

(Bounded) group cohomology via resolutions

Computing group cohomology by means of its definition is usually very hard. The topological interpretation of group cohomology already provides a powerful tool for computations: for example, one may estimate the cohomological dimension of a group Γ (i.e. the maximal $n \in \mathbb{N}$ such that $H^n(\Gamma, R) \neq 0$) in terms of the dimension of a $K(\Gamma, 1)$ as a CW-complex. We have already mentioned that a topological interpretation for the bounded cohomology of a group is still available, but in order to prove this fact more machinery has to be introduced. Before going into the bounded case, we describe some well-known results which hold in the classical case: namely, we will show that the cohomology of Γ may be computed by looking at several complexes of cochains. This crucial fact was already observed in the pioneering works on group cohomology of the 1940s and the 1950s.

There are several ways to define group cohomology in terms of resolutions. We privilege here an approach that better extends to the case of bounded cohomology. We briefly compare our approach to more traditional ones in Section 4.3.

4.1. Relative injectivity

We begin by introducing the notions of relatively injective $R[\Gamma]$ -module and of strong Γ -resolution of an $R[\Gamma]$ -module. The counterpart of these notions in the context of normed $R[\Gamma]$ -modules will play an important role in the theory of bounded cohomology of groups. The importance of these notions is due to the fact that the cohomology of Γ may be computed by looking at *any* strong Γ -resolution of the coefficient module by relatively injective modules (see Corollary 4.6 below).

A Γ -map $\iota: A \rightarrow B$ between $R[\Gamma]$ -modules is *strongly injective* if there is an R -linear map $\sigma: B \rightarrow A$ such that $\sigma \circ \iota = \text{Id}_A$ (in particular, ι is injective). We emphasize that, even if A and B are Γ -modules, the map σ is *not* required to be Γ -equivariant.

DEFINITION 4.1. An $R[\Gamma]$ -module U is *relatively injective* (over $R[\Gamma]$) if the following holds: whenever A, B are Γ -modules, $\iota: A \rightarrow B$ is a strongly injective Γ -map and $\alpha: A \rightarrow U$ is a Γ -map, there exists a Γ -map $\beta: B \rightarrow U$ such that $\beta \circ \iota = \alpha$.

$$\begin{array}{ccccc}
 0 & \longrightarrow & A & \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\iota} \end{array} & B & . \\
 & & \downarrow \alpha & \swarrow \beta & & \\
 & & U & & &
 \end{array}$$

In the case when $R = \mathbb{R}$, a map is strongly injective if and only if it is injective, so in the context of $\mathbb{R}[\Gamma]$ -modules the notions of relative injectivity and the traditional notion of injectivity coincide. However, relative injectivity is weaker

than injectivity in general. For example, if $R = \mathbb{Z}$ and $\Gamma = \{1\}$, then $C^i(\Gamma, R)$ is isomorphic to the trivial Γ -module \mathbb{Z} , which of course is not injective over $\mathbb{Z}[\Gamma] = \mathbb{Z}$. On the other hand, we have the following:

LEMMA 4.2. *For every $R[\Gamma]$ -module V and every $n \in \mathbb{N}$, the $R[\Gamma]$ -module $C^n(\Gamma, V)$ is relatively injective.*

PROOF. Let us consider the extension problem described in Definition 4.1, with $U = C^n(\Gamma, V)$. Then we define β as follows:

$$\begin{aligned} \beta(b)(g_0, \dots, g_n) &= \alpha(g_0 \sigma(g_0^{-1} b))(g_0, \dots, g_n) \\ &= g_0(\alpha(\sigma(g_0^{-1} b))(1, g_0^{-1} g_1, \dots, g_0^{-1} g_n)) . \end{aligned}$$

The fact that β is R -linear and that $\beta \circ \iota = \alpha$ is straightforward, so we just need to show that β is Γ -equivariant. But for every $g \in \Gamma$, $b \in B$ and $(g_0, \dots, g_n) \in \Gamma^{n+1}$ we have

$$\begin{aligned} g(\beta(b))(g_0, \dots, g_n) &= g(\beta(b)(g^{-1} g_0, \dots, g^{-1} g_n)) \\ &= g(\alpha(g^{-1} g_0 \sigma(g_0^{-1} g b))(g^{-1} g_0, \dots, g^{-1} g_n)) \\ &= g(g^{-1}(\alpha(g_0 \sigma(g_0^{-1} g b)))(g^{-1} g_0, \dots, g^{-1} g_n)) \\ &= \alpha(g_0 \sigma(g_0^{-1} g b))(g_0, \dots, g_n) \\ &= \beta(g b)(g_0, \dots, g_n) . \end{aligned}$$

□

The following lemma shows that relatively injective modules allow to solve also generalized extension problems:

LEMMA 4.3. *Let U be a relatively injective $R[\Gamma]$ -module, let $j: A \rightarrow B$ be an $R[\Gamma]$ -map and suppose that there exists a (not necessarily Γ -equivariant) R -linear map $\sigma: B \rightarrow A$ such that $j\sigma j = j$. Let also $\alpha: A \rightarrow U$ be an $R[\Gamma]$ -map and suppose that $\ker j \subseteq \ker \alpha$. Then there exists an $R[\Gamma]$ -map $\beta: B \rightarrow U$ such that $\beta \circ j = \alpha$.*

$$\begin{array}{ccc} 0 & \longrightarrow & A \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{j} \end{array} B \\ & & \downarrow \alpha \quad \swarrow \beta \\ & & U \end{array} .$$

PROOF. Let $\bar{A} = A/\ker j$ and denote by $\pi: A \rightarrow \bar{A}$ the quotient map. Then \bar{A} admits an obvious structure of $R[\Gamma]$ -module, the map π is an $R[\Gamma]$ -map and the map $j: A \rightarrow B$ induces a quotient $R[\Gamma]$ -map $\bar{j}: \bar{A} \rightarrow B$. Observe that, since $j\sigma j = j$, for every $a \in A$ the element $\sigma(j(a)) - a$ belongs to $\ker j$. Therefore, if we denote by $\bar{\sigma}: B \rightarrow \bar{A}$ the composition $\bar{\sigma} = \pi \circ \sigma$, then we have $\bar{\sigma} \circ \bar{j} = \text{Id}_{\bar{A}}$, i.e. the map \bar{j} is strongly injective.

Let us now denote by $\bar{\alpha}: \bar{A} \rightarrow U$ the $R[\Gamma]$ -map induced by α (this maps exists because $\ker j \subseteq \ker \alpha$). Then, thanks to the relative injectivity of U , the extension problem

$$\begin{array}{ccc} 0 & \longrightarrow & \bar{A} \begin{array}{c} \xrightarrow{\bar{\sigma}} \\ \xrightarrow{\bar{j}} \end{array} B \\ & & \downarrow \bar{\alpha} \quad \swarrow \beta \\ & & U \end{array} .$$

admits a solution β . By construction we have $\alpha = \beta \circ j$, and this concludes the proof. \square

4.2. Resolutions of Γ -modules

An $R[\Gamma]$ -*complex* (or simply a Γ -*complex* or a *complex*) is a sequence of $R[\Gamma]$ -modules E^i and Γ -maps $\delta^i: E^i \rightarrow E^{i+1}$, $i \in \mathbb{N}$, such that $\delta^{i+1} \circ \delta^i = 0$ for every i :

$$0 \longrightarrow E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \dots$$

Such a sequence will be denoted by $(E^\bullet, \delta^\bullet)$. Moreover, we set $Z^n(E^\bullet) = \ker \delta^n \cap (E^n)^\Gamma$, $B^n(E^\bullet) = \delta^{n-1}((E^{n-1})^\Gamma)$ (where again we understand that $B^0(E^\bullet) = 0$), and we define the cohomology of the complex E^\bullet by setting

$$H^n(E^\bullet) = Z^n(E^\bullet)/B^n(E^\bullet).$$

A *chain map* between Γ -complexes $(E^\bullet, \delta_E^\bullet)$ and $(F^\bullet, \delta_F^\bullet)$ is a sequence of Γ -maps $\{\alpha^i: E^i \rightarrow F^i \mid i \in \mathbb{N}\}$ such that $\delta_F^i \circ \alpha^i = \alpha^{i+1} \circ \delta_E^i$ for every $i \in \mathbb{N}$. If $\alpha^\bullet, \beta^\bullet$ are chain maps between $(E^\bullet, \delta_E^\bullet)$ and $(F^\bullet, \delta_F^\bullet)$, a Γ -*homotopy* between α^\bullet and β^\bullet is a sequence of Γ -maps $\{T^i: E^i \rightarrow F^{i-1} \mid i \geq 0\}$ such that $T^1 \circ \delta_E^0 = \alpha^0 - \beta^0$ and $\delta_F^{i-1} \circ T^i + T^{i+1} \circ \delta_E^i = \alpha^i - \beta^i$ for every $i \geq 1$. Every chain map induces a well-defined map in cohomology, and Γ -homotopic chain maps induce the same map in cohomology.

If E is an $R[\Gamma]$ -module, an augmented Γ -complex $(E, E^\bullet, \delta^\bullet)$ with augmentation map $\varepsilon: E \rightarrow E^0$ is a complex

$$0 \longrightarrow E \xrightarrow{\varepsilon} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \dots$$

A *resolution* of E (over Γ) is an exact augmented complex $(E, E^\bullet, \delta^\bullet)$ (over Γ). A resolution $(E, E^\bullet, \delta^\bullet)$ is *relatively injective* if E^n is relatively injective for every $n \geq 0$. It is well-known that any map between modules extends to a chain map between *injective* resolutions of the modules. Unfortunately, the same result for relatively injective resolutions does not hold. The point is that relative injectivity guarantees the needed extension property only for *strongly* injective maps. Therefore, we need to introduce the notion of *strong* resolution.

A *contracting homotopy* for a resolution $(E, E^\bullet, \delta^\bullet)$ is a sequence of R -linear maps $k^i: E^i \rightarrow E^{i-1}$ such that $\delta^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \text{Id}_{E^i}$ if $i \geq 0$, and $k^0 \circ \varepsilon = \text{Id}_E$:

$$0 \longrightarrow E \xleftarrow[\varepsilon]{k^0} E^0 \xleftarrow[\delta^0]{k^1} E^1 \xleftarrow[\delta^1]{k^2} \dots \xleftarrow[\delta^{n-1}]{k^n} E^n \xleftarrow[\delta^n]{k^{n+1}} \dots \dots$$

Note however that it is not required that k^i be Γ -equivariant. A resolution is *strong* if it admits a contracting homotopy.

The following proposition shows that the chain complex $C^\bullet(\Gamma, V)$ provides a relatively injective strong resolution of V :

PROPOSITION 4.4. *Let $\varepsilon: V \rightarrow C^0(\Gamma, V)$ be defined by $\varepsilon(v)(g) = v$ for every $v \in V$, $g \in \Gamma$. Then the augmented complex*

$$0 \longrightarrow V \xrightarrow{\varepsilon} C^0(\Gamma, V) \xrightarrow{\delta^0} C^1(\Gamma, V) \rightarrow \dots \rightarrow C^n(\Gamma, V) \rightarrow \dots$$

provides a relatively injective strong resolution of V .

PROOF. We already know that each $C^i(\Gamma, V)$ is relatively injective. In order to show that the augmented complex $(V, C^\bullet(\Gamma, V), \delta^\bullet)$ is a strong resolution it is sufficient to observe that the maps

$$k^{n+1}: C^{n+1}(\Gamma, V) \rightarrow C^n(\Gamma, V) \quad k^{n+1}(f)(g_0, \dots, g_n) = f(1, g_0, \dots, g_n)$$

provide the required contracting homotopy. \square

The resolution of V described in the previous proposition is obtained just by augmenting the homogeneous complex associated to (Γ, V) , and it is usually called the *standard* resolution of V (over Γ).

The following result implies that any relatively injective strong resolution of a Γ -module V may be used to compute the cohomology modules $H^\bullet(\Gamma, V)$ (see Corollary 4.6).

THEOREM 4.5. *Let $\alpha: E \rightarrow F$ be a Γ -map between $R[\Gamma]$ -modules, let $(E, E^\bullet, \delta_E^\bullet)$ be a strong resolution of E , and suppose that $(F, F^\bullet, \delta_F^\bullet)$ is an augmented complex such that F^i is relatively injective for every $i \geq 0$. Then α extends to a chain map α^\bullet , and any two extensions of α to chain maps are Γ -homotopic.*

PROOF. We denote by $\varepsilon_E: E \rightarrow E^0$ and $\varepsilon_F: F \rightarrow F_0$ the augmentation maps for the resolutions $(E, E^\bullet, \delta_E^\bullet)$ and $(F, F^\bullet, \delta_F^\bullet)$, and by k^i , $i \in \mathbb{N}$ a contracting homotopy for $(E, E^\bullet, \delta_E^\bullet)$.

We construct the map α^\bullet inductively, starting in degree 0. By definition of contracting homotopy we have $k_0 \circ \varepsilon_E = \text{Id}_E$, so we can consider the extension problem

$$\begin{array}{ccc} 0 & \longrightarrow & E & \xleftarrow{k^0} & E^0 & . \\ & & \downarrow \varepsilon_F \circ \alpha & \swarrow \alpha^0 & \downarrow \varepsilon_E & \\ & & F^0 & & & \end{array}$$

By the relative injectivity of F^0 , there exists $\alpha^0: E^0 \rightarrow F^0$ such that $\alpha^0 \circ \varepsilon_E = \varepsilon_F \circ \alpha$, and this settles the case $n = 0$ of the induction.

