Introduction

Bounded cohomology of groups was first defined by Johnson [**Joh72**] and Trauber during the seventies in the context of Banach algebras. As an independent and very active research field, however, bounded cohomology started to develop in 1982, thanks to the pioneering paper "Volume and Bounded Cohomology" by M. Gromov [**Gro82**], where the definition of bounded cohomology was extended to deal also with topological spaces.

Let $C^\bullet(M, \mathbb{R})$ denote the complex of real singular cochains with values in the topological space M. A cochain $\varphi \in C^n(M, \mathbb{R})$ is *bounded* if it takes uniformly bounded values on the set of singular n-simplices. Bounded cochains provide a subcomplex $C_b^\bullet(M, \mathbb{R})$ of singular cochains, and the bounded cohomology $H_b^\bullet(M, \mathbb{R})$ of M (with trivial real coefficients) is just the (co)homology of the complex $C_b^\bullet(M, \mathbb{R})$. An analogous definition of boundedness applies to group cochains with (trivial) real coefficients, and the bounded cohomology $H_b^\bullet(\Gamma, \mathbb{R})$ of a group Γ (with trivial real coefficients) is the (co)homology of the complex $C_b^\bullet(\Gamma, \mathbb{R})$ of the bounded cochains on Γ. A classical result which dates back to the forties ensures that the singular cohomology of an aspherical CW-complex is canonically isomorphic to the cohomology of its fundamental group. In the context of bounded cohomology a stronger result holds: the bounded cohomology of a topological space is canonically isomorphic to the bounded cohomology of its fundamental group, even without any assumption on the asphericity of the space [**Gro82, Bro81, Iva87, FM11, Iva**]. For example, the bounded cohomology of spheres is trivial in positive degree. On the other hand, the bounded cohomology of the wedge of two circles is infinite-dimensional in degrees 2 and 3, and still unknown in any degree bigger than 3. As we will see in this monograph, this phenomenon eventually depends on the fact that higher homotopy groups are abelian, and abelian groups are invisible to bounded cohomology, while "negatively curved" groups, such as non-abelian free groups, tend to have very big bounded cohomology modules.

The bounded cohomology of a group Γ can be defined with coefficients in any normed (e.g. Banach) Γ-module, where the norm is needed to make sense of the notion of boundedness of cochains. Moreover, if Γ is a topological group, then one may restrict to considering only *continuous* bounded cochains, thus obtaining the *continuous* bounded cohomology of Γ. In this monograph, we will often consider arbitrary normed Γ-modules, but we will restrict our attention to bounded cohomology of discrete groups. The reason for this choice is twofold. First, the (very powerful) theory of continuous bounded cohomology is based on a quite sophisticated machinery, which is not needed in the case of discrete groups. Secondly, Monod's book [**Mon01**] and Burger-Monod's paper [**BM02**] already provide an excellent introduction to continuous bounded cohomology, while to the author's knowledge no reference is available where the fundamental properties of bounded cohomology of discrete groups are collected and proved in detail. However, we should emphasize that the theory of continuous bounded cohomology is essential in proving important results also in the context of discrete groups: many vanishing theorems for the bounded cohomology of lattices in Lie groups can be obtained by comparing the bounded cohomology of the lattice with the continuous bounded cohomology of the ambient group.

This monograph is devoted to provide a self-contained introduction to bounded cohomology of discrete groups and topological spaces. Several (by now classical) applications of the theory will be described in detail, while many others will be completely omitted. Of course, the choice of the topics discussed here is largely arbitrary, and based on the taste (and on the knowledge) of the author.

Before describing the content of each chapter, let us provide a brief overview on the relationship between bounded cohomology and other research fields.

