in a simplicial arrangement. If $H_1C = D$ and $H_2C = D$, then $(H_1 \wedge H_2)C = D$.*

Distance functions and Varchenko matrix

We introduce an abstract notion of distance function on chambers of an arrangement. The motivating example arises by assigning a weight to each half-space, and letting the distance between C and D to be the product of weights of all half-spaces that one has to move out of while going from C to D . An important special case is when all half-spaces have the same weight, say q , in which case, the distance between C and D is q power the number of hyperplanes separating C and D . (Recall that the latter is the gallery distance between C and D .)

A distance function gives rise to a matrix indexed by chambers whose entry in position (C, D) is the distance between C and D . This is the Varchenko matrix. For distance functions arising from weight functions on half-spaces, the determinant of this matrix has a nice factorization. The same is true, more generally, for the Varchenko matrix indexed by chambers of any top-cone.

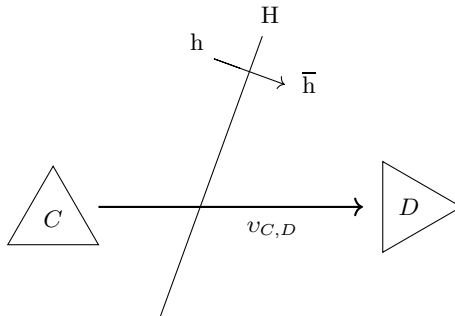
8.1. Weights on half-spaces

Let \mathcal{A} be an arrangement. We begin with distance functions on \mathcal{A} which arise from weight functions on its half-spaces. This material builds on the discussion on separating hyperplanes, minimal galleries and their basic properties given in Section 1.10. An abstract approach to distance functions is given in Section 8.3.

8.1.1. Distance function on chambers. A *weight function* assigns a number (weight) to each half-space of \mathcal{A} . We write $\text{wt}(h)$ for the weight on the half-space h . Given a weight function, for any chambers C and D , let

$$(8.1) \quad v_{C,D} := \prod_{h \in r(C,D)} \text{wt}(h),$$

where recall that $r(C, D)$ consists of half-spaces h which contain C but do not contain D . This defines a function v on the set of pairs of chambers. We call it a (multiplicative) distance function.



Birkhoff algebra and Tits algebra

We study the Birkhoff algebra and the Tits algebra. They are obtained by linearizing the Birkhoff monoid of flats and the Tits monoid of faces, respectively. Since the poset of flats is a lattice, the Birkhoff algebra is a split-semisimple commutative algebra. Thus, its simple modules are one-dimensional, and there is one for each flat. Further, any module is a direct sum of simple modules. The Tits algebra is a non-commutative elementary algebra. Its split-semisimple quotient is precisely the Birkhoff algebra (under the support map). Thus, its simple modules are also one-dimensional and indexed by flats. However, an arbitrary module (for instance, the module of chambers) is not a direct sum of simple modules. A consequence of knowing the simple modules is that one can compute the eigenvalues and multiplicities of the action of any given element of the Tits algebra on any module. For the module of chambers, this yields the Bidigare-Hanlon-Rockmore (BHR) theorem.

A general framework for studying the Birkhoff algebra is given by algebras obtained by linearizing lattices. We give three more examples of this nature, namely, the algebra of charts, the algebra of dicharts and the algebra of cones. These are obtained by linearizing the lattices of charts, dicharts and cones, respectively. Further, the four algebras relate to one another by linearizations of join-preserving maps. In the same vein, a general framework for studying the Tits algebra would be a “noncommutative lattice” such as a left regular band, but we do not pursue this idea in detail.

Modules over the Birkhoff algebra and Tits algebra have a primitive part. For instance, the algebra of charts can be viewed as a module over the Birkhoff algebra, and its primitive part is spanned by connected charts. The primitive part of the algebra of cones is spanned by cones whose base is the minimum flat.

The Janus algebra is obtained by linearizing the Janus monoid of bi-faces. Just like the Tits algebra, it is elementary and its split-semisimple quotient is the Birkhoff algebra. It admits a deformation by a parameter q which we call the q -Janus algebra. When q is not a root of unity, this algebra is split-semisimple and Morita equivalent to the Birkhoff algebra.