Let now $n \geq 1$ and suppose we have a collection of Γ -maps $\alpha^i: E^i \rightarrow F^i$, $i = 0, \dots, n-1$, such that $\alpha^k \delta_E^{k-1} = \delta_F^{k-1} \alpha^{k-1}$ for every $k = 0, \dots, n-1$ (where we understand that $\alpha^{-1} = \alpha$ and $\delta_E^{-1} = \varepsilon_E$, $\delta_F^{-1} = \varepsilon_F$). We would like to apply Lemma 4.3 to the extension problem

$$\begin{array}{ccc} 0 & \longrightarrow & E^{n-1} & \xleftarrow{k^n} & E^n & . \\ & & \downarrow \delta_F^{n-1} \alpha^{n-1} & \swarrow \alpha^n & \downarrow \delta_E^{n-1} & \\ & & F^n & & & \end{array}$$

To this aim we need to check that $\delta_E^n k^{n+1} \delta_E^n = \delta_E^n$ and $\ker \delta_E^{n-1} \subseteq \ker(\delta_F^{n-1} \alpha^{n-1})$. The first equality is obtained by composing with δ_E^n the equality $\delta_E^{n-1} k^n + k^{n+1} \delta_E^n = \text{Id}_{E^n}$. For the latter fact, take $v \in \ker \delta_E^{n-1}$. We have

$$v = \delta_E^{n-2}(k^{n-1}(v)) + k^n(\delta_E^{n-1}(v)) = \delta_E^{n-2}(k^{n-1}(v)) ,$$

hence

$$\delta_F^{n-1}(\alpha^{n-1}(v)) = \delta_F^{n-1}(\alpha^{n-1}(\delta_E^{n-2}(k^{n-1}(v)))) ,$$

which by induction hypothesis is equal to

$$\delta_F^{n-1}(\delta_F^{n-2}(\alpha_E^{n-2}(k^{n-1}(v)))) = 0 .$$

This shows that Lemma 4.3 can be applied to get a map $\alpha^n: E^n \rightarrow F^n$ such that $\alpha^n \delta_E^{n-1} = \delta_F^{n-1} \alpha^{n-1}$, and concludes the construction of the desired chain map α^\bullet .

Let us now prove that distinct extensions of α are Γ -homotopic. Considering the difference between two such extensions, we see that it is sufficient to show that any extension α^\bullet of the null map $\alpha: E \rightarrow F$ is Γ -homotopic to the null chain map between E^\bullet and F^\bullet . To this aim, we construct inductively a sequence of $R[\Gamma]$ -maps $\sigma^i: E^i \rightarrow F^{i-1}$, $i = -1, 0, 1, \dots$, such that

$$(8) \quad \delta_F^{i-1} \sigma^i + \sigma^{i+1} \delta_E^i = \alpha^i$$

for every i , where we understand that $E^{-1} = E$, $F^{-1} = F$, and $E^i = F^i = 0$ for every $i \leq -2$. We set $\sigma^{-1} = \sigma^0 = 0$, so that condition (8) holds for $n = -1$.

For the induction step, we let $n \geq 0$ and we assume to have constructed $\sigma^i: E^i \rightarrow F^{i-1}$, $i = -1, \dots, n-1$ in such a way that (8) holds for all $-1 \leq i \leq n-1$. Let us now consider the extension problem

$$\begin{array}{ccc} 0 & \longrightarrow & E^n & \xleftarrow{k^{n+1}} & E^{n+1} & . \\ & & \downarrow \alpha^n - \delta_F^{n-1} \sigma^n & \searrow \delta_E^n & \swarrow ? & \\ & & F^n & & & \end{array}$$

We have already shown that $\delta_E^n k^{n+1} \delta_E^n = \delta_E^n$, so in order to apply Lemma 4.3 we need to check that $\ker \delta_E^n \subseteq \ker(\alpha^n - \delta_F^{n-1} \sigma^n)$. Let $v \in \ker \delta_E^n$, so that $v = \delta_E^{n-1}(k^n(v))$, and compute

$$\begin{aligned} (\alpha^n - \delta_F^{n-1} \sigma^n)(v) &= (\alpha^n - \delta_F^{n-1} \sigma^n)(\delta_E^{n-1}(k^n(v))) \\ &= \delta_F^{n-1}(\alpha^{n-1}(k^n(v))) - \delta_F^{n-1}(\sigma^n(\delta_E^{n-1}(k^n(v)))) , \end{aligned}$$

where we used that $\alpha^n \delta_E^{n-1} = \delta_F^{n-1} \alpha^{n-1}$. Since by induction $\sigma^n \delta_E^{n-1} = \alpha^{n-1} - \delta_F^{n-2} \sigma^{n-1}$, the second term above is equal to

$$\delta_F^{n-1}(\alpha^{n-1}(k^n(v))) - \delta_F^{n-1}(\delta_F^{n-2}(\sigma^{n-1}(k^n(v)))) = \delta_F^{n-1}(\alpha^{n-1}(k^n(v))) ,$$

so that $(\alpha^n - \delta_F^{n-1} \sigma^n)(v) = 0$, and all the conditions of Lemma 4.3 are satisfied. Since $n \geq 0$, the module F^n is relatively injective, so the extension problem above admits a solution $\sigma^{n+1}: E^{n+1} \rightarrow F^n$, and the induction step is proved. \square

COROLLARY 4.6. *Let $(V, V^\bullet, \delta_V^\bullet)$ be a relatively injective strong resolution of V . Then for every $n \in \mathbb{N}$ there is a canonical isomorphism*

$$H^n(\Gamma, V) \cong H^n(V^\bullet) .$$

PROOF. By Proposition 4.4, both $(V, V^\bullet, \delta_V^\bullet)$ and the standard resolution of V are relatively injective strong resolutions of V over Γ . Therefore, Theorem 4.5 provides chain maps between $C^\bullet(\Gamma, V)$ and V^\bullet , which are one the Γ -homotopy inverse of the other. Therefore, these chain maps induce isomorphisms in cohomology. \square

4.3. The classical approach to group cohomology via resolutions

As far as the author knows, the standard homological algebraic approach to group cohomology involves injective (or projective) resolutions, rather than *relatively* injective ones. As already mentioned above, our apparently exotic choice is due to the fact that relative injectivity plays an important role in the context of bounded cohomology, and to our purposes it is useful to develop both the theory of classical cohomology and the theory of bounded cohomology within (basically) the same framework. However, we feel that readers who are already used to group cohomology could benefit from a brief comparison between the description of group cohomology via resolutions given here and more traditional approaches to the subject (see e.g. [Bro82]). The reader who is not interested can safely skip the section, since the results cited below will not be used elsewhere in this monograph. We restrict our attention to the case when $R = \mathbb{Z}$, the case when $R = \mathbb{R}$ being similar (and easier).

If V, W are $\mathbb{Z}[\Gamma]$ -modules, then the space $\text{Hom}_{\mathbb{Z}}(V, W)$ is endowed with the structure of a $\mathbb{Z}[\Gamma]$ -module by setting $(g \cdot f)(v) = g(f(g^{-1}v))$ for every $g \in \Gamma$, $f \in \text{Hom}_{\mathbb{Z}}(V, W)$, $v \in V$. Then, the cohomology $H^\bullet(\Gamma, V)$ is often defined in the following equivalent ways (we refer e.g. to [Bro82] for the definition of projective module (over a ring R); for our discussion, it is sufficient to know that any free R -module is projective over R , so any free resolution is projective):

- (1) **(Via injective resolutions of V):** Let $(V, V^\bullet, \delta^\bullet)$ be an injective resolution of V over $\mathbb{Z}[\Gamma]$, and take the complex $W^\bullet = \text{Hom}_{\mathbb{Z}}(V^\bullet, \mathbb{Z})$, endowed with the action $g \cdot f(v) = f(g^{-1}v)$. Then $(W^\bullet)^\Gamma = \text{Hom}_{\mathbb{Z}[\Gamma]}(V^\bullet, \mathbb{Z})$ (where \mathbb{Z} is endowed with the structure of trivial Γ -module), and one may define $H^\bullet(\Gamma, V)$ as the homology of the Γ -invariants of W^\bullet .
- (2) **(Via projective resolutions of \mathbb{Z}):** Let $(\mathbb{Z}, P_\bullet, d_\bullet)$ be a *projective* resolution of \mathbb{Z} over $\mathbb{Z}[\Gamma]$, and take the complex $Z^\bullet = \text{Hom}_{\mathbb{Z}}(P_\bullet, V)$. We have again that $(Z^\bullet)^\Gamma = \text{Hom}_{\mathbb{Z}[\Gamma]}(P_\bullet, V)$, and again we may define $H^\bullet(\Gamma, V)$ as the homology of the Γ -invariants of the complex Z^\bullet .

The fact that these two definitions are indeed equivalent is proved e.g. in [Bro82].

We have already observed that, if V is a $\mathbb{Z}[\Gamma]$ -module, the module $C^n(\Gamma, V)$ is not injective in general. However, this is not really a problem, since the complex $C^\bullet(\Gamma, V)$ may be recovered from a projective resolution of \mathbb{Z} over $\mathbb{Z}[\Gamma]$. Namely, let $C_n(\Gamma, \mathbb{Z})$ be the free \mathbb{Z} -module admitting the set Γ^{n+1} as a basis. The diagonal action of Γ onto Γ^{n+1} endows $C_n(\Gamma, \mathbb{Z})$ with the structure of a $\mathbb{Z}[\Gamma]$ -module. The modules $C_n(\Gamma, \mathbb{Z})$ may be arranged into a resolution

$$0 \longleftarrow \mathbb{Z} \xleftarrow{\varepsilon} C_0(\Gamma, \mathbb{Z}) \xleftarrow{d_1} C_1(\Gamma, \mathbb{Z}) \xleftarrow{d_2} \dots \xleftarrow{d_n} C_n(\Gamma, \mathbb{Z}) \xleftarrow{d_{n+1}} \dots$$

of the trivial $\mathbb{Z}[\Gamma]$ -module \mathbb{Z} over $\mathbb{Z}[\Gamma]$ (the homology of the Γ -coinvariants of this resolution is by definition the *homology* of Γ , see Section 6.2). Now, it is easy to check that $C_n(\Gamma, \mathbb{Z})$ is free, whence projective, as a $\mathbb{Z}[\Gamma]$ -module. Moreover, the module $\text{Hom}_{\mathbb{Z}}(C_n(\Gamma, \mathbb{Z}), V)$ is $\mathbb{Z}[\Gamma]$ -isomorphic to $C^n(\Gamma, V)$. This shows that, in the context of the traditional definition of group cohomology via resolutions, the complex $C^\bullet(\Gamma, V)$ arises from a projective resolution of \mathbb{Z} , rather than from an injective resolution of V .

4.4. The topological interpretation of group cohomology revisited

Corollary 4.6 may be exploited to prove that the cohomology of Γ is isomorphic to the singular cohomology of any path-connected topological space X satisfying conditions (1), (2) and (3) described in Section 1.2, which we recall here for the reader's convenience:

- (1) the fundamental group of X is isomorphic to Γ ,
- (2) the space X admits a universal covering \tilde{X} , and
- (3) \tilde{X} is R -acyclic, i.e. $H_n(\tilde{X}, R) = 0$ for every $n \geq 1$.

We fix an identification between Γ and the group of the covering automorphisms of the universal covering \tilde{X} of X . The action of Γ on \tilde{X} induces an action of Γ on $C_\bullet(\tilde{X}, R)$, whence an action of Γ on $C^\bullet(\tilde{X}, R)$, which is defined by $(g \cdot \varphi)(c) = \varphi(g^{-1}c)$ for every $c \in C_\bullet(\tilde{X}, R)$, $\varphi \in C^\bullet(\tilde{X}, R)$, $g \in \Gamma$. Therefore, for $n \in \mathbb{N}$ both $C_n(\tilde{X}, R)$ and $C^n(\tilde{X}, R)$ are endowed with the structure of $R[\Gamma]$ -modules. We have a natural identification $C^\bullet(\tilde{X}, R)^\Gamma \cong C^\bullet(X, R)$.

LEMMA 4.7. *For every $n \in \mathbb{N}$, the singular cochain module $C^n(\tilde{X}, R)$ is relatively injective.*

PROOF. For every topological space Y , let us denote by $S_n(Y)$ the set of singular simplices with values in Y .

We denote by $L_n(\tilde{X})$ a set of representatives for the action of Γ on $S_n(\tilde{X})$ (for example, if F is a set of representatives for the action of Γ on \tilde{X} , we may define $L_n(\tilde{X})$ as the set of singular n -simplices whose first vertex lies in F). Then, for every n -simplex $s \in S_n(\tilde{X})$, there exist a unique $g_s \in \Gamma$ and a unique $\bar{s} \in L_n(\tilde{X})$ such that $g_s \cdot \bar{s} = s$. Let us now consider the extension problem:

$$\begin{array}{ccc}
 0 & \longrightarrow & A & \xrightleftharpoons[\iota]{\sigma} & B & . \\
 & & \downarrow \alpha & & \swarrow \beta & \\
 & & C^n(\tilde{X}, R) & & &
 \end{array}$$

We define the desired extension β by setting

$$\beta(b)(s) = \alpha(g_s \sigma(g_s^{-1} \cdot b))(s) = \alpha(\sigma(g_s^{-1} \cdot b))(\bar{s})$$

for every $s \in S_n(\tilde{X})$. It is easy to verify that the map β is an $R[\Gamma]$ -map, and that $\alpha = \beta \circ \iota$.