Geometric group theory and quasification. The bounded cohomology of a closed manifold is strictly related to the curvature of the metrics that the manifold can support. For example, if the closed manifold M is flat or positively curved, then the fundamental group of M is amenable, and $H_b^n(M,\mathbb{R}) = H_b^n(\pi_1(M),\mathbb{R}) = 0$ for every $n \geq 1$. On the other hand, if M is negatively curved, then it is well-known that the comparison map $H_b^\bullet(M,\mathbb{R}) \to H^\bullet(M,\mathbb{R})$ induced by the inclusion $C_b^\bullet(M,\mathbb{R}) \to C^\bullet(M,\mathbb{R})$ is surjective in every degree bigger than one.

In fact, it turns out that the surjectivity of the comparison map is related in a very clean way to the notion of Gromov hyperbolicity, which represents the coarse geometric version of negative curvature. Namely, a group Γ is Gromov hyperbolic if and only if the comparison map $H_b^n(\Gamma,V) \to H^n(\Gamma,V)$ is surjective for every $n \geq 2$ and for every Banach Γ-module V [**Min01, Min02**].

Coarse geometry comes into play also when studying the (non-)injectivity of the comparison map. In fact, let $EH_b^n(\Gamma,V)$ denote the kernel of the comparison map in degree n. It follows by the very definitions that an element of $H_b^n(\Gamma,V)$ lies in $EH_b^n(\Gamma,V)$ if and only if any of its representatives is the coboundary of a (possibly unbounded) cocycle. More precisely, a cochain is usually called a *quasi-cocycle* if its differential is bounded, and a quasi-cocycle is *trivial* if it is the sum of a cocycle and a bounded cochain. Then $EH_b^n(\Gamma,V)$ is canonically isomorphic to the space of $(n-1)$-quasi-cocycles modulo trivial $(n-1)$-quasi-cocycles. When $V=\mathbb{R}$, quasi-cocycles of degree one are usually called *quasimorphisms*. There exists a large literature which concerns the construction of non-trivial quasimorphisms in presence of (weaker and weaker versions of) negative curvature. Brooks [**Bro81**] first constructed infinitely many quasimorphisms on the free group with two generators F_2, that were shown to define linearly indepenedent elements in $EH^2(F_2,\mathbb{R})$ by Mitsumatsu [**Mit84**]. In particular, this proved that $EH_b^2(F_2,\mathbb{R})$ (which coincides with $H_b^2(F_2,\mathbb{R})$) is infinite-dimensional.

Quasi-cocycles are cochains which satisfy the cocycle equation only up to a finite error, and geometric group theory provides tools that are particularly well-suited to study notions which involve finite errors in their definition. Therefore, it is not surprising that Brooks' and Mitsumatsu's result has been generalized to larger and larger classes of groups, which now include the class of non-elementary relatively hyperbolic groups, and most mapping class groups. We refer the reader to Section 2.9 for a more detailed account on this issue. It is maybe worth mentioning that, even if in the cases cited above $EH_b^2(\Gamma,\mathbb{R})$ is always infinite-dimensional, there exist lattices Γ in non-linear Lie groups for which $EH_b^2(\Gamma,\mathbb{R})$ is of finite non-zero dimension [**MR06**].

We have mentioned the fact that $H_b^2(\Gamma,\mathbb{R})$ is infinite-dimensional for negatively curved groups according to a suitable notion of negative curvature for groups. On the other hand, bounded cohomology vanishes for "positively curved" (i.e. finite) or "flat" (i.e. virtually abelian) groups. In fact, $H_b^n(\Gamma,\mathbb{R})=0$ for any $n\geq 1$ and any Γ belonging to the distinguished class of *amenable* groups. The class of amenable groups contains all virtually solvable groups, it is closed under quasi-isometries, and it admits a nice characterization in terms of bounded cohomology (see Section 3.4). These facts provide further evidence that bounded cohomology detects coarse non-positive curvature as well as coarse non-negative curvature.