Background material on algebras and modules is given in Appendix D. This includes split-semisimple commutative algebras, simple modules, characters, complete systems of idempotents, radical of an algebra, diagonalizable elements and elementary algebras. Algebras obtained by linearizing lattices are treated in Section D.9.

We will continue to follow Convention 1.1. For instance, $\Sigma[\mathcal{A}]$ denotes the Tits algebra of \mathcal{A} and $\Pi[\mathcal{A}]$ denotes the Birkhoff algebra of \mathcal{A} . When \mathcal{A} is understood from the context, we may simply write Σ , Π , and so on.

Lie and Zie elements

We introduce Lie elements. Recall that left modules over the Tits algebra have a primitive part. Lie elements are the primitive part of the left module of chambers. These elements can also be expressed as solutions to a linear system of equations whose variables are chambers. The set of chambers involved in any given equation form a top-lune. We pay special attention to Lie elements of rank-one and rank-two arrangements; the antisymmetry relation appears in rank-one and the Jacobi identity in rank-two arrangements.

We also introduce Zie elements. They are the primitive part of the Tits algebra viewed as a left module over itself. They can also be expressed as solutions to a linear system of equations whose variables are faces. In fact, we consider two such linear systems. The set of faces involved in any given equation form either the interior or the closure of a lune. A Zie element is special if the central face appears in it with coefficient 1. A special Zie element projects any left module over the Tits algebra onto its primitive part. Using this principle, we derive formulas for the dimensions of the spaces of Lie elements and Zie elements. Both formulas involve the Möbius function of the lattice of flats.

Lie elements carry a substitution product which specifies a way to multiply Lie elements in arrangements under a flat with Lie elements in arrangements over a flat. It is obtained by restricting the substitution product of chambers. Similarly, Lie can be multiplied with Zie on the right and with chambers on the left. This is the restriction of the substitution product of chambers with faces on the right and with top-lunes on the left.

10.1. Lie elements

We introduce Lie elements as solutions to a linear system of equations whose variables are chambers. We then discuss various characterizations for it named after Friedrichs, Ree and Garsia. They involve the notions of primitive part, top-lunes and descents, respectively. The Garsia criterion is also intimately connected to the Witt identity.

10.1.1. Lie elements. Recall the left module of chambers $\Gamma[\mathcal{A}]$. We write a typical element as

$$z = \sum_C x^C \mathbf{H}_C.$$

An element $z \in \Gamma[\mathcal{A}]$ is a *Lie element* if

$$(10.1) \quad \sum_{C: HC=D} x^C = 0 \text{ for all } O < H \leq D.$$

This is a linear system in the variables x^C .

Eulerian idempotents

We saw that the Tits algebra is elementary and its split-semisimple quotient is the Birkhoff algebra. The quotient map is the support map. Let us refer to a complete system of primitive orthogonal idempotents of the Tits algebra as an Eulerian family. (In the exposition, an Eulerian family is defined slightly differently and later shown to be equivalent to a complete system.) Any Eulerian family lifts the primitive idempotents of the Birkhoff algebra. There are two theoretically significant methods to construct and characterize Eulerian families. The first method starts with a homogeneous section of the support map. This is the Saliola construction. The second method starts with a family of special Zie elements. Each Eulerian family gives rise to a \mathbb{Q} -basis of the Tits algebra. This is in contrast to the Birkhoff algebra which has a unique complete system which also serves as the unique \mathbb{Q} -basis. These ideas are further developed in Chapter 15 through consideration of the lune-incidence algebra with a summary given in Section 15.5.

The Saliola construction is recursive in nature and involves alternating sums. Hence it is nontrivial to write down closed formulas for the Eulerian idempotents in general. For a good reflection arrangement, we give cancellation-free formulas for the Eulerian idempotents associated to the uniform section.

As an application, we discuss the extension problem for chambers. Any chamber element of \mathcal{A} induces a chamber element of \mathcal{A}_F by Tits projection on the face F . The extension problem is to start with chamber elements of \mathcal{A}_F for each noncentral face $F > O$ which are “mutually compatible”, and construct chamber elements of \mathcal{A} whose Tits projections are these given elements. We show that the solution space is a translate of the space of Lie elements.