An alternative proof (producing the same solution to the extension problem) is the following. Using that the action of Γ on \tilde{X} is free, it is easy to show that $L_n(\tilde{X})$, when considered as a subset of $C_n(\tilde{X}, R)$, is a free basis of $C_n(\tilde{X}, R)$ over $R[\Gamma]$. In other words, every $c \in C_n(\tilde{X}, R)$ may be expressed uniquely as a sum of the form $c = \sum_{i=1}^k a_i g_i s_i$, $a_i \in R$, $g_i \in \Gamma$, $s_i \in L_n(\tilde{X})$. Therefore, the map

$$\psi: C^0(\Gamma, C^0(L_n(\tilde{X}), R)) \rightarrow C^n(\tilde{X}, R), \quad \psi(f) \left(\sum_{i=1}^k a_i g_i s_i \right) = \sum_{i=1}^k a_i f(g_i)(s_i)$$

is well defined. If we endow $C^0(L_n(\tilde{X}), R)$ with the structure of trivial Γ -module, then a straightforward computation shows that ψ is in fact a Γ -isomorphism, so the conclusion follows from Lemma 4.2. \square

PROPOSITION 4.8. *Let $\varepsilon: R \rightarrow C^0(\tilde{X}, R)$ be defined by $\varepsilon(t)(s) = t$ for every singular 0-simplex s in \tilde{X} . Suppose that $H_i(\tilde{X}, R) = 0$ for every $i \geq 1$. Then the augmented complex*

$$0 \longrightarrow R \xrightarrow{\varepsilon} C^0(\tilde{X}, R) \xrightarrow{\delta^1} C^1(\tilde{X}, R) \rightarrow \dots \xrightarrow{\delta^{n-1}} C^n(\tilde{X}, R) \xrightarrow{\delta^n} C^{n+1}(\tilde{X}, R)$$

is a relatively injective strong resolution of the trivial $R[\Gamma]$ -module R .

PROOF. Since \tilde{X} is path-connected, we have that $\text{Im } \varepsilon = \ker \delta^0$. Observe now that $C_n(\tilde{X}, R)$ is R -free for every $n \in \mathbb{N}$. As a consequence, the obvious augmented complex associated to $C_\bullet(\tilde{X}, R)$, being acyclic, is homotopically trivial over R . Since $C^n(\tilde{X}, R) \cong \text{Hom}_R(C_n(\tilde{X}, R), R)$, we may conclude that the augmented complex described in the statement is a strong resolution of R over $R[\Gamma]$. Then the conclusion follows from Lemma 4.7. \square

Putting together Propositions 4.4, 4.8 and Corollary 4.6 we can provide the following topological description of $H^n(\Gamma, R)$:

COROLLARY 4.9. *Let X be a path-connected space admitting a universal covering \tilde{X} , and suppose that $H_i(\tilde{X}, R) = 0$ for every $i \geq 1$. Then $H^i(X, R)$ is canonically isomorphic to $H^i(\pi_1(X), R)$ for every $i \in \mathbb{N}$.*

4.5. Bounded cohomology via resolutions

Just as in the case of classical cohomology, it is often useful to have alternative ways to compute the bounded cohomology of a group. This section is devoted to an approach to bounded cohomology which closely follows the traditional approach to classical cohomology via homological algebra. The circle of ideas we are going to describe first appeared (in the case with trivial real coefficients) in a paper by Brooks [Bro81], where it was exploited to give an independent proof of Gromov's result that the isomorphism type of the bounded cohomology of a space (with real coefficients) only depends on its fundamental group [Gro82]. Brooks' theory was then developed by Ivanov in his foundational paper [Iva87] (see also [Nos91] for the case of coefficients in general normed Γ -modules). Ivanov gave a new proof of the vanishing of the bounded cohomology (with real coefficients) of a simply connected space (this result played an important role in Brooks' argument, and was originally due to Gromov [Gro82]), and managed to incorporate the seminorm into the homological algebra approach to bounded cohomology, thus proving that the bounded cohomology of a space is *isometrically* isomorphic to the bounded cohomology of its fundamental group (in the case with real coefficients). Ivanov-Noskov's theory was further developed by Burger and Monod [BM99, BM02, Mon01], who paid a particular attention to the *continuous* bounded cohomology of topological groups.

Both Ivanov-Noskov's and Monod's theory are concerned with *Banach* Γ -modules, which are in particular $\mathbb{R}[\Gamma]$ -modules. For the moment, we prefer to consider also the (quite different) case with integral coefficients. Therefore, we let R be either \mathbb{Z} or \mathbb{R} , and we concentrate our attention on the category of normed $R[\Gamma]$ -modules introduced in Section 1.3. In the next sections we will see that relative injective modules and strong resolutions may be defined in this context just by adapting to normed $R[\Gamma]$ -modules the analogous definitions for generic $R[\Gamma]$ -modules.

4.6. Relatively injective normed Γ -modules

Throughout the whole section, unless otherwise stated, we will deal only with *normed* $R[\Gamma]$ -modules. Therefore, Γ -morphisms will be always assumed to be bounded.

The following definitions are taken from [Iva87] (where only the case when $R = \mathbb{R}$ and V is Banach is dealt with). A bounded linear map $\iota: A \rightarrow B$ of normed R -modules is *strongly injective* if there is an R -linear map $\sigma: B \rightarrow A$ with $\|\sigma\| \leq 1$ and $\sigma \circ \iota = \text{Id}_A$ (in particular, ι is injective). We emphasize that, even when A and B are $R[\Gamma]$ -modules, the map σ is *not* required to be Γ -equivariant.

DEFINITION 4.10. A normed $R[\Gamma]$ -module E is *relatively injective* if for every strongly injective Γ -morphism $\iota: A \rightarrow B$ of normed $R[\Gamma]$ -modules and every Γ -morphism $\alpha: A \rightarrow E$ there is a Γ -morphism $\beta: B \rightarrow E$ satisfying $\beta \circ \iota = \alpha$ and $\|\beta\| \leq \|\alpha\|$.

$$\begin{array}{ccccc}
 0 & \longrightarrow & A & \begin{array}{c} \xrightarrow{\sigma} \\ \xrightarrow{\iota} \end{array} & B & . \\
 & & \downarrow \alpha & \searrow \beta & & \\
 & & E & & &
 \end{array}$$

REMARK 4.11. Let E be a normed $R[\Gamma]$ -module, and let \hat{E} be the underlying $R[\Gamma]$ -module. Then no obvious implication exists between the fact that E is relatively injective (in the category of normed $R[\Gamma]$ -modules, i.e. according to Definition 4.10), and the fact that \hat{E} is (in the category of $R[\Gamma]$ -modules, i.e. according to Definition 4.1). This could suggest that the use of the same name for these different notions could indeed be an abuse. However, unless otherwise stated, henceforth we will deal with relatively injective modules only in the context of normed $R[\Gamma]$ -modules, so the reader may safely take Definition 4.10 as the only definition of relative injectivity.

The following result is due to Ivanov [Iva87] in the case of real coefficients, and to Monod [Mon01] in the general case (see also Remark 4.14), and shows that the modules involved in the definition of bounded cohomology are relatively injective.

LEMMA 4.12. *Let V be a normed $R[\Gamma]$ -module. Then the normed $R[\Gamma]$ -module $C_b^n(\Gamma, V)$ is relatively injective.*

PROOF. Let us consider the extension problem described in Definition 4.10, with $E = C_b^n(\Gamma, V)$. Then we define β as follows:

$$\beta(b)(g_0, \dots, g_n) = \alpha(g_0 \sigma(g_0^{-1} b))(g_0, \dots, g_n) .$$

It is immediate to check that $\beta \circ \iota = \alpha$. Moreover, since $\|\sigma\| \leq 1$, we have $\|\beta\| \leq \|\alpha\|$. Finally, the fact that β commutes with the actions of Γ has been shown in the proof of Lemma 4.2. \square

4.7. Resolutions of normed Γ -modules

A normed $R[\Gamma]$ -complex is an $R[\Gamma]$ -complex whose modules are normed $R[\Gamma]$ -spaces, and whose differential is a bounded $R[\Gamma]$ -map in every degree. A chain map between two normed $R[\Gamma]$ -complexes $(E^\bullet, \delta_E^\bullet)$, $(F^\bullet, \delta_F^\bullet)$ is a chain map between the underlying $R[\Gamma]$ -complexes which is bounded in every degree, and a Γ -homotopy between two such chain maps is just a Γ -homotopy between the underlying maps

of $R[\Gamma]$ -modules, which is bounded in every degree. The cohomology $H_b^\bullet(E^\bullet)$ of the normed Γ -complex $(E^\bullet, \delta_E^\bullet)$ is defined as usual by taking the cohomology of the subcomplex of Γ -invariants. The norm on E^n restricts to a norm on Γ -invariant cocycles, which induces in turn a seminorm on $H_b^n(E^\bullet)$ for every $n \in \mathbb{N}$.

An augmented normed Γ -complex $(E, E^\bullet, \delta^\bullet)$ with augmentation map $\varepsilon: E \rightarrow E^0$ is a Γ -complex

$$0 \longrightarrow E \xrightarrow{\varepsilon} E^0 \xrightarrow{\delta^0} E^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^n} E^{n+1} \xrightarrow{\delta^{n+1}} \dots$$

We also ask that ε is an isometric embedding. A resolution of E (as a normed $R[\Gamma]$ -complex) is an exact augmented normed complex $(E, E^\bullet, \delta^\bullet)$. It is *relatively injective* if E^n is relatively injective for every $n \geq 0$. From now on, we will call simply *complex* any normed Γ -complex.

Let $(E, E^\bullet, \delta_E^\bullet)$ be a resolution of E , and suppose that $(F, F^\bullet, \delta_F^\bullet)$ is a relatively injective resolution of F . We would like to be able to extend any Γ -map $E \rightarrow F$ to a chain map between E^\bullet and F^\bullet . As observed in the preceding section, to this aim we need to require the resolution $(E, E^\bullet, \delta_E^\bullet)$ to be strong, according to the following definition.

A *contracting homotopy* for a resolution $(E, E^\bullet, \delta^\bullet)$ is a sequence of linear maps $k^i: E^i \rightarrow E^{i-1}$ such that $\|k^i\| \leq 1$ for every $i \in \mathbb{N}$, $\delta^{i-1} \circ k^i + k^{i+1} \circ \delta^i = \text{Id}_{E^i}$ if $i \geq 0$, and $k^0 \circ \varepsilon = \text{Id}_E$:

$$0 \longrightarrow E \xrightleftharpoons[\varepsilon]{k^0} E^0 \xrightleftharpoons[\delta^0]{k^1} E^1 \xrightleftharpoons[\delta^1]{k^2} \dots \xrightleftharpoons[\delta^{n-1}]{k^n} E^n \xrightleftharpoons[\delta^n]{k^{n+1}} \dots$$

Note however that it is not required that k^i be Γ -equivariant. A resolution is *strong* if it admits a contracting homotopy.

PROPOSITION 4.13. *Let V be a normed $R[\Gamma]$ -space, and let $\varepsilon: V \rightarrow C_b^0(\Gamma, V)$ be defined by $\varepsilon(v)(g) = v$ for every $v \in V$, $g \in \Gamma$. Then the augmented complex*

$$0 \longrightarrow V \xrightarrow{\varepsilon} C_b^0(\Gamma, V) \xrightarrow{\delta^0} C_b^1(\Gamma, V) \rightarrow \dots \rightarrow C_b^n(\Gamma, V) \rightarrow \dots$$

provides a relatively injective strong resolution of V .

PROOF. We already know that each $C_b^i(\Gamma, V)$ is relatively injective, so in order to conclude it is sufficient to observe that the map

$$k^{n+1}: C_b^{n+1}(\Gamma, V) \rightarrow C_b^n(\Gamma, V) \quad k^{n+1}(f)(g_0, \dots, g_n) = f(1, g_0, \dots, g_n)$$

provides a contracting homotopy for the resolution $(V, C_b^\bullet(\Gamma, V), \delta^\bullet)$. \square

The resolution described in Proposition 4.13 is the *standard resolution* of V as a normed $R[\Gamma]$ -module.

REMARK 4.14. Let us briefly compare our notion of standard resolution with Ivanov's and Monod's ones. In [Iva87], for every $n \in \mathbb{N}$ the set $C_b^n(\Gamma, \mathbb{R})$ is denoted by $B(\Gamma^{n+1})$, and it is endowed with the structure of a *right* Banach Γ -module by the action $g \cdot f(g_0, \dots, g_n) = f(g_0, \dots, g_n \cdot g)$. Moreover, the sequence of modules $B(\Gamma^n)$, $n \in \mathbb{N}$, is equipped with a structure of Γ -complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{d_{-1}} B(\Gamma) \xrightarrow{d_0} B(\Gamma^2) \xrightarrow{d_1} \dots \xrightarrow{d_n} B(\Gamma^{n+2}) \xrightarrow{d_{n+1}} \dots,$$

where $d_{-1}(t)(g) = t$ and

$$\begin{aligned} d_n(f)(g_0, \dots, g_{n+1}) &= (-1)^{n+1} f(g_1, \dots, g_{n+1}) \\ &\quad + \sum_{i=0}^n (-1)^{n-i} f(g_0, \dots, g_i g_{i+1}, \dots, g_{n+1}) \end{aligned}$$

for every $n \geq 0$ (here we are using Ivanov's notation also for the differential). Now, it is readily seen that (in the case with trivial real coefficients) Ivanov's resolution is isomorphic to our standard resolution via the isometric Γ -chain isomorphism $\varphi^\bullet: B^\bullet(\Gamma) \rightarrow C_b^\bullet(\Gamma, \mathbb{R})$ defined by

$$\varphi^n(f)(g_0, \dots, g_n) = f(g_n^{-1}, g_n^{-1} g_{n-1}^{-1}, \dots, g_n^{-1} g_{n-1}^{-1} \cdots g_1^{-1} g_0^{-1})$$

(with inverse $(\varphi^n)^{-1}(f)(g_0, \dots, g_n) = f(g_n^{-1} g_{n-1}, g_{n-1}^{-1} g_{n-2}, \dots, g_1^{-1} g_0, g_0^{-1})$). We also observe that the contracting homotopy described in Proposition 4.13 is conjugated by φ^\bullet into Ivanov's contracting homotopy for the complex $(B(\Gamma^\bullet), d_\bullet)$ (which is defined in [Iva87]).