Simplicial volume. The ℓ^∞-norm of an n-cochain is the supremum of the values it takes on single singular n-simplices (or on single $(n+1)$-tuples of elements of the group, when dealing with group cochains rather than with singular cochains). So a cochain φ is bounded if and only if it has a finite ℓ^∞-norm, and the ℓ^∞-norm induces a natural quotient ℓ^∞-seminorm on bounded cohomology. The ℓ^∞-norm on singular cochains arises as the dual of a natural ℓ^1-norm on singular *chains*. This ℓ^1-norm induces an ℓ^1-seminorm on homology. If M is a closed oriented manifold, then the *simplicial volume* $\|M\|$ of M is the ℓ^1-seminorm of the real fundamental class of M [**Gro82**]. Even if it depends only on the homotopy type of the manifold, the simplicial volume is deeply related to the geometric structures that a manifold can carry. As one of the main motivations for its definition, Gromov himself showed that the simplicial volume provides a lower bound for the *minimal volume* of a manifold, which is the infimum of the volumes of the Riemannian metrics that are supported by the manifold and that satisfy suitable curvature bounds.

An elementary duality result relates the simplicial volume of an n-dimensional manifold M to the bounded cohomology module $H_b^n(M,\mathbb{R})$. For example, $\|M\|=0$ provided that $H_b^n(M,\mathbb{R})=0$. In particular, the simplicial volume of simply connected manifolds (or, more in general, of manifolds with an amenable fundamental group) is vanishing. It is worth stressing that no homological proof of this statement is available: in many cases, the fact that $\|M\|=0$ cannot be proved by exhibiting fundamental cycles with arbitrarily small norm. Moreover, the exact value of non-vanishing simplicial volumes (of compact manifolds) is known only in the following very few cases: hyperbolic manifolds [**Gro82, Thu79**], some classes of 3-manifolds with boundary [**BFP15**], and the product of two surfaces [**BK08b**]. In the last case, it is not known to the author any description of a sequence of fundamental cycles whose ℓ^1-norms approximate the simplicial volume. In fact, Bucher's computation of the simplicial volume of the product of surfaces heavily relies on deep properties of bounded cohomology that have no counterpart in the context of singular homology.

Characteristic classes. A fundamental theorem by Gromov [**Gro82**] states that, if G is an algebraic subgroup of $\mathrm{GL}_n(\mathbb{R})$, then every characteristic class of flat G-bundles lies in the image of the comparison map (i.e. it can be represented by a bounded cocycle). (See [**Buc04**] for an alternative proof of a stronger result.) Several natural questions arise from this result. First of all, one may ask whether such characteristic classes admit a canonical representative in bounded cohomology: since the comparison map is often non-injective, this would produce more refined invariants. Even when one is not able to find natural bounded representatives for a characteristic class, the seminorm on bounded cohomology (which induces a seminorm on the image of the comparison map by taking the infimum over the bounded representatives) can be used to produce numerical invariants, or to provide useful estimates.

In this context, the most famous example is certainly represented by the Euler class. To every oriented circle bundle there is associated its Euler class, which arises as an obstruction to the existence of a section, and completely classifies the topological type of the bundle. When restricting to *flat* circle bundles, a *bounded* Euler class can be defined, which now depends on the flat structure of the bundle (and, in fact, classifies the isomorphism type of flat bundles with minimal holonomy [**Ghy87, Ghy01**]). Moreover, the seminorm of the bounded Euler class is equal to $1/2$. Now the Euler number of a circle bundle over a surface is obtained by evaluating the Euler class on the fundamental class of the surface. As a consequence, the Euler number of any flat circle bundle over a surface is bounded above by the product of $1/2$ times the simplicial volume of the surface. This yields the celebrated *Milnor-Wood inequalities* [**Mil58a, Woo71**], which provided the first explicit and easily computable obstructions for a circle bundle to admit a flat structure. Of course, Milnor's and Wood's original proofs did not explicitly use bounded cohomology, which was not defined yet. However, their arguments introduced ideas and techniques which are now fundamental in the theory of quasimorphisms, and they were implicitly based on the construction of a bounded representative for the Euler class.