11.1. Homogeneous sections of the support map

Fix an arrangement \mathcal{A} . We define a class of (linear) sections of the support map of \mathcal{A} . We call them homogeneous sections.

11.1.1. Homogeneous sections. Recall the Birkhoff algebra $\Pi[\mathcal{A}]$, the Tits algebra $\Sigma[\mathcal{A}]$ and the support map (9.31) relating them. Let

$$\mathbf{u} : \Pi[\mathcal{A}] \rightarrow \Sigma[\mathcal{A}]$$

be any section of the support map. (The section is only required to be a linear map, not an algebra map.) For each flat X , let $\mathbf{u}_X := \mathbf{u}(\mathbb{H}_X)$ denote the value of \mathbf{u} on \mathbb{H}_X . Thus

$$(11.1) \quad s(\mathbf{u}_X) = \mathbb{H}_X.$$

Diagonalizability and characteristic elements

For any element of the Tits algebra, the sum of all its coefficients is an eigenvalue. When the element satisfies an additional hypothesis which we call the top-separating condition, this eigenvalue has a unique eigenvector (up to scaling) in the left module of chambers. There is an explicit formula for this eigenvector which we call the Brown-Diaconis formula. While working over the reals, the eigenvector can be interpreted as the stationary distribution of a random walk associated to the given element. From this point of view, elements with nonnegative coefficients are of interest. We also discuss the Billera-Brown-Diaconis formula which treats a special case in rank-three arrangements.

Next, we consider the general problem of diagonalizability of elements of the Tits algebra. An important sufficient condition for diagonalizability is that the element be separating. The separating condition is similar to but stronger than the top-separating condition. The key step in the proof is to determine the homogeneous section, called the eigensection, whose associated Eulerian family will diagonalize the given element. The eigensection can be built by a separate computation in each flat, and in each case, one can employ the Brown-Diaconis formula. This leads to the Brown formulas for the eigensection of a separating element, and also for the associated Eulerian family.

We study in detail the Takeuchi element. It is defined via an “alternating” sum of faces (similar to the Euler characteristic). Its square is the unit of the Tits algebra. The Takeuchi element is projective and commutes with every projective element. It is neither separating nor does it have nonnegative coefficients, yet it is diagonalizable. Its eigenvalues are ± 1 , and any projective section is an eigensection. Further, it acts on the left module of chambers by sending a chamber to its opposite (up to sign). We also consider a more general class of elements called the Fulman elements. For a good reflection arrangement, the uniform section is an eigensection for the Fulman elements which then leads to an explicit diagonalization (using the cancellation-free formulas for the corresponding Eulerian idempotents). We specialize to the braid arrangement and discuss in particular the Adams elements. We also give a similar discussion for the arrangement of type B .

Takeuchi and Fulman elements are examples of characteristic elements. The eigenvalues of such elements are powers of a fixed parameter. This interesting class of elements is intimately connected to the characteristic polynomial of the arrangement.

12.1. Stationary distribution

Let w be an element of the Tits algebra. Theorem 9.44 gives the eigenvalues of w for the action on any module h . We would now like to work towards finding the eigenvectors of w . It is particularly important to understand the eigenvectors of λ_{\top}

Loewy series and Peirce decompositions

Recall from Section D.5 that for any module over an algebra, one can define its radical series and socle series with the former contained termwise in the latter. These are two extreme examples of Loewy series. For left modules over the Tits algebra, we introduce a third series called the primitive series. It is a Loewy series and hence trapped between the radical series and the socle series. The first nontrivial term (from the bottom) in the primitive series is the primitive part of the module. For the left module of chambers, all three series coincide. Dually, for right modules, we introduce the decomposable series which is also a Loewy series. The first nontrivial term (from the top) in the decomposable series is the decomposable part of the module. For the right module of Zie elements, the radical, decomposable and socle series all coincide.