Our notation is much closer to Monod's. In fact, in [Mon01] the more general case of a topological group Γ is addressed, and the n -th module of the standard Γ -resolution of an $\mathbb{R}[\Gamma]$ -module V is inductively defined by setting

$$C_b^0(\Gamma, V) = C_b(\Gamma, V), \quad C_b^n(\Gamma, V) = C_b(\Gamma, C_b^{n-1}(\Gamma, V)),$$

where $C_b(\Gamma, E)$ denotes the space of *continuous* bounded maps from Γ to the Banach space E . However, as observed in [Mon01, Remarks 6.1.2 and 6.1.3], the case when Γ is an abstract group may be recovered from the general case just by equipping Γ with the discrete topology. In that case, our notion of standard resolution coincides with Monod's (see also [Mon01, Remark 7.4.9]).

The following result implies that any relatively injective strong resolution of a normed Γ -module V may be used to compute the cohomology modules $H_b^\bullet(\Gamma, V)$.

THEOREM 4.15 ([Mon01, Lemmas 7.2.4 and 7.2.6]). *Let $\alpha: E \rightarrow F$ be a Γ -map between normed $R[\Gamma]$ -modules, let $(E, E^\bullet, \delta_E^\bullet)$ be a strong resolution of E , and suppose $(F, F^\bullet, \delta_F^\bullet)$ is an augmented complex such that F^i is relatively injective for every $i \geq 0$. Then α extends to a chain map α^\bullet , and any two extensions of α to chain maps are Γ -homotopic.*

PROOF. The constructions both of an extension α^\bullet of α and of a Γ -homotopy between any two such extensions were carried out in the proof of Theorem 4.5. In fact, using that the contracting homotopy of the complex $(E, E^\bullet, \delta_E^\bullet)$ has operator norm bounded by 1 in every degree, one can observe that the proof of Theorem 4.5 applies *verbatim* to the context of normed $R[\Gamma]$ -modules to provide the required bounded Γ -maps. \square

COROLLARY 4.16. *Let V be a normed $R[\Gamma]$ -module, and let $(V, V^\bullet, \delta_V^\bullet)$ be a relatively injective strong resolution of V . Then for every $n \in \mathbb{N}$ there is a canonical isomorphism*

$$H_b^n(\Gamma, V) \cong H_b^n(V^\bullet).$$

Moreover, this isomorphism is bi-Lipschitz with respect to the seminorms of $H_b^n(\Gamma, V)$ and $H^n(V^\bullet)$.

PROOF. By Proposition 4.13, the standard resolution of V is a relatively injective strong resolution of V over Γ . Therefore, Theorem 4.15 provides chain maps between $C_b^\bullet(\Gamma, V)$ and V^\bullet , which are one the Γ -homotopy inverse of the other. Therefore, these chain maps induce isomorphisms in cohomology. The conclusion follows from the fact that bounded chain maps induce bounded maps in cohomology. \square

By Corollary 4.16, every relatively injective strong resolution of V induces a seminorm on $H_b^\bullet(\Gamma, V)$. Moreover, the seminorms defined in this way are pairwise equivalent. However, in many applications, it is important to be able to compute the exact *canonical* seminorm of elements in $H_b^n(\Gamma, V)$, i.e. the seminorm induced on $H_b^n(\Gamma, V)$ by the standard resolution $C_b^\bullet(\Gamma, V)$. Unfortunately, it is not possible to capture the isometry type of $H_b^n(\Gamma, V)$ via arbitrary relatively injective strong resolutions. Therefore, a special role is played by those resolutions which compute the canonical seminorm. The following fundamental result is due to Ivanov, and implies that these distinguished resolutions are in some sense extremal:

THEOREM 4.17. *Let V be a normed $R[\Gamma]$ -module, and let $(V, V^\bullet, \delta^\bullet)$ be any strong resolution of V . Then the identity of V can be extended to a chain map α^\bullet between V^\bullet and the standard resolution of V , in such a way that $\|\alpha^n\| \leq 1$ for every $n \geq 0$. In particular, the canonical seminorm of $H_b^\bullet(\Gamma, V)$ is not bigger than the seminorm induced on $H_b^\bullet(\Gamma, V)$ by any relatively injective strong resolution.*

PROOF. One can inductively define α^n by setting, for every $v \in E^n$ and $g_j \in \Gamma$:

$$\alpha^n(v)(g_0, \dots, g_n) = \alpha^{n-1} \left(g_0 \left(k^n(g_0^{-1}(v)) \right) \right) (g_1, \dots, g_n),$$

where k^\bullet is a contracting homotopy for the given resolution $(V, V^\bullet, \delta^\bullet)$. It is not difficult to prove by induction that α^\bullet is indeed a norm non-increasing chain Γ -map (see [Iva87], [Mon01, Theorem 7.3.1] for the details). \square

COROLLARY 4.18. *Let V be a normed $R[\Gamma]$ -module, let $(V, V^\bullet, \delta^\bullet)$ be a relatively injective strong resolution of V , and suppose that the identity of V can be extended to a chain map $\alpha^\bullet: C_b^\bullet(\Gamma, V) \rightarrow V^\bullet$ such that $\|\alpha^n\| \leq 1$ for every $n \in \mathbb{N}$. Then α^\bullet induces an isometric isomorphism between $H_b^\bullet(\Gamma, V)$ and $H^\bullet(V^\bullet)$. In particular, the seminorm induced by the resolution $(V, V^\bullet, \delta^\bullet)$ coincides with the canonical seminorm on $H_b^\bullet(\Gamma, V)$.*

4.8. More on amenability

The following result establishes an interesting relationship between the amenability of Γ and the relative injectivity of normed $\mathbb{R}[\Gamma]$ -modules.

PROPOSITION 4.19. *The following facts are equivalent:*

- (1) *The group Γ is amenable.*
- (2) *Every dual normed $\mathbb{R}[\Gamma]$ -module is relatively injective.*
- (3) *The trivial $\mathbb{R}[\Gamma]$ -module \mathbb{R} is relatively injective.*

PROOF. (1) \Rightarrow (2): Let W be a normed $\mathbb{R}[\Gamma]$ -module, and let $V = W'$ be the dual normed $\mathbb{R}[\Gamma]$ -module of W . We first construct a left inverse (over $\mathbb{R}[\Gamma]$) of the augmentation map $\varepsilon: V \rightarrow C_b^0(\Gamma, V)$. We fix an invariant mean m on Γ . For $f \in C_b^0(\Gamma, V)$ and $w \in W$ we consider the function

$$f_w: \Gamma \rightarrow \mathbb{R}, \quad f_w(g) = f(g)(w).$$

It follows from the definitions that f_w is an element of $\ell^\infty(\Gamma)$, so we may define a map $r: C_b^0(\Gamma, V) \rightarrow V$ by setting $r(f)(w) = m(f_w)$. It is immediate to check that $r(f)$ is indeed a bounded functional on W , whose norm is bounded by $\|f\|_\infty$. In other words, the map r is well defined and norm non-increasing. The Γ -invariance of the mean m implies that r is Γ -equivariant, and an easy computation shows that $r \circ \varepsilon = \text{Id}_V$.

Let us now consider the diagram

$$\begin{array}{ccc}
 0 & \longrightarrow & A & \xrightleftharpoons[\iota]{\sigma} & B & . \\
 & & \downarrow \alpha & & \swarrow \beta & \\
 & & V & & & \\
 & & \downarrow \varepsilon & & \swarrow \beta' & \\
 & & C_b^0(\Gamma, V) & & &
 \end{array}$$

By Lemma 4.12, $C_b^0(\Gamma, V)$ is relatively injective, so there exists a bounded $\mathbb{R}[\Gamma]$ -map β' such that $\|\beta'\| \leq \|\varepsilon \circ \alpha\| \leq \|\alpha\|$ and $\beta' \circ \iota = \varepsilon \circ \alpha$. The $\mathbb{R}[\Gamma]$ -map $\beta := r \circ \beta'$ satisfies $\|\beta\| \leq \|\alpha\|$ and $\beta \circ \iota = \alpha$. This shows that V is relatively injective.

(2) \Rightarrow (3) is obvious, so we are left to show that (3) implies (1). If \mathbb{R} is relatively injective and $\sigma: \ell^\infty(\Gamma) \rightarrow \mathbb{R}$ is the map defined by $\sigma(f) = f(1)$, then there exists an $\mathbb{R}[\Gamma]$ -map $\beta: \ell^\infty(\Gamma) \rightarrow \mathbb{R}$ such that $\|\beta\| \leq 1$ and the following diagram commutes:

$$\begin{array}{ccc}
 \mathbb{R} & \xrightleftharpoons[\varepsilon]{\sigma} & \ell^\infty(\Gamma) & . \\
 \text{Id} \downarrow & & \swarrow \beta & \\
 \mathbb{R} & & &
 \end{array}$$

By construction, since $\beta(\mathbf{1}_\Gamma) = 1$, the map β is an invariant continuous non-trivial functional on $\ell^\infty(\Gamma)$, so Γ is amenable by Lemma 3.2. \square

The previous proposition allows us to provide an alternative proof of Theorem 3.6, which we recall here for the convenience of the reader:

THEOREM 3.6. Let Γ be an amenable group, and let V be a dual normed $\mathbb{R}[\Gamma]$ -module. Then $H_b^n(\Gamma, V) = 0$ for every $n \geq 1$.

PROOF. The complex

$$0 \longrightarrow V \xrightarrow{\text{Id}} V \longrightarrow 0$$

provides a relatively injective strong resolution of V , so the conclusion follows from Corollary 4.16. \square

4.9. Amenable spaces

The notion of amenable space was introduced by Zimmer [Zim78] in the context of actions of topological groups on standard measure spaces (see e.g. [Mon01, Section 5.3] for several equivalent definitions). In our case of interest, i.e. when Γ is a discrete group acting on a set S (which may be thought as endowed with the discrete topology), the amenability of S as a Γ -space is equivalent to the amenability of the stabilizers in Γ of elements of S [AEG94, Theorem 5.1]. Therefore, we may take this characterization as a definition:

DEFINITION 4.20. A left action $\Gamma \times S \rightarrow S$ of a group Γ on a set S is *amenable* if the stabilizer of every $s \in S$ is an amenable subgroup of Γ . In this case, we equivalently say that S is an amenable Γ -set.

The importance of amenable Γ -sets is due to the fact that they may be exploited to isometrically compute the bounded cohomology of Γ . If S is any Γ -set and V is any normed $\mathbb{R}[\Gamma]$ -module, then we denote by $\ell^\infty(S^{n+1}, V)$ the space of bounded functions from S^{n+1} to V . This space may be endowed with the structure of a normed $\mathbb{R}[\Gamma]$ -module via the action

$$(g \cdot f)(s_0, \dots, s_n) = g \cdot (f(g^{-1}s_0, \dots, g^{-1}s_n)) .$$

The differential $\delta^n: \ell^\infty(S^{n+1}, V) \rightarrow \ell^\infty(S^{n+2}, V)$ defined by

$$\delta^n(f)(s_0, \dots, s_{n+1}) = \sum_{i=0}^n (-1)^i f(s_0, \dots, \hat{s}_i, \dots, s_{n+1})$$

endows the pair $(\ell^\infty(S^{\bullet+1}, V), \delta^\bullet)$ with the structure of a normed $\mathbb{R}[\Gamma]$ -complex. Together with the augmentation $\varepsilon: V \rightarrow \ell^\infty(S, V)$ given by $\varepsilon(v)(s) = v$ for every $s \in S$, such a complex provides a strong resolution of V :

LEMMA 4.21. *The augmented complex*

$$0 \longrightarrow V \longrightarrow \ell^\infty(S, V) \xrightarrow{\delta^0} \ell^\infty(S^2, V) \xrightarrow{\delta^1} \ell^\infty(S^3, V) \xrightarrow{\delta^2} \dots$$

provides a strong resolution of V .

PROOF. Let s_0 be a fixed element of S . Then the maps

$$k^n: \ell^\infty(S^{n+1}, V) \rightarrow \ell^\infty(S^n, V), \quad k^n(f)(s_1, \dots, s_n) = f(s_0, s_1, \dots, s_n)$$

provide the required contracting homotopy. \square

LEMMA 4.22. *Suppose that S is an amenable Γ -set, and that V is a dual normed $\mathbb{R}[\Gamma]$ -module. Then $\ell^\infty(S^{n+1}, V)$ is relatively injective for every $n \geq 0$.*

PROOF. Since any intersection of amenable subgroups is amenable, the Γ -set S is amenable if and only if S^n is. Therefore, it is sufficient to deal with the case $n = 0$.

Let W be a normed $\mathbb{R}[\Gamma]$ -module such that $V = W'$, and consider the extension problem described in Definition 4.10, with $E = \ell^\infty(S, V)$:

$$\begin{array}{ccc} 0 & \longrightarrow & A & \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\iota} \end{array} & B & . \\ & & \downarrow \alpha & & \swarrow \beta & \\ & & \ell^\infty(S, V) & & & \end{array}$$

We denote by $R \subseteq S$ a set of representatives for the action of Γ on S , and for every $r \in R$ we denote by Γ_r the stabilizer of r , endowed with the invariant mean μ_r . Moreover, for every $s \in S$ we choose an element $g_s \in \Gamma$ such that $g_s^{-1}(s) = r_s \in R$. Then g_s is uniquely determined up to right multiplication by elements in Γ_{r_s} .

Let us fix an element $b \in B$. In order to define $\beta(b)$, for every $s \in S$ we need to know the value taken by $\beta(b)(s)$ on every $w \in W$. Therefore, we fix $s \in S$, $w \in W$, and we set

$$(\beta(b)(s))(w) = \mu_{r_s}(f_b) ,$$

where $f_b \in \ell^\infty(\Gamma_{r_s}, \mathbb{R})$ is defined by

$$f_b(g) = ((g_s g) \cdot \alpha(\sigma(g^{-1} g_s^{-1} b))) (s) (w) .$$

Since $\|\sigma\| \leq 1$ we have that $\|\beta\| \leq \|\alpha\|$, and the behaviour of means on constant functions implies that $\beta \circ \iota = \alpha$.