These results also extend to higher dimensions. It was proved by Sullivan [**Sul76**] that the ℓ^∞-seminorm of the (simplicial) Euler class of an oriented flat vector bundle is bounded above by 1. A clever trick due to Smillie allowed to sharpen this bound from 1 to 2^{-n}, where n is the rank of the bundle. Then, Ivanov and Turaev [**IT82**] gave a different proof of Smillie's result, also constructing a natural bounded Euler class in every dimension. A very clean characterization of this bounded cohomology class, as well as the proof that its seminorm is equal to 2^{-n}, have recently been provided by Bucher and Monod in [**BM12**].

Actions on the circle. If X is a topological space with fundamental group Γ, then any orientation-preserving topological action of Γ on the circle gives rise to a flat circle bundle over X. Therefore, we can associate to every such action a bounded Euler class. It is a fundamental result of Ghys [**Ghy87, Ghy01**] that the bounded Euler class encodes the most essential features of the dynamics of an action. For example, an action admits a global fixed point if and only if its bounded Euler class vanishes. Furthermore, in the almost opposite case of *minimal* actions (i.e. of actions every orbit of which is dense), the bounded Euler class provides a complete conjugacy invariant: two minimal circle actions share the same bounded Euler class if and only if they are topologically conjugate [**Ghy87, Ghy01**]. These results establish a deep connection between bounded cohomology and a fundamental problem in one-dimensional dynamics.

Representations and Rigidity. Bounded cohomology has been very useful in proving rigidity results for representations. It is known that an epimorphism between discrete groups induces an injective map on 2-dimensional bounded cohomology with real coefficients (see Theorem 2.17). As a consequence, if $H_b^2(\Gamma, \mathbb{R})$ is finite-dimensional and $\rho\colon \Gamma \to G$ is any representation, then the second bounded cohomology of the image of ρ must also be finite-dimensional. In some cases, this information suffices to ensure that ρ is almost trivial. For example, if Γ is a uniform irreducible lattice in a higher rank semisimple Lie group, then by work of Burger and Monod we have that $H_b^2(\Gamma, \mathbb{R})$ is finite-dimensional [**BM99**]. On the contrary, non-virtually-abelian subgroups of mapping class groups of hyperbolic surfaces admit many non-trivial quasimorphisms [**BF02**], whence an infinite-dimensional second bounded cohomology group. As a consequence, the image of any representation of a higher rank lattice into a mapping class group is virtually abelian, whence finite. This provides an independent proof of a result by Farb, Kaimanovich and Masur [**FM98, KM96**].

Rigidity results of a different nature arise when exploiting bounded cohomology to get more refined invariants with respect to the ones provided by classical cohomology. For example, we have seen that the norm of the Euler class can be used to bound the Euler number of flat circle bundles. When considering representations into $PSL(2, \mathbb{R})$ of the fundamental group of closed surfaces of negative Euler characteristic, a celebrated result by Goldman [**Gol80**] implies that a representation has maximal Euler number if and only if it is faithful and discrete, i.e. if and only if it is the holonomy of a hyperbolic structure (in this case, one usually says that the representation is *geometric*). A new proof of this fact (in the more general case of surfaces with punctures) has been recently provided in [**BIW10**] via a clever use of the bounded Euler class of a representation (see also [**Ioz02**] for another proof of Goldman's Theorem based on bounded cohomology). In fact, one may define maximal representations of surface groups into a much more general class of Lie groups (see e.g. [**BIW10**]). This strategy has been followed e.g. in [**BI07, BI09**] to establish deformation rigidity for representations into $SU(m, 1)$ of lattices in $SU(n, 1)$ also in the non-uniform case (the uniform case having being settled by Goldman and Millson in [**GM87**]). Finally, bounded cohomology has proved very useful in studying rigidity phenomena also in the context of orbit equivalence and measure equivalence for countable groups (see e.g. [**MS06**] and [**BFS13**]).

Content of the book. Let us now briefly outline the content of each chapter. In the **first chapter** we introduce the basic definitions about the cohomology and the bounded cohomology of groups. We introduce the comparison map between the bounded cohomology and the classical cohomology of a group, and we briefly discuss the relationship between the classical cohomology of a group and the singular cohomology of its classifying space, postponing to Chapter 5 the investigation of the same topic in the context of bounded cohomology.