Recall that decompositions arising from a system of orthogonal idempotents are called Peirce decompositions (left, right, two-sided). Any Eulerian family yields a left Peirce decomposition of a left module over the Tits algebra with components indexed by flats. We provide formulas for the dimensions of the components and relate them to terms in the primitive series. For instance, the component for the minimum flat coincides with the primitive part of the module. For the left module of chambers, the component indexed by a flat identifies with the space of Lie elements in the arrangement over that flat. This can be viewed as an algebraic form of the Zaslavsky formula. Similarly, for the Tits algebra viewed as a left module over itself, the component indexed by a flat identifies with the space of Zie elements in the arrangement over that flat. There are similar results for the right Peirce decompositions of right modules. For the right module of Zie elements, the component indexed by a flat identifies with the space of Lie elements in the arrangement under that flat. Similarly, for the Tits algebra viewed as a right module over itself, the component indexed by a flat identifies with the space of chamber elements in the arrangement under that flat.

Since the Tits algebra is a bimodule over itself, any Eulerian family yields a two-sided Peirce decomposition (obtained by combining the left and right Peirce decompositions). The components are indexed by nested flats. A typical component identifies with the space of Lie elements in the arrangement over the first flat and under the second flat (taken from the nested flat). Further, this identification is compatible with the substitution product of Lie. This can be used to describe the powers of the radical of the Tits algebra and also compute its quiver.

All modules are assumed to be finite dimensional as per Convention 9.1.

Notation 13.1. In this chapter, r denotes the rank of the arrangement \mathcal{A} .

Dynkin idempotents

We discuss a construction of special Zie elements of an arrangement \mathcal{A} . The starting data is a generic half-space h . The Zie element is defined as an alternating sum of faces contained in h . We call this the Dynkin element associated to h . Its action on chambers gives rise to a basis of the space of Lie elements which we call the Dynkin basis. This yields another proof of the fact that the dimension of the space of Lie elements is the absolute value of the Möbius number of \mathcal{A} . We establish certain key properties of Dynkin elements and present two applications to the theory of finite affine hyperplane arrangements. One is a derivation of the Zaslavsky formula for the number of bounded chambers of an affine arrangement. The other is a more surprising result: that the semigroup algebra of such an arrangement possesses a unit element.

We discuss the notion of orientation of an arrangement in terms of maximal chains in its poset of faces. We then prove the Joyal-Klyachko-Stanley (JKS) theorem which identifies, up to orientation, the top-cohomology of the lattice of flats with the space of Lie elements. In effect, it says that the space of Lie elements is freely generated by the orientation space in rank one subject to the Jacobi identities in rank two. (There is also an analogue of the JKS theorem which relates the top-cohomology of the poset of faces with the space of chambers.) The dual of the Dynkin basis is, up to orientation, a basis for the top-homology of the lattice of flats. This is the Björner-Wachs basis. We also discuss another pair of dual bases, the Björner basis for top-homology and the Lyndon basis for top-cohomology. The starting data for this is an ordered coordinate chart. A summary is given below.

Starting data	Top-homology	Lie
Generic half-space	Björner-Wachs basis	Dynkin basis
Ordered coordinate chart	Björner basis	Lyndon basis

We discuss examples in the later sections of the chapter. We begin with rank-two and coordinate arrangements. We then treat the important case of the braid arrangement, which motivates most of our terminology. Contact with the classical Lie bracket, antisymmetry, Jacobi identity, Lie operad, binary trees, the classical JKS theorem, the Dynkin-Specht-Wever theorem and the Lyndon basis is made here. A similar discussion is given for the arrangement of type B .

14.1. Dynkin elements

A Dynkin element is a special Zie element constructed from a generic half-space.

14.1.1. Dynkin element. Generic half-spaces are discussed in Section 1.9.1. Let h be a generic half-space wrt \mathcal{A} , and let H denote its bounding hyperplane. Now

Incidence algebras

Incidence algebras are reviewed in Section C.1. We begin with the flat-incidence algebra. It is the incidence algebra of the poset of flats. It is related to the Birkhoff algebra in the sense that the zeta function and the Möbius function intervene in the change of basis formulas between the \mathbb{H} - and \mathbb{Q} -bases.