Observe that the element $\beta(b)(s)$ does not depend on the choice of $g_s \in \Gamma$. In fact, if we replace g_s by $g_s g'$ for some $g' \in \Gamma_{r_s}$, then the function f_b defined above is replaced by the function

$$f'_b(g) = ((g_s g' g) \cdot \alpha(\sigma(g^{-1} (g')^{-1} g_s^{-1} b))) (s) (w) = f_b(g' g) ,$$

and $\mu_{r_s}(f'_b) = \mu_{r_s}(f_b)$ by the invariance of the mean μ_{r_s} . This fact allows us to prove that β is a Γ -map. In fact, let us fix $\bar{g} \in \Gamma$ and let $\bar{s} = \bar{g}^{-1}(s)$. Then we may assume that $g_{\bar{s}} = \bar{g}^{-1} g_s$, so

$$\bar{g}(\beta(b))(s)(w) = \bar{g}(\beta(b)(\bar{g}^{-1}s))(w) = \beta(b)(\bar{s})(\bar{g}^{-1}w) = \mu_{r_s}(\bar{f}_b) ,$$

where $\bar{f}_b \in \ell^\infty(\Gamma_{r_s}, \mathbb{R})$ is given by

$$\begin{aligned} \bar{f}_b(g) &= ((\bar{g}^{-1} g_s g) \cdot \alpha(\sigma(g^{-1} g_s^{-1} \bar{g} b))) (\bar{s}) (\bar{g}^{-1} w) \\ &= ((g_s g) \cdot \alpha(\sigma(g^{-1} g_s^{-1} \bar{g} b))) (s) (w) \\ &= f_{\bar{g} b}(g) . \end{aligned}$$

Therefore,

$$\bar{g}(\beta(b))(s)(w) = \mu_{r_s}(f_{\bar{g} b}) = \beta(\bar{g} b)(s)(w)$$

for every $s \in S$, $w \in W$, i.e. $\bar{g}\beta(b) = \beta(\bar{g}b)$, and we are done. \square

As anticipated above, we are now able to show that amenable spaces may be exploited to compute bounded cohomology:

THEOREM 4.23. *Let S be an amenable Γ -set and let V be a dual normed $\mathbb{R}[\Gamma]$ -module. Let also $(V, V^\bullet, \delta_V^\bullet)$ be a strong resolution of V . Then, there exists a Γ -chain map $\alpha^\bullet: V^\bullet \rightarrow \ell^\infty(S^{\bullet+1}, V)$ which extends Id_V and is norm non-increasing in every degree. In particular, the homology of the complex*

$$0 \longrightarrow \ell^\infty(S, V)^\Gamma \xrightarrow{\delta^0} \ell^\infty(S^2, V)^\Gamma \xrightarrow{\delta^1} \ell^\infty(S^3, V)^\Gamma \xrightarrow{\delta^2} \dots$$

is canonically isometrically isomorphic to $H_b^\bullet(\Gamma, V)$.

PROOF. By Theorem 4.17, in order to prove the first statement of the theorem it is sufficient to assume that $V^\bullet = C_b^\bullet(\Gamma, V)$, so we are left to construct a norm non-increasing chain map

$$\alpha^\bullet: C_b^\bullet(\Gamma, V) \rightarrow \ell^\infty(S^{\bullet+1}, V) .$$

We keep notation from the proof of the previous lemma, i.e. we fix a set of representatives R for the action of Γ on S , and for every $s \in S$ we choose an element $g_s \in \Gamma$ such that $g_s^{-1}s = r_s \in R$. For every $r \in R$ we also fix an invariant mean μ_r on the stabilizer Γ_r of r .

Let us fix an element $f \in C_b^n(\Gamma, V)$, and take $(s_0, \dots, s_n) \in S^{n+1}$. If $V = W'$, we also fix an element $w \in W$. For every i we denote by $r_i \in R$ the representative of the orbit of s_i , and we consider the invariant mean $\mu_{r_0} \times \dots \times \mu_{r_n}$ on $\Gamma_{r_0} \times \dots \times \Gamma_{r_n}$ (see Remark 3.5). Then we consider the function $f_{s_0, \dots, s_n} \in \ell^\infty(\Gamma_{r_0} \times \dots \times \Gamma_{r_n}, \mathbb{R})$ defined by

$$f_{s_0, \dots, s_n}(g_0, \dots, g_n) = f(g_{s_0} g_0, \dots, g_{s_n} g_n)(w) .$$

By construction we have $\|f_{s_0, \dots, s_n}\|_\infty \leq \|f\|_\infty \cdot \|w\|_W$, so we may set

$$(\alpha^n(f)(s_0, \dots, s_n))(w) = (\mu_{r_0} \times \dots \times \mu_{r_n})(f_{s_0, \dots, s_n}),$$

thus defining an element $\alpha^n(f)(s_0, \dots, s_n) \in W' = V$ such that $\|\alpha^n(f)\|_V \leq \|f\|_\infty$. We have thus shown that $\alpha^n: C_b^n(\Gamma, V) \rightarrow \ell^\infty(S^{n+1}, V)$ is a well-defined norm non-increasing linear map. The fact that α^n commutes with the action of Γ follows from the invariance of the means μ_r , $r \in R$, and the fact that α^\bullet is a chain map is obvious.

The fact that the complex $\ell^\infty(S^{\bullet+1}, V)$ isometrically computes the bounded cohomology of Γ with coefficients in V is now straightforward. Indeed, the previous lemmas imply that the augmented complex $(V, \ell^\infty(S^{\bullet+1}, V), \delta^\bullet)$ provides a relatively injective strong resolution of V , so Corollary 4.16 implies that the homology of the Γ -invariants of $(V, \ell^\infty(S^{\bullet+1}, V), \delta^\bullet)$ is isomorphic to the bounded cohomology of Γ with coefficients in V . Thanks to the existence of the norm non-increasing chain map α^\bullet , the fact that this isomorphism is isometric is then a consequence of Corollary 4.18. \square

Let us prove some direct corollaries of the previous results. Observe that, if W is a dual normed $\mathbb{R}[K]$ -module and $\psi: \Gamma \rightarrow K$ is a homomorphism, then the induced module $\psi^{-1}(W)$ is a dual normed $\mathbb{R}[\Gamma]$ -module.

THEOREM 4.24. *Let $\psi: \Gamma \rightarrow K$ be a surjective homomorphism with amenable kernel, and let W be a dual normed $\mathbb{R}[K]$ -module. Then the induced map*

$$H_b^\bullet(K, W) \rightarrow H_b^\bullet(\Gamma, \psi^{-1}W)$$

is an isometric isomorphism.

PROOF. The action $\Gamma \times K \rightarrow K$ defined by $(g, k) \mapsto \psi(g)k$ endows K with the structure of an amenable Γ -set. Therefore, Theorem 4.23 implies that the bounded cohomology of Γ with coefficients in $\psi^{-1}W$ is isometrically isomorphic to the cohomology of the complex $\ell^\infty(K^{\bullet+1}, \psi^{-1}W)^\Gamma$. However, since ψ is surjective, we have a (tautological) isometric identification between $\ell^\infty(K^{\bullet+1}, \psi^{-1}W)^\Gamma$ and $C_b^\bullet(K, W)^K$, whence the conclusion. \square

COROLLARY 4.25. *Let $\psi: \Gamma \rightarrow K$ be a surjective homomorphism with amenable kernel. Then the induced map*

$$H_b^n(K, \mathbb{R}) \rightarrow H_b^n(\Gamma, \mathbb{R})$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

4.10. Alternating cochains

Let V be a normed $\mathbb{R}[\Gamma]$ -module. A cochain $\varphi \in C^n(\Gamma, V)$ is *alternating* if it satisfies the following condition: for every permutation $\sigma \in \mathfrak{S}_{n+1}$ of the set $\{0, \dots, n\}$, if we denote by $\text{sgn}(\sigma) = \pm 1$ the sign of σ , then the equality

$$\varphi(g_{\sigma(0)}, \dots, g_{\sigma(n)}) = \text{sgn}(\sigma) \cdot \varphi(g_0, \dots, g_n)$$

holds for every $(g_0, \dots, g_n) \in \Gamma^{n+1}$. We denote by $C_{\text{alt}}^n(\Gamma, V) \subseteq C^n(\Gamma, V)$ the subset of alternating cochains, and we set $C_{b, \text{alt}}^n(\Gamma, V) = C_{\text{alt}}^n(\Gamma, V) \cap C_b^n(\Gamma, V)$. It is well-known that (bounded) alternating cochains provide a Γ -subcomplex of general (bounded) cochains. In fact, it turns out that, in the case of real coefficients, one can compute the (bounded) cohomology of Γ via the complex of alternating cochains:

PROPOSITION 4.26. *The complex $C_{\text{alt}}^{\bullet}(\Gamma, V)$ (resp. $C_{b,\text{alt}}^{\bullet}(\Gamma, V)$) isometrically computes the cohomology (resp. the bounded cohomology) of Γ with real coefficients.*

PROOF. We concentrate our attention on bounded cohomology, the case of ordinary cohomology being identical. The inclusion $j^{\bullet}: C_{b,\text{alt}}^{\bullet}(\Gamma, V) \rightarrow C_b^{\bullet}(\Gamma, V)$ induces a norm non-increasing map on bounded cohomology, so in order to prove the proposition it is sufficient to construct a norm non-increasing Γ -chain map $\text{alt}_b^{\bullet}: C_b^{\bullet}(\Gamma, V) \rightarrow C_{b,\text{alt}}^{\bullet}(\Gamma, V)$ which satisfies the following properties:

- (1) alt_b^n is a retraction onto the subcomplex of alternating cochains, i.e. $\text{alt}_b^n \circ j^n = \text{Id}$ for every $n \geq 0$;
- (2) $j^{\bullet} \circ \text{alt}_b^{\bullet}$ is Γ -homotopic to the identity of $C_b^{\bullet}(\Gamma, V)$ (as usual, via a homotopy which is bounded in every degree).

For every $\varphi \in C_b^n(\Gamma, V)$, $(g_0, \dots, g_n) \in \Gamma^{n+1}$ we set

$$\text{alt}_b^n(\varphi)(g_0, \dots, g_n) = \frac{1}{(n+1)!} \sum_{\sigma \in \mathfrak{S}_{n+1}} \text{sgn}(\sigma) \cdot \varphi(g_{\sigma(0)}, \dots, g_{\sigma(n)}).$$

It is easy to check that alt_b^{\bullet} satisfies the required properties (the fact that it is indeed homotopic to the identity of $C_b^{\bullet}(\Gamma, V)$ may be deduced, for example, by the computations carried out in the context of singular chains in [FM11, Appendix B]). \square

Let S be an amenable Γ -set. We have seen in Theorem 4.23 that the bounded cohomology of Γ with coefficients in the normed $\mathbb{R}[\Gamma]$ -module V is isometrically isomorphic to the cohomology of the complex $\ell^{\infty}(S^{\bullet+1}, V)^{\Gamma}$. The very same argument described in the proof of Proposition 4.26 shows that the bounded cohomology of Γ is computed also by the subcomplex of alternating elements of $\ell^{\infty}(S^{\bullet+1}, V)^{\Gamma}$. More precisely, let us denote by $\ell_{\text{alt}}^{\infty}(S^n, V)$ the submodule of alternating elements of $\ell^{\infty}(S^n, V)$ (the definition of alternating being obvious). Then we have the following:

THEOREM 4.27. *Let S be an amenable Γ -set and let V be a dual normed $\mathbb{R}[\Gamma]$ -module. Then the homology of the complex*

$$0 \longrightarrow \ell_{\text{alt}}^{\infty}(S, V)^{\Gamma} \xrightarrow{\delta^0} \ell_{\text{alt}}^{\infty}(S^2, V)^{\Gamma} \xrightarrow{\delta^1} \ell_{\text{alt}}^{\infty}(S^3, V)^{\Gamma} \xrightarrow{\delta^2} \dots$$

is canonically isometrically isomorphic to $H_b^{\bullet}(\Gamma, V)$.

4.11. Further readings

The study of bounded cohomology of groups by means of homological algebra was initiated by Brooks [Bro81] and developed by Ivanov [Iva87], who first managed to construct a theory that allowed the use of resolutions for the *isometric* computation of bounded cohomology (see also [Nos91] for the case with twisted coefficients). Ivanov's ideas were then generalized and further developed by Burger and Monod [BM99, BM02, Mon01] in order to deal with topological groups (their theory lead also to a deeper understanding of the bounded cohomology of discrete groups, for example in the case of lattices in Lie groups). In the case of discrete groups, an introduction to bounded cohomology following Ivanov's approach may also be found in [Löh].

From the point of view of homological algebra, bounded cohomology is a pretty exotic theory, since it fails excision and thus it cannot be easily studied via any (generalized) Mayer-Vietoris principle. As observed in the introduction of [Büh11], this seemed to suggest that (continuous) bounded cohomology could not be interpreted as a derived functor and that triangulated methods could not apply to its study. On the contrary, it is proved in [Büh11] that the formalism of exact categories and their derived categories can be exploited to construct a classical derived functor on the category of Banach Γ -modules with values in Waelbroeck’s abelian category. Building on this fact, Bühler provided an axiomatic characterization of bounded cohomology, and illustrated how the theory of bounded cohomology can be interpreted in the standard framework of homological and homotopical algebra.