In **Chapter 2** we study the bounded cohomology of groups in low degrees. When working with trivial coefficients, bounded cohomology in degree 0 and 1 is completely understood, so our attention is mainly devoted to degree 2. We recall that degree-2 classical cohomology is related to group extensions, while degree-2 bounded cohomology is related to the existence of quasimorphisms. We exhibit many non-trivial quasimorphisms on the free group, thus showing that $H_b^2(F, \mathbb{R})$ is infinite-dimensional for every non-abelian free group F. We also describe a somewhat neglected result by Bouarich, which states that every class in $H_b^2(\Gamma, \mathbb{R})$ admits a canonical "homogeneous" representative, and, following [**Bou04**], we use this fact to show that any group epimorphism induces an injective map on bounded cohomology (with trivial real coefficients) in degree 2. A stronger result (with more general coefficients allowed, and where the induced map is shown to be an isometric embedding) may be found in [**Hub12**].

Chapter 3 is devoted to amenability, which represents a fundamental notion in the theory of bounded cohomology. We briefly review some results on amenable groups, also providing a complete proof of von Neumann's Theorem, which ensures that abelian groups are amenable. Then we show that bounded cohomology of amenable groups vanishes (at least for a very wide class of coefficients), and we describe Johnson's characterization of amenability in terms of bounded cohomology.

In **Chapter 4** we introduce the tools from homological algebra which are best suited to deal with bounded cohomology. Following [**Iva87**] and [**BM99, BM02**], we define the notion of relatively injective Γ-module, and we establish the basic results that allow to compute bounded cohomology via relatively injective strong resolutions. We also briefly discuss how this part of the theory iteracts with amenability, also defining the notion of amenable action (in the very restricted context of discrete groups acting on discrete spaces).

We come back to the topological interpretation of bounded cohomology of groups in **Chapter 5**. By exploiting the machinery developed in the previous chapter, we describe Ivanov's proof of a celebrated result due to Gromov, which asserts that the bounded cohomology of a space is isometrically isomorphic to the bounded cohomology of its fundamental group. We also introduce the relative bounded cohomology of topological pairs, and prove that $H_b^n(X, Y)$ is isometrically isomorphic to $H_b^n(X)$ whenever every component of Y has an amenable fundamental group.

In **Chapter 6** we introduce ℓ^1-homology of groups and spaces. Bounded cohomology naturally provides a dual theory to ℓ^1-homology. Following some works by Löh, we prove several statements on duality. As an application, we describe Löh's proof of the fact that the ℓ^1-homology of a space is canonically isomorphic to the ℓ^1-homology of its fundamental group. We also review some results by Matsumoto and Morita, showing for example that in degree 2 the seminorm on bounded cohomology is always a norm, and providing a characterization of the injectivity of the comparison map in terms of the so-called *uniform boundary condition*. Following [**BBF$^+$14**], we also make use of duality (and of the results on relative bounded cohomology proved in the previous chapter) to obtain a proof of Gromov equivalence Theorem.

Chapter 7 is devoted to an important application of bounded cohomology: the computation of simplicial volume. The simplicial volume of a closed oriented manifold is equal to the ℓ^1-seminorm of its real fundamental class. Thanks to the duality between ℓ^1-homology and bounded cohomology, the study of the simplicial volume of a manifold often benefits from the study of its bounded cohomology. Here we introduce the basic definitions and the most elementary properties of the simplicial volume, and we state several results concerning it, postponing the proofs to the subsequent chapters.

Gromov's proportionality principle states that, for closed Riemannian manifolds admitting the same Riemannian universal covering, the ratio between the simplicial volume and the Riemannian volume is a constant only depending on the universal covering. In **Chapter 8** we prove a simplified version of the proportionality principle, which applies only to non-positively curved manifolds. The choice of restricting to the non-positively curved context allows us to avoid a lot of technicalities, while introducing the most important ideas on which also the proof of the general statement is based. In particular, we introduce a bit of continuous cohomology (of topological spaces) and the trasfer map, which relates the (bounded) cohomology of a lattice in a Lie group to the continuous (bounded) cohomology of the ambient group. Even if our use of the transfer map is very limited, we feel worth introducing it, since this map plays a very important role in the theory of continuous bounded cohomology of topological groups, as developed by Burger and Monod [**BM99, BM02**]. As an application, we carry out the computation of the simplicial volume of closed hyperbolic manifolds following the strategy described in [**BK08a**].