Next, we introduce the lune-incidence algebra, which is a certain reduced incidence algebra of the poset of faces. It can also be viewed as the incidence algebra of the category of lunes. In particular, it has a basis indexed by lunes. In this algebra, one can define noncommutative zeta functions and noncommutative Möbius functions, which are inverse to each other. This algebra connects to the Tits algebra. Recall that homogeneous sections and special Zie families can be used to construct and characterize Eulerian families. The punchline is that homogeneous sections correspond to noncommutative zeta functions, while special Zie families correspond to noncommutative Möbius functions. In effect, noncommutative zeta and Möbius functions intervene in the change of basis formulas between the \mathbb{H} - and \mathbb{Q} -bases of the Tits algebra.

There are three important subspaces of the lune-incidence algebra that we consider, namely, the Lie-incidence algebra, the space of additive functions and the space of Weisner functions. All three subspaces have the same dimension given by the number of faces in the arrangement. The Lie-incidence algebra is a subalgebra of the lune-incidence algebra. Its elements can be identified with Lie elements in arrangements over and under various flats. In fact, it is isomorphic to the Tits algebra. The space of additive functions contains the space of noncommutative zeta functions. It is a right module over the Lie-incidence algebra with action induced from the product in the lune-incidence algebra. In fact, it is isomorphic to the right regular representation of the Lie-incidence algebra. Similarly, the space of Weisner functions contains the space of noncommutative Möbius functions. It is a left module over the Lie-incidence algebra isomorphic to the left regular representation.

There are similarities between the flat-incidence algebra, lune-incidence algebra and Lie-incidence algebra. All three algebras are elementary, with the Birkhoff algebra as their split-semisimple quotient. Their quivers have flats as their vertices, and arrows can go only from a flat to a smaller flat which it covers.

An encompassing picture for these observations involves a generalization of the classical notion of operad.

15.1. Flat-incidence algebra

The flat-incidence algebra is the incidence algebra of the poset of flats. We discuss this briefly.

Invariant Birkhoff algebra and invariant Tits algebra

We studied the Birkhoff algebra and Tits algebra of an arrangement in detail. Recall that a reflection arrangement is acted upon by its Coxeter group. Hence, in this situation, it makes sense to consider the invariant part of these algebras. We call these the invariant Birkhoff algebra and invariant Tits algebra. The former is a split-semisimple commutative algebra whose primitive idempotents are indexed by flat-types. The latter is elementary and has a basis indexed by face-types. Its radical is the invariant part of the radical of the Tits algebra, and the quotient by this is the invariant Birkhoff algebra. Thus, the simple modules over the invariant Tits algebra are one-dimensional and indexed by flat-types. Complete systems of primitive orthogonal idempotents for the invariant Tits algebra are the same as invariant Eulerian families. They correspond to invariant sections and to invariant special Zie families, and similarly to invariant noncommutative zeta and Möbius functions. The latter belong to the invariant lune-incidence algebra. For a good reflection arrangement, for the uniform section, there are cancellation-free formulas for the Eulerian idempotents. The two-sided Peirce decomposition of the invariant Tits algebra can be used to shed light on its quiver. This necessitates the study of invariant Lie and Zie elements.

Recall that there is an injective map from the Tits algebra to the space indexed by pairs of chambers. Taking invariants induces an injective map from the invariant Tits algebra to the Coxeter group algebra. The image of this map is a subalgebra of the Coxeter group algebra which is known as the Solomon descent algebra. This induces an anti-isomorphism of algebras between the invariant Tits algebra and the Solomon descent algebra. This result makes it possible to study the Solomon descent algebra using the invariant Tits algebra.

The structure constants of the invariant Tits algebra are of great theoretical significance. They intervene in the invariant formulation of lune-additivity and the noncommutative Weisner formula. They are also intimately connected to enumeration of face-types.

We illustrate some of the above ideas for the braid arrangement. This makes contact with the Garsia-Reutenauer idempotents and the Bayer-Diaconis-Garsia-Loday formula. We also briefly discuss the arrangement of type B .

Notation 16.1. For a reflection arrangement \mathcal{A} , for any face-types $T \leq U$, we let \mathcal{A}_T^U refer to any arrangement \mathcal{A}_H^G , with $t(H) = T$ and $t(G) = U$. Similarly, we let c_T^U denote the number of chambers in \mathcal{A}_T^U . In particular, c^T is the number of chambers in \mathcal{A}^T .