Amenable actions. There exist at least two quite different notions of amenable actions in the literature. Namely, suppose that the group Γ acts via Borel isomorphisms on a topological space X . Then one can say that the action is amenable if X supports a Γ -invariant Borel probability measure. This is the case, for example, whenever the action has a finite orbit O , because in that case a Γ -invariant probability measure is obtained just by taking the (normalized) counting measure supported on O . This definition of amenable action dates back to [Gre69], and it is not the best suited to bounded cohomology. Indeed, our definition of amenable action (for a discrete group on a discrete space) is a very particular instance of a more general notion due to Zimmer [Zim78, Zim84]. Zimmer’s original definition of amenable action is quite involved, but it turns out to be equivalent to the existence of a Γ -invariant conditional expectation $L^\infty(\Gamma \times X) \rightarrow L^\infty(X)$, or to the fact that the equivalence relation on X defined by the action is amenable, together with the amenability of the stabilizer of $x \in X$ for almost every $x \in X$ (this last characterization readily implies that our Definition 4.20 indeed coincides with Zimmer’s one in the case of actions on discrete spaces). For a discussion of the equivalence of these definitions of Zimmer’s amenability we refer the reader to [Mon01, II.5.3], which summarizes and discusses results from [AEG94]. Amenable actions also admit a characterization in the language of homological algebra: an action of Γ on a space X is amenable if and only if the Γ -module $L^\infty(X)$ is relatively injective. This implies that amenable spaces may be exploited to compute (and, in fact, to isometrically compute) the bounded cohomology of groups (in the case of discrete spaces, this is just Theorem 4.23).

Both definitions of amenable actions we have just discussed may be generalized to the case when Γ is a locally compact topological group. As noted in [GM07], these two notions are indeed quite different, and in fact somewhat “dual” one to the other: for example, a trivial action is always amenable in the sense of Greenleaf, while it is amenable in the sense of Zimmer precisely when the group Γ is itself amenable.

An important example of amenable action is given by the action of any countable group Γ on its Poisson boundary [Zim78]. Such an action is also doubly ergodic [Kai03], and this readily implies that the second bounded cohomology of a group Γ may be isometrically identified with the space of Γ -invariant 2-cocycles on its Poisson boundary. This fact has proved to be extremely powerful for the computation of low-dimensional bounded cohomology modules for discrete and locally compact groups (see e.g. [BM99, BM02]). As an example, let us recall that

Bouarich proved that, if $\varphi: \Gamma \rightarrow H$ is a surjective homomorphism, then the induced map $H_b^2(\varphi): H_b^2(H, \mathbb{R}) \rightarrow H_b^2(\Gamma, \mathbb{R})$ is injective (see Theorem 2.17). Indeed, the morphism φ induces a measurable map from a suitably chosen Poisson boundary of Γ to a suitably chosen Poisson boundary of H , and this implies in turn that the map $H_b^2(\varphi)$ is in fact an isometric isomorphism [**Hub12**, Theorem 2.14].

Bounded cohomology of topological spaces

Let X be a topological space, and let $R = \mathbb{Z}, \mathbb{R}$. Recall that $C_\bullet(X, R)$ (resp. $C^\bullet(X, R)$) denotes the usual complex of singular chains (resp. cochains) on X with coefficients in R , and $S_i(X)$ is the set of singular i -simplices in X . We also regard $S_i(X)$ as a subset of $C_i(X, R)$, so that for any cochain $\varphi \in C^i(X, R)$ it makes sense to consider its restriction $\varphi|_{S_i(X)}$. For every $\varphi \in C^i(X, R)$, we set

$$\|\varphi\| = \|\varphi\|_\infty = \sup \{|\varphi(s)| \mid s \in S_i(X)\} \in [0, \infty].$$

We denote by $C_b^\bullet(X, R)$ the submodule of bounded cochains, i.e. we set

$$C_b^\bullet(X, R) = \{\varphi \in C^\bullet(X, R) \mid \|\varphi\| < \infty\} .$$

Since the differential takes bounded cochains into bounded cochains, $C_b^\bullet(X, R)$ is a subcomplex of $C^\bullet(X, R)$. We denote by $H^\bullet(X, R)$ (resp. $H_b^\bullet(X, R)$) the homology of the complex $C^\bullet(X, R)$ (resp. $C_b^\bullet(X, R)$). Of course, $H^\bullet(X, R)$ is the usual singular cohomology module of X with coefficients in R , while $H_b^\bullet(X, R)$ is the *bounded cohomology module* of X with coefficients in R . Just as in the case of groups, the norm on $C^i(X, R)$ descends (after taking the suitable restrictions) to a seminorm on each of the modules $H^\bullet(X, R)$, $H_b^\bullet(X, R)$. More precisely, if $\varphi \in H$ is a class in one of these modules, which is obtained as a quotient of the corresponding module of cocycles Z , then we set

$$\|\varphi\|_\infty = \inf \{\|\psi\|_\infty \mid \psi \in Z, [\psi] = \varphi \text{ in } H\} .$$

This seminorm may take the value ∞ on elements in $H^\bullet(X, R)$ and may be null on non-zero elements in $H_b^\bullet(X, R)$ (but not on non-zero elements in $H^\bullet(X, R)$): this is clear in the case with integer coefficients, and it is a consequence of the Universal Coefficient Theorem in the case with real coefficients, since a real cohomology class with vanishing seminorm has to be null on any cycle, whence null in $H^\bullet(X, \mathbb{R})$.

The inclusion of bounded cochains into possibly unbounded cochains induces the *comparison map*

$$c^\bullet : H_b^\bullet(X, R) \rightarrow H^\bullet(X, R) .$$

5.1. Basic properties of bounded cohomology of spaces

Bounded cohomology enjoys some of the fundamental properties of classical singular cohomology: for example, $H_b^i(\{\text{pt.}\}, R) = 0$ if $i > 0$, and $H_b^0(\{\text{pt.}\}, R) = R$ (more in general, $H_b^0(X, R)$ is canonically isomorphic to $\ell^\infty(S)$, where S is the set of path connected components of X). The usual proof of the homotopy invariance of singular homology is based on the construction of an algebraic homotopy which maps every n -simplex to the sum of at most $n + 1$ simplices of dimension $n + 1$. As a consequence, the homotopy operator induced in cohomology preserves bounded cochains. This implies that bounded cohomology is a homotopy invariant of topological spaces. Moreover, if (X, Y) is a topological pair, then there is an

obvious definition of $H_b^\bullet(X, Y)$, and it is immediate to check that the analogous of the long exact sequence of the pair in classical singular cohomology also holds in the bounded case.

Perhaps the most important phenomenon that distinguishes bounded cohomology from classical singular cohomology is the lacking of any Mayer-Vietoris sequence (or, equivalently, of any excision theorem). In particular, spaces with finite-dimensional bounded cohomology may be tamely glued to each other to get spaces with infinite-dimensional bounded cohomology (see Remark 5.6 below).

Recall from Section 1.2 that $H^n(X, R) \cong H^n(\pi_1(X), R)$ for every aspherical CW-complex X . As anticipated in Section 1.6, a fundamental result by Gromov provides an isometric isomorphism $H_b^n(X, \mathbb{R}) \cong H_b^n(\pi_1(X), \mathbb{R})$ even without any assumption on the asphericity of X . This section is devoted to a proof of Gromov's result. We will closely follow Ivanov's argument [Iva87], which deals with the case when X is (homotopically equivalent to) a countable CW-complex. Before going into Ivanov's argument, we will concentrate our attention on the easier case of aspherical spaces.

5.2. Bounded singular cochains as relatively injective modules

Henceforth, we assume that X is a path connected topological space admitting the universal covering \tilde{X} , we denote by Γ the fundamental group of X , and we fix an identification of Γ with the group of covering automorphisms of \tilde{X} . Just as we did in Section 4.4 for $C^\bullet(\tilde{X}, R)$, we endow $C_b^\bullet(\tilde{X}, R)$ with the structure of a normed $R[\Gamma]$ -module. Our arguments are based on the obvious but fundamental isometric identification

$$C_b^\bullet(X, R) \cong C_b^\bullet(\tilde{X}, R)^\Gamma,$$

which induces a canonical isometric identification

$$H_b^\bullet(X, R) \cong H^\bullet(C_b^\bullet(\tilde{X}, R)^\Gamma).$$

As a consequence, in order to prove the isomorphism $H_b^\bullet(X, R) \cong H_b^\bullet(\Gamma, R)$ it is sufficient to show that the complex $C_b^\bullet(\tilde{X}, R)$ provides a relatively injective strong resolution of R (an additional argument shows that this isomorphism is also isometric). The relative injectivity of the modules $C_b^n(\tilde{X}, R)$ can be easily deduced from the argument described in the proof of Lemma 4.7, which applies verbatim in the context of bounded singular cochains:

LEMMA 5.1. *For every $n \in \mathbb{N}$, the bounded cochain module $C_b^n(\tilde{X}, R)$ is relatively injective.*

Therefore, in order to show that the bounded cohomology of X is isomorphic to the bounded cohomology of Γ we need to show that the (augmented) complex $C_b^\bullet(\tilde{X}, R)$ provides a strong resolution of R . We will show that this is the case if X is aspherical. In the general case, it is not even true that $C_b^\bullet(\tilde{X}, R)$ is acyclic (see Remark 5.12). However, in the case when $R = \mathbb{R}$ a deep result by Ivanov shows that the complex $C_b^\bullet(\tilde{X}, \mathbb{R})$ indeed provides a strong resolution of \mathbb{R} . A sketch of Ivanov's proof will be given in Section 5.4.

Before going on, we point out that we always have a norm non-increasing map from the bounded cohomology of Γ to the bounded cohomology of X :

LEMMA 5.2. *Let us endow the complexes $C_b^\bullet(\Gamma, R)$ and $C_b^\bullet(\tilde{X}, R)$ with the obvious augmentations. Then, there exists a norm non-increasing Γ -equivariant chain map*

$$r^\bullet : C_b^\bullet(\Gamma, R) \rightarrow C_b^\bullet(\tilde{X}, R)$$

extending the identity of R .

PROOF. Let us choose a set of representatives F for the action of Γ on \tilde{X} . We consider the map $r_0 : S_0(\tilde{X}) = \tilde{X} \rightarrow \Gamma$ taking a point x to the unique $g \in \Gamma$ such that $x \in g(F)$. For $n \geq 1$, we define $r_n : S_n(\tilde{X}) \rightarrow \Gamma^{n+1}$ by setting $r_n(s) = (r_0(s(e_0)), \dots, r_0(s(e_n)))$, where $s(e_i)$ is the i -th vertex of s . Finally, we extend r_n to $C_n(\tilde{X}, R)$ by R -linearity and define r^n as the dual map of r_n . Since r_n takes every single simplex to a single $(n+1)$ -tuple, it is readily seen that r^n is norm non-increasing (in particular, it takes bounded cochains into bounded cochains). The fact that r^n is a Γ -equivariant chain map is obvious. \square

COROLLARY 5.3. *Let X be a path connected topological space. Then, for every $n \in \mathbb{N}$ there exists a natural norm non-increasing map*

$$H_b^n(\Gamma, R) \rightarrow H_b^n(X, R).$$

PROOF. By Lemma 5.2, there exists a norm non-increasing chain map

$$r^\bullet : C_b^\bullet(\Gamma, R) \rightarrow C_b^\bullet(\tilde{X}, R)$$

extending the identity of R , so we may define the desired map $H_b^n(\Gamma, R) \rightarrow H_b^n(X, R)$ to be equal to $H_b^n(r^\bullet)$. We know that every $C_b^n(\tilde{X}, R)$ is relatively injective as an $R[\Gamma]$ -module, while $C_b^\bullet(\Gamma, R)$ provides a strong resolution of R over $R[\Gamma]$, so Theorem 4.15 ensures that $H_b^n(r^\bullet)$ does not depend on the choice of the particular chain map r^\bullet which extends the identity of R . \square

A very natural question is whether the map provided by Corollary 5.3 is an (isometric) isomorphism. In the following sections we will see that this holds true when X is aspherical or when R is equal to the field of real numbers. The fact that $H_b^n(\Gamma, R)$ is isometrically isomorphic to $H_b^n(X, R)$ when X is aspherical is not surprising, and just generalizes the analogous result for classical cohomology: the (bounded) cohomology of a group Γ may be defined as the (bounded) cohomology of any aspherical CW-complex having Γ as fundamental group; this is a well-posed definition since any two such spaces are homotopically equivalent, and (bounded) cohomology is a homotopy invariant. On the other hand, the fact that the isometric isomorphism $H_b^n(\Gamma, \mathbb{R}) \cong H_b^n(X, \mathbb{R})$ holds even without the assumption that X is aspherical is a very deep result due to Gromov [Gro82].

We begin by proving the following:

LEMMA 5.4. *Let Y be a path connected topological space. If Y is aspherical, then the augmented complex $C_b^\bullet(Y, R)$ admits a contracting homotopy (so it is a strong resolution of R). If $\pi_i(Y) = 0$ for every $i \leq n$, then there exists a partial contracting homotopy*

$$R \xleftarrow{k^0} C_b^0(Y, R) \xleftarrow{k^1} C_b^1(Y, R) \xleftarrow{k^2} \dots \xleftarrow{k^n} C_b^n(Y, R) \xleftarrow{k^{n+1}} C_b^{n+1}(Y, R)$$

(where we require that the equality $\delta^{m-1}k^m + k^{m+1}\delta^m = \text{Id}_{C_b^m(Y,R)}$ holds for every $m \leq n$, and $\delta^{-1} = \varepsilon$ is the usual augmentation).

PROOF. For every $-1 \leq m \leq n$ we construct a map

$$T_m: C_m(Y, R) \rightarrow C_{m+1}(Y, R)$$

sending every single simplex into a single simplex in such a way that $d_{m+1}T_m + T_{m-1}d_m = \text{Id}_{C_m(Y,R)}$, where we understand that $C_{-1}(Y, R) = R$ and $d_0: C_0(Y, R) \rightarrow R$ is the augmentation map $d_0(\sum r_i y_i) = \sum r_i$. Let us fix a point $y_0 \in Y$, and define $T_{-1}: R \rightarrow C_0(Y, R)$ by setting $T_{-1}(r) = r y_0$. For $m \geq 0$ we define T_m as the R -linear extension of a map $T_m: S_m(Y) \rightarrow S_{m+1}(Y)$ having the following property: for every $s \in S_m(Y)$, the 0-th vertex of $T_m(s)$ is equal to y_0 , and has s as opposite face. We proceed by induction, and suppose that T_i has been defined for every $-1 \leq i \leq m$. Take $s \in S_m(Y)$. Then, using the fact that $\pi_m(Y) = 0$ and the properties of T_{m-1} , it is not difficult to show that a simplex $s' \in S_{m+1}(Y)$ exists which satisfies both the equality $d_{m+1}s' = s - T_{m-1}(d_m s)$ and the additional requirement described above. We set $T_{m+1}(s) = s'$, and we are done.