In **Chapter 9** we prove that the simplicial volume is additive with respect to gluings along π_1-injective boundary components with an amenable fundamental group. Our proof of this fundamental theorem (which is originally due to Gromov) is based on a slight variation of the arguments described in [**BBF$^+$14**]. In fact, we deduce additivity of the simplicial volume from a result on bounded cohomology, together with a suitable application of duality.

As mentioned above, bounded cohomology has found interesting applications in the study of the dynamics of homemorphisms of the circle. In **Chapter 10** we introduce the Euler class and the bounded Euler class of a circle action, and we review a fundamental result due to Ghys [**Ghy87, Ghy99, Ghy01**], who proved that semi-conjugacy classes of circle actions are completely classified by their bounded Euler class. The bounded Euler class is thus a much finer invariant than the classical Euler class, and this provides a noticeable instance of a phenomenon mentioned above: passing from classical to bounded cohomology often allows to refine classical invariants. We also relate the bounded Euler class of a cyclic group of homeomorphisms of the circle to the classical *rotation number* of the generator of the group, and prove some properties of the rotation number that will be used in the next chapters. Finally, following Matsumoto [**Mat86**] we describe the canonical representative of the *real* bounded Euler class, also proving a characterization of semi-conjugacy in terms of the real bounded Euler class.

Chapter 11 is devoted to a brief digression from the theory of bounded cohomology. The main aim of the chapter is a detailed description of the Euler class of a sphere bundle. However, our treatment of the subject is a bit different from the usual one, in that we define the Euler cocycle as a *singular* (rather than cellular) cocycle. Of course, cellular and singular cohomology are canonically isomorphic for every CW-complex. However, cellular cochains are not useful to compute the *bounded* cohomology of a cellular complex: for example, it is not easy to detect whether the singular coclass corresponding to a cellular one admits a bounded representative or not, and this motivates us to work directly in the singular context, even if this choice could seem a bit costly at first glance.

In **Chapter 12** we specialize our study of sphere bundles to the case of *flat* bundles. The theory of flat bundles builds a bridge between the theory of fiber bundles and the theory of representations. For example, the Euler class of a flat circle bundle corresponds (under a canonical morphism) to the Euler class of a representation canonically associated to the bundle. This leads to the definition of the *bounded* Euler class of a flat circle bundle. By putting together an estimate on the norm of the bounded Euler class and the computation of the simplicial volume of surfaces carried out in the previous chapters, we are then able to prove *Milnor-Wood inequalities*, which provide sharp estimates on the possible

Euler numbers of flat circle bundles over surfaces. We then concentrate our attention on *maximal* representations, i.e. on representations of surface groups which attain the extremal value allowed by Milnor-Wood inequalities. A celebrated result by Goldman [**Gol80**] states that maximal representations of surface groups into the group of orientation-preserving isometries of the hyperbolic plane are exactly the holonomies of hyperbolic structures. Following [**BIW10**], we will give a proof of Goldman's theorem based on the use of bounded cohomology. In doing this, we will describe the Euler number of flat bundles over surfaces with boundary, as defined in [**BIW10**].

Chapter 13 is devoted to higher-dimensional generalizations of Milnor-Wood inequalities. We introduce Ivanov-Turaev's bounded Euler cocycle in dimension $n \geq 2$, and from an estimate (proved to be optimal in [**BM12**]) on its norm we deduce several inequalities on the possible Euler numbers of flat vector bundles over closed manifolds. We also discuss the relationship between this topic and the Chern conjecture, which predicts that the Euler characteristic of a closed affine manifold should vanish.

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