Since T_{m-1} sends every single simplex to a single simplex, its dual map k^m sends bounded cochains into bounded cochains, and has operator norm equal to one. Therefore, the maps $k^m: C_b^m(Y, R) \rightarrow C_b^{m-1}(Y, R)$, $m \leq n+1$, provide the desired (partial) contracting homotopy. \square

5.3. The aspherical case

We are now ready to show that, under the assumption that X is aspherical, the bounded cohomology of X is isometrically isomorphic to the bounded cohomology of Γ (both with integral and with real coefficients):

THEOREM 5.5. *Let X be an aspherical space, i.e. a path connected topological space such that $\pi_n(X) = 0$ for every $n \geq 2$. Then $H_b^n(X, R)$ is isometrically isomorphic to $H_b^n(\Gamma, R)$ for every $n \in \mathbb{N}$.*

PROOF. Lemmas 5.1 and 5.4 imply that the complex

$$0 \longrightarrow R \xrightarrow{\varepsilon} C_b^0(\tilde{X}, R) \xrightarrow{\delta^0} C_b^1(\tilde{X}, R) \xrightarrow{\delta^1} \dots$$

provides a relatively injective strong resolution of R as a trivial $R[\Gamma]$ -module, so $H_b^n(X, R)$ is canonically isomorphic to $H_b^n(\Gamma, R)$ for every $n \in \mathbb{N}$. The fact the the isomorphism $H_b^n(X, R) \cong H_b^n(\Gamma, R)$ is isometric is a consequence of Corollary 4.18 and Lemma 5.2. \square

REMARK 5.6. Let $S^1 \vee S^1$ be the wedge of two copies of the circle. Then Theorem 5.5 implies that $H_b^2(S^1 \vee S^1, \mathbb{R}) \cong H_b^2(F_2, \mathbb{R})$ is infinite-dimensional, while $H_b^2(S^1, \mathbb{R}) \cong H_b^2(\mathbb{Z}, \mathbb{R}) = 0$. This shows that bounded cohomology of spaces cannot satisfy any Mayer-Vietoris principle.

5.4. Ivanov's contracting homotopy

We now come back to the general case, i.e. we do not assume that X is aspherical. In order to show that $H_b^n(X, R)$ is isometrically isomorphic to $H_b^n(\Gamma, R)$ we need to prove that the complex of singular bounded cochains on \tilde{X} provides a strong resolution of R . In the case when $R = \mathbb{Z}$, this is false in general, since the complex $C_b^\bullet(\tilde{X}, \mathbb{Z})$ may even be non-exact (see Remark 5.12). On the other hand,

a deep result by Ivanov ensures that $C_b^\bullet(\tilde{X}, \mathbb{R})$ indeed provides a strong resolution of \mathbb{R} .

Ivanov's argument makes use of sophisticated techniques from algebraic topology, which work under the assumption that X , whence \tilde{X} , is a countable CW-complex (but see Remark 5.10). We begin with the following:

LEMMA 5.7 ([Iva87], Theorem 2.2). *Let $p: Z \rightarrow Y$ be a principal H -bundle, where H is an abelian topological group. Then there exists a chain map $A^\bullet: C_b^\bullet(Z, \mathbb{R}) \rightarrow C_b^\bullet(Y, \mathbb{R})$ such that $\|A^n\| \leq 1$ for every $n \in \mathbb{N}$ and $A^\bullet \circ p^\bullet$ is the identity of $C_b^\bullet(Y, \mathbb{R})$.*

PROOF. For $\varphi \in C_b^n(Z, \mathbb{R})$, $s \in S_n(Y)$, the value $A(\varphi)(s)$ is obtained by suitably averaging the value of $\varphi(s')$, where s' ranges over the set

$$P_s = \{s' \in S_n(Z) \mid p \circ s' = s\}.$$

More precisely, let $K_n = S_n(H)$ be the space of continuous functions from the standard n -simplex to H , and define on K_n the operation given by pointwise multiplication of functions. With this structure, K_n is an abelian group, so it admits an invariant mean μ_n . Observe that the permutation group \mathfrak{S}_{n+1} acts on Δ^n via affine transformations. This action induces an action on K_n , whence on $\ell^\infty(K_n)$, and on the space of means on K_n , so there is an obvious notion of \mathfrak{S}_{n+1} -invariant mean on K_n . By averaging over the action of \mathfrak{S}_{n+1} , the space of K_n -invariant means may be retracted onto the space \mathcal{M}_n of \mathfrak{S}_{n+1} -invariant K_n -invariant means on K_n , which, in particular, is non-empty. Finally, observe that any affine identification of Δ^{n-1} with a face of Δ^n induces a map $\mathcal{M}_n \rightarrow \mathcal{M}_{n-1}$. Since elements of \mathcal{M}_n are \mathfrak{S}_{n+1} -invariant, this map does not depend on the chosen identification. Therefore, we get a sequence of maps

$$\mathcal{M}_0 \longleftarrow \mathcal{M}_1 \longleftarrow \mathcal{M}_2 \longleftarrow \mathcal{M}_3 \longleftarrow \dots$$

Recall now that the Banach-Alaouglu Theorem implies that every \mathcal{M}_n is compact (with respect to the weak- $*$ topology on $\ell^\infty(K_n)'$). This easily implies that there exists a sequence $\{\mu_n\}$ of means such that $\mu_n \in \mathcal{M}_n$ and $\mu_n \mapsto \mu_{n-1}$ under the map $\mathcal{M}_n \rightarrow \mathcal{M}_{n-1}$. We say that such a sequence is *compatible*.

Let us now fix $s \in S_n(Y)$, and observe that there is a bijection between P_s and K_n . This bijection is uniquely determined up to the choice of an element in P_s , i.e. up to left multiplication by an element of K_n . In particular, if $f \in \ell^\infty(P_s)$, and μ is a left-invariant mean on K_n , then there is a well-defined value $\mu(f)$.

Let us now choose a compatible sequence of means $\{\mu_n\}_{n \in \mathbb{N}}$. We define the operator A^n by setting

$$A^n(\varphi)(s) = \mu_n(\varphi|_{P_s}) \quad \text{for every } \varphi \in C_b^n(Z), s \in S_n(Y).$$

Since the sequence $\{\mu_n\}$ is compatible, the sequence of maps A^\bullet is a chain map. The inequality $\|A^n\| \leq 1$ is obvious, and the fact that $A^\bullet \circ p^\bullet$ is the identity of $C_b^\bullet(Y, \mathbb{R})$ may be deduced from the behaviour of means on constant functions. \square

THEOREM 5.8 ([Iva87]). *Let X be a path connected countable CW-complex with universal covering \tilde{X} . Then the (augmented) complex $C_b^\bullet(\tilde{X}, \mathbb{R})$ provides a relatively injective strong resolution of \mathbb{R} .*

PROOF. We only sketch Ivanov's argument, referring the reader to [Iva87] for full details. Building on results by Dold and Thom [DT58], Ivanov constructs an infinite tower of bundles

$$X_1 \xleftarrow{p_1} X_2 \xleftarrow{p_2} X_3 \longleftarrow \dots \longleftarrow X_n \longleftarrow \dots$$

where $X_1 = \tilde{X}$, $\pi_i(X_m) = 0$ for every $i \leq m$, $\pi_i(X_m) = \pi_i(X)$ for every $i > m$ and each map $p_m: X_{m+1} \rightarrow X_m$ is a principal H_m -bundle for some topological connected abelian group H_m , which has the homotopy type of a $K(\pi_{m+1}(X), m)$.

By Lemma 5.4, for every n we may construct a partial contracting homotopy

$$\mathbb{R} \xleftarrow{k_n^0} C_b^0(X_n, \mathbb{R}) \xleftarrow{k_n^1} C_b^1(X_n, \mathbb{R}) \xleftarrow{k_n^2} \dots \xleftarrow{k_n^{n+1}} C_b^{n+1}(X_n, \mathbb{R}).$$

Moreover, Lemma 5.7 implies that for every $m \in \mathbb{N}$ the chain map

$$p_m^\bullet: C_b^\bullet(X_m, \mathbb{R}) \rightarrow C_b^\bullet(X_{m+1}, \mathbb{R})$$

admits a left inverse chain map $A_m^\bullet: C_b^\bullet(X_{m+1}, \mathbb{R}) \rightarrow C_b^\bullet(X_m, \mathbb{R})$ which is norm non-increasing. This allows us to define a partial contracting homotopy

$$\mathbb{R} \xleftarrow{k^0} C_b^0(X, \mathbb{R}) \xleftarrow{k^1} C_b^1(X, \mathbb{R}) \xleftarrow{k^2} \dots \xleftarrow{k^{n+1}} C_b^{n+1}(X, \mathbb{R}) \longleftarrow \dots$$

via the formula

$$k^i = A_1^{i-1} \circ \dots \circ A_{n-1}^{i-1} \circ k_n^i \circ p_{n-1}^i \circ \dots \circ p_2^i \circ p_1^i \quad \text{for every } i \leq n+1.$$

The existence of such a partial homotopy is sufficient for all our applications. In order to construct a complete contracting homotopy one should check that the definition of k^i does not depend on n . This is not automatically true, and it is equivalent to the fact that, for $i \leq n+1$, the equality $A_n^i \circ k_{n+1}^i \circ p_n^i = k_n^i$ holds. In order to get this, one has to coherently choose both the basepoints and the cones involved in the construction providing the (partial) contracting homotopy k_n^\bullet (see Lemma 5.4). The fact that this can be achieved ultimately depends on the fact that the fibers of the bundle $X_{n+1} \rightarrow X_n$ have the homotopy type of a $K(\pi_{n+1}(X), n)$. We refer the reader to [Iva87] for the details. \square

5.5. Gromov's Theorem

The discussion in the previous section implies the following:

THEOREM 5.9 ([Gro82, Iva87]). *Let X be a countable CW-complex. Then $H_b^n(X, \mathbb{R})$ is canonically isometrically isomorphic to $H_b^n(\Gamma, \mathbb{R})$. An explicit isometric isomorphism is induced by the map $r^\bullet: C_b^\bullet(\Gamma, \mathbb{R})^\Gamma \rightarrow C_b^\bullet(\tilde{X}, \mathbb{R})^\Gamma = C_b^\bullet(X, \mathbb{R})$ described in Lemma 5.2.*

PROOF. Observe that, if X is a countable CW-complex, then $\Gamma = \pi_1(X)$ is countable, so \tilde{X} is also a countable CW-complex. Therefore, Lemma 5.1 and Theorem 5.8 imply that the complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{\varepsilon} C_b^0(\tilde{X}, \mathbb{R}) \xrightarrow{\delta^0} C_b^1(\tilde{X}, \mathbb{R}) \xrightarrow{\delta^1} \dots$$

provides a relatively injective strong resolution of \mathbb{R} as a trivial $\mathbb{R}[\Gamma]$ -module, so $H_b^n(X, \mathbb{R})$ is canonically isomorphic to $H_b^n(\Gamma, \mathbb{R})$ for every $n \in \mathbb{N}$. The fact the the isomorphism $H_b^n(X, \mathbb{R}) \cong H_b^n(\Gamma, \mathbb{R})$ induced by r^\bullet is isometric is a consequence of Corollary 4.18 and Lemma 5.2. \square

REMARK 5.10. In a recent preprint, Ivanov extended Theorem 5.8 to show that $C_b^\bullet(X, \mathbb{R})$ is a strong resolution of \mathbb{R} whenever X is any (not necessarily countable) CW-complex [Iva]. Using the fact that weak homotopy equivalences induce isometric isomorphisms in bounded cohomology (see again [Iva]), he was then able to prove Theorem 5.9 without any assumption on the topological space X . A completely different proof of Theorem 5.9, based on Gromov's theory of multicomplexes, is given in [FM11] for path connected spaces with infinitely many points.

COROLLARY 5.11 (Gromov Mapping Theorem). *Let X, Y be path connected countable CW-complexes and let $f: X \rightarrow Y$ be a continuous map inducing an epimorphism $f_*: \pi_1(X) \rightarrow \pi_1(Y)$ with amenable kernel. Then*

$$H_b^n(f): H_b^n(Y, \mathbb{R}) \rightarrow H_b^n(X, \mathbb{R})$$

is an isometric isomorphism for every $n \in \mathbb{N}$.

PROOF. The explicit description of the isomorphism between the bounded cohomology of a space and the one of its fundamental group implies that the diagram

$$\begin{array}{ccc} H_b^n(Y) & \xrightarrow{H_b^n(f)} & H_b^n(X) \\ \updownarrow & & \updownarrow \\ H_b^n(\pi_1(Y)) & \xrightarrow{H_b^n(f_*)} & H_b^n(\pi_1(X)) \end{array}$$

is commutative, where the vertical arrows represent the isometric isomorphisms of Theorem 5.9. Therefore, the conclusion follows from Corollary 4.25. \square

REMARK 5.12. Theorem 5.9 does not hold for bounded cohomology with integer coefficients. In fact, if X is any topological space, then the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$ induces an exact sequence

$$H_b^n(X, \mathbb{R}) \longrightarrow H^n(X, \mathbb{R}/\mathbb{Z}) \longrightarrow H_b^{n+1}(X, \mathbb{Z}) \longrightarrow H_b^{n+1}(X, \mathbb{R})$$

(see the proof of Proposition 2.13). If X is a simply connected CW-complex, then $H_b^n(X, \mathbb{R}) = 0$ for every $n \geq 1$, so we have

$$H_b^n(X, \mathbb{Z}) \cong H_b^{n-1}(X, \mathbb{R}/\mathbb{Z}) \quad \text{for every } n \geq 2.$$

For example, in the case of the 2-dimensional sphere we have

$$H_b^3(S^2, \mathbb{Z}) \cong H^2(S^2, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}.$$

5.6. Alternating cochains

We have seen in Section 4.10 that (bounded) cohomology of groups may be computed via the complex of alternating cochains. The same holds true also in the context of (bounded) singular cohomology of topological spaces.

If $\sigma \in \mathfrak{S}_{n+1}$ is any permutation of $\{0, \dots, n\}$, then we denote by $\bar{\sigma}: \Delta^n \rightarrow \Delta^n$ the affine automorphism of the standard simplex which induces the permutation σ on the vertices of Δ^n . Then we say that a cochain $\varphi \in C^n(X, \mathbb{R})$ is *alternating* if

$$\varphi(s) = \text{sgn}(\sigma) \cdot \varphi(s \circ \bar{\sigma})$$

for every $s \in S_n(X)$, $\sigma \in \mathfrak{S}_{n+1}$. We also denote by $C_{\text{alt}}^\bullet(X, \mathbb{R}) \subseteq C^\bullet(X, \mathbb{R})$ the sub-complex of alternating cochains, and we set $C_{b, \text{alt}}^\bullet(X, \mathbb{R}) = C_{\text{alt}}^\bullet(X, \mathbb{R}) \cap C_b^\bullet(X, \mathbb{R})$.

Once a generic cochain $\varphi \in C^n(X, \mathbb{R})$ is given, we may alternate it by setting

$$\text{alt}^n(\varphi)(s) = \frac{1}{(n+1)!} \sum_{\sigma \in \mathfrak{S}_{n+1}} \text{sgn}(\sigma) \cdot \varphi(s \circ \bar{\sigma})$$

for every $s \in S_n(X)$. Then the very same argument exploited in the proof of Proposition 4.26 applies in this context to give the following:

PROPOSITION 5.13. *The complex $C_{\text{alt}}^\bullet(X, \mathbb{R})$ (resp. $C_{b,\text{alt}}^\bullet(X, \mathbb{R})$) isometrically computes the cohomology (resp. the bounded cohomology) of X with real coefficients.*

5.7. Relative bounded cohomology

When dealing with manifolds with boundary, it is often useful to study *relative* homology and cohomology. For example, in Section 7.5 we will show how the simplicial volume of a manifold with boundary M can be computed via the analysis of the *relative* bounded cohomology module of the pair $(M, \partial M)$. This will prove useful to show that the simplicial volume is additive with respect to gluings along boundary components with amenable fundamental groups.

Until the end of the chapter, all the cochain and cohomology modules will be assumed to be with real coefficients. Let Y be a subspace of the topological space X . We denote by $C^n(X, Y)$ the submodule of cochains which vanish on simplices supported in Y . In other words, $C^n(X, Y)$ is the kernel of the map $C^n(X) \rightarrow C^n(Y)$ induced by the inclusion $Y \hookrightarrow X$. We also set $C_b^n(X, Y) = C^n(X, Y) \cap C_b^n(X)$, and we denote by $H^\bullet(X, Y)$ (resp. $H_b^\bullet(X, Y)$) the cohomology of the complex $C^\bullet(X, Y)$ (resp. $C_b^\bullet(X, Y)$). The well-known short exact sequence of the pair for ordinary cohomology also holds for bounded cohomology: the short exact sequence of complexes

$$0 \longrightarrow C_b^\bullet(X, Y) \longrightarrow C_b^\bullet(X) \longrightarrow C_b^\bullet(Y) \longrightarrow 0$$

induces the long exact sequence

$$(9) \quad \dots \longrightarrow H_b^n(Y) \longrightarrow H_b^{n+1}(X, Y) \longrightarrow H_b^{n+1}(X) \longrightarrow H_b^{n+1}(Y) \longrightarrow \dots$$

Recall now that, if the fundamental group of every component of Y is amenable, then $H_b^n(Y) = 0$ for every $n \geq 1$, so the inclusion $j^n: C_b^n(X, Y) \rightarrow C_b^n(X)$ induces a norm non-increasing isomorphism

$$H_b^n(j^n): H_b^n(X, Y) \rightarrow H_b^n(X)$$

for every $n \geq 2$. The following result is proved in [BBF⁺14], and shows that this isomorphism is in fact isometric:

THEOREM 5.14. *Let (X, Y) be a pair of countable CW-complexes, and suppose that the fundamental group of each component of Y is amenable. Then the map*

$$H_b^n(j^n): H_b^n(X, Y) \rightarrow H_b^n(X)$$

is an isometric isomorphism for every $n \geq 2$.

The rest of this section is devoted to the proof of Theorem 5.14.

Henceforth we assume that (X, Y) is a pair of countable CW-complexes such that the fundamental group of every component of Y is amenable. Let $p: \tilde{X} \rightarrow X$ be a universal covering of X , and set $\tilde{Y} = p^{-1}(Y)$. As usual, we denote by Γ the

fundamental group of X , we fix an identification of Γ with the group of the covering automorphisms of p , and we consider the induced identification

$$C_b^n(X) = C_b^n(\tilde{X})^\Gamma .$$

DEFINITION 5.15. We say that a cochain $\varphi \in C_b^n(\tilde{X})$ is *special* (with respect to \tilde{Y}) if the following conditions hold:

- φ is alternating;
- let s, s' be singular n -simplices with values in \tilde{X} and suppose that, for every $i = 0, \dots, n$, if $s(w_i) \neq s'(w_i)$ then $s(w_i)$ and $s'(w_i)$ belong to the same connected component of \tilde{Y} , where w_0, \dots, w_n are the vertices of the standard n -simplex (in other words, the projections of s and s' onto the space obtained from \tilde{X} by collapsing each connected component of \tilde{Y} to a point share the same vertices). Then $\varphi(s) = \varphi(s')$.

We denote by $C_{bs}^\bullet(\tilde{X}, \tilde{Y}) \subseteq C_b^\bullet(\tilde{X})$ the subcomplex of special cochains, and we set

$$C_{bs}^\bullet(X, Y) = C_{bs}^\bullet(\tilde{X}, \tilde{Y}) \cap C_b^\bullet(\tilde{X})^\Gamma \subseteq C_b^\bullet(\tilde{X})^\Gamma = C_b^\bullet(X) .$$

REMARK 5.16. Any cochain $\varphi \in C_{bs}^\bullet(\tilde{X}, \tilde{Y})$ vanishes on every simplex having two vertices on the same connected component of \tilde{Y} . In particular

$$C_{bs}^n(X, Y) \subseteq C_b^n(X, Y) \subseteq C_b^n(X)$$

for every $n \geq 1$.

We denote by $l^\bullet: C_{bs}^\bullet(X, Y) \rightarrow C_b^\bullet(X)$ the natural inclusion.

PROPOSITION 5.17. *There exists a norm non-increasing chain map*

$$\eta^\bullet: C_b^\bullet(X) \rightarrow C_{bs}^\bullet(X, Y)$$

such that the composition $l^\bullet \circ \eta^\bullet$ is chain-homotopic to the identity of $C_b^\bullet(X)$.

PROOF. Let us briefly describe the strategy of the proof. First of all, we will define a Γ -set S which provides a sort of discrete approximation of the pair (\tilde{X}, \tilde{Y}) . As usual, the group Γ already provides an approximation of \tilde{X} . However, in order to prove that Y is completely irrelevant from the point of view of bounded cohomology, we need to approximate every component of \tilde{Y} by a single point, and this implies that the set S cannot coincide with Γ itself. Basically, we add one point for each component of \tilde{Y} . Since the fundamental group of every component of Y is amenable, the so obtained Γ -set S is amenable: therefore, the bounded cohomology of Γ , whence of X , may be isometrically computed using the complex of alternating cochains on S . Finally, alternating cochains on S can be isometrically translated into special cochains on X .

Let us now give some more details. Let $Y = \sqcup_{i \in I} C_i$ be the decomposition of Y into the union of its connected components. If \tilde{C}_i is a choice of a connected component of $p^{-1}(C_i)$ and Γ_i denotes the stabilizer of \tilde{C}_i in Γ then

$$p^{-1}(C_i) = \bigsqcup_{\gamma \in \Gamma/\Gamma_i} \gamma \tilde{C}_i .$$

We endow the set

$$S = \Gamma \sqcup \bigsqcup_{i \in I} \Gamma/\Gamma_i$$

with the obvious structure of Γ -set and we choose a set of representatives $\mathcal{F} \subset \tilde{X} \setminus \tilde{Y}$ for the Γ -action on $\tilde{X} \setminus \tilde{Y}$. We define a Γ -equivariant map $r: \tilde{X} \rightarrow S$ as follows:

$$r(\gamma x) = \begin{cases} \gamma \in \Gamma & \text{if } x \in \mathcal{F}, \\ \gamma \Gamma_i \in \Gamma/\Gamma_i & \text{if } x \in \tilde{C}_i. \end{cases}$$

For every $n \geq 0$ we set $\ell_{\text{alt}}^\infty(S^{n+1}) = \ell_{\text{alt}}^\infty(S^{n+1}, \mathbb{R})$ and we define

$$r^n: \ell_{\text{alt}}^\infty(S^{n+1}) \rightarrow C_{bs}^\bullet(\tilde{X}, \tilde{Y}), \quad r^n(f)(s) = f(r(s(e_0)), \dots, r(s(e_n))).$$

The fact that r^n takes values in the module of special cochains is immediate, and clearly r^\bullet is a norm non-increasing Γ -equivariant chain map extending the identity on \mathbb{R} .

Recall now that, by Theorem 5.8, the complex $C_b^\bullet(\tilde{X})$ provides a relatively injective strong resolution of \mathbb{R} , so there exists a norm non-increasing Γ -equivariant chain map $C_b^\bullet(\tilde{X}) \rightarrow C_b^\bullet(\Gamma, \mathbb{R})$. Moreover, by composing the map provided by Theorem 4.23 and the obvious altering operator we get a norm non-increasing Γ -equivariant chain map $C_b^\bullet(\Gamma, \mathbb{R}) \rightarrow \ell_{\text{alt}}^\infty(S^{\bullet+1})$. By composing these morphisms of normed Γ -complexes we finally get a norm non-increasing Γ -chain map

$$\zeta^\bullet: C_b^\bullet(\tilde{X}) \rightarrow \ell_{\text{alt}}^\infty(S^{\bullet+1})$$

which extends the identity of \mathbb{R} .

Let us now consider the composition $\theta^\bullet = r^\bullet \circ \zeta^\bullet: C_b^\bullet(\tilde{X}) \rightarrow C_{bs}^\bullet(\tilde{X}, \tilde{Y})$: it is a norm non-increasing Γ -chain map which extends the identity of \mathbb{R} . It is now easy to check that the chain map $\eta^\bullet: C_b^\bullet(X) \rightarrow C_{bs}^\bullet(X, Y)$ induced by θ^\bullet satisfies the required properties. In fact, η^\bullet is obviously norm non-increasing. Moreover, the composition of θ^\bullet with the inclusion $C_{bs}^\bullet(\tilde{X}, \tilde{Y}) \rightarrow C_b^\bullet(\tilde{X})$ extends the identity of \mathbb{R} . Since $C_b^\bullet(\tilde{X})$ is a relatively injective strong resolution of \mathbb{R} , this implies in turn that this composition is Γ -homotopic to the identity of \mathbb{R} , thus concluding the proof. \square

COROLLARY 5.18. *Let $n \geq 1$, take $\alpha \in H_b^n(X)$ and let $\varepsilon > 0$ be given. Then there exists a special cocycle $f \in C_{bs}^n(X, Y)$ such that*

$$[f] = \alpha, \quad \|f\|_\infty \leq \|\alpha\|_\infty + \varepsilon.$$

Recall now that $C_{bs}^n(X, Y) \subseteq C_b^n(X, Y)$ for every $n \geq 1$. Therefore, Corollary 5.18 implies that, for $n \geq 1$, the norm of every coclass in $H_b^n(X)$ may be computed by taking the infimum over relative cocycles. Since we already know that the inclusion $C_b^\bullet(X, Y) \hookrightarrow C_b^\bullet(X)$ induces an isomorphism in bounded cohomology in degree greater than one, this concludes the proof of Theorem 5.14.

5.8. Further readings

It would be interesting to extend to the relative case Gromov's and Ivanov's results on the coincidence of bounded cohomology of (pairs of) spaces with the bounded cohomology of their (pairs of) fundamental groups. The case when the subspace is path connected corresponds to the case when the pair of groups indeed consists of a group and one of its subgroups, and it is somewhat easier than the general case. It was first treated by Park in [Par03], and then by Pagliantini and the author in [FP12]. Rather disappointingly, Ivanov's cone construction runs into some difficulties in the relative case, so some extra assumptions on higher homotopy groups is needed in order to get the desired isometric isomorphism (if

one only requires a bi-Lipschitz isomorphism, then the request that the subspace is π_1 -injective in the whole space is sufficient).

In the case of a disconnected subspace it is first necessary to define bounded cohomology for pairs (Γ, Γ') , where Γ' is a family of subgroups of Γ . This was first done for classical cohomology of groups by Bieri and Eckmann [BE78], and extended to the case of bounded cohomology by Mineyev and Yaman [MY07] (see also [Fra]). Probably the best suited approach to bounded cohomology of generic pairs is via the theory of bounded cohomology for *groupoids*, as introduced and developed by Blank in [Bla16]. Many results from [FP12] are extended in [Bla16] to the case of disconnected subspaces (under basically the same hypotheses on higher homotopy groups that were required in [FP12]).

The theory of multicomplexes initiated by Gromov in [Gro82] may also be exploited to study (relative) bounded cohomology of (pairs of) spaces. We refer the reader to [Kue15, FM] for some results in this direction.