Preface

In representation theory of Lie algebras, *Casimir operators* are commonly understood as certain expressions constructed from generators of a Lie algebra which commute with its action. Their spectra are useful for understanding the representation. In particular, finite-dimensional irreducible representations of a simple Lie algebra $\mathfrak{g}$ over the field of complex numbers are characterized by the eigenvalues of the Casimir operators. This fact is based on a theorem of Harish-Chandra describing the center $\mathrm{Z}(\mathfrak{g})$ of the associated universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. The center is isomorphic to an algebra of polynomials via the *Harish-Chandra isomorphism*

$$\mathrm{Z}(\mathfrak{g}) \cong \mathbb{C}\big[L_1, \dots, L_n\big]. \tag{0.1}$$

Here n is the rank of $\mathfrak{g}$ and $L_1, \dots, L_n$ are polynomial functions in the highest weights of the representations, each L_i is invariant under a certain action of the Weyl group of $\mathfrak{g}$. The isomorphism (0.1) relies on a theorem of Chevalley which can also be recovered as a 'classical limit' of (0.1). Namely, the symmetric algebra $\mathrm{S}(\mathfrak{g})$ is isomorphic to the graded algebra gr $\mathrm{U}(\mathfrak{g})$, and the subalgebra of $\mathfrak{g}$-invariants in $\mathrm{S}(\mathfrak{g})$ is isomorphic to gr $\mathrm{Z}(\mathfrak{g})$. Taking the symbols M_i of the polynomials L_i, we get the *Chevalley isomorphism*

$$\mathrm{S}(\mathfrak{g})^{\mathfrak{g}} \cong \mathbb{C}\big[M_1, \dots, M_n\big]. \tag{0.2}$$

The respective degrees $d_1, \dots, d_n$ of the Weyl group invariants $M_1, \dots, M_n$ coincide with the exponents of $\mathfrak{g}$ increased by 1.

A vast amount of literature both in mathematical physics and representation theory has been devoted to understanding the correspondence in (0.1) in terms of concrete generators on both sides, especially for the Lie algebras $\mathfrak{g}$ of classical types A, B, C and D. Various families of generators of the center $\mathrm{Z}(\mathfrak{g})$ were discovered together with their Harish-Chandra images.

The simple Lie algebras $\mathfrak{g}$ can be regarded as a part of the family of *Kac–Moody algebras* parameterized by generalized Cartan matrices. Of particular importance is the class of *affine Kac–Moody algebras* which admits a simple presentation. The (untwisted) affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ is the central extension $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ of the Lie algebra of Laurent polynomials with coefficients in $\mathfrak{g}$. Basic results of representation theory of these Lie algebras together with applications to conformal field theory, modular forms and soliton equations can be found in the book by V. Kac [**86**]. Motivated by the significance of the Lie algebras $\widehat{\mathfrak{g}}$, one comes to wonder what the center of $\mathrm{U}(\widehat{\mathfrak{g}})$ looks like. However, this straight question turns out to be too naive to have a meaningful answer. First of all, the enveloping algebra is 'too small' to contain central elements beyond polynomials in K. The canonical quadratic Casimir element is already a formal series of elements of the algebra $\mathrm{U}(\widehat{\mathfrak{g}})$, so it is necessary to consider its completion. As a natural choice, one requires that the action of elements for such a completion is well-defined on certain *smooth modules* over $\widehat{\mathfrak{g}}$. Secondly, the central element K must be given a unique constant value known as the *critical level*. With a standard choice of the invariant bilinear form on $\mathfrak{g}$, this value is the negative of the dual Coxeter number, $K = -h^\vee$. The suitably completed universal enveloping algebra $\widetilde{\mathrm{U}}_{-h^\vee}(\widehat{\mathfrak{g}})$ at the critical level does contain a large center $\mathrm{Z}(\widehat{\mathfrak{g}})$, and the qualified question has a remarkably comprehensive answer which is explained in detail in the book by E. Frenkel [**46**]. Namely, similar to (0.1), the center $\mathrm{Z}(\widehat{\mathfrak{g}})$ is a completion of the algebra of polynomials

$$\mathbb{C}\big[S_{1\,[r]}, \dots, S_{n\,[r]} \mid r \in \mathbb{Z}\big]$$

in infinitely many variables. Moreover, the elements $S_{i\,[r]}$ which are known as *Sugawara operators*, can be produced from a family of generators $S_1, \dots, S_n$ of a commutative differential algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ by employing instruments of the *vertex algebra theory*: the vacuum module at the critical level over $\widehat{\mathfrak{g}}$ is equipped with a vertex algebra structure, and $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the *center* of this vertex algebra.

Thus the key to understanding the center $\mathrm{Z}(\widehat{\mathfrak{g}})$ lies within the smaller object $\mathfrak{z}(\widehat{\mathfrak{g}})$. Its structure was described by a theorem of B. Feigin and E. Frenkel [**39**] and hence is known as the *Feigin–Frenkel center*. The theorem states that $\mathfrak{z}(\widehat{\mathfrak{g}})$ is an algebra of polynomials

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \mathbb{C}\big[T^r S_1, \dots, T^r S_n \mid r = 0, 1, \dots\big],$$

where T is a derivation defined as the *translation operator* of the vertex algebra. For type A this theorem can be derived from a previous work of R. Goodman and N. Wallach [**58**], and for types A, B, C from an independent work of T. Hayashi [**65**]. Both papers were concerned with a derivation of the character formula for the irreducible quotient $L(\lambda)$ of the Verma module $M(\lambda)$ at the critical level over $\widehat{\mathfrak{g}}$. The Sugawara operators form a commuting family of $\widehat{\mathfrak{g}}$-endomorphisms of $M(\lambda)$ which leads to a computation of the character and thus proves the Kac–Kazhdan conjecture [**89**]. Our choice for the title of the book was motivated by the terminology used in both pioneering papers [**58**] and [**65**], although the term *Segal–Sugawara operators* is also common in the literature. The origins of the terminology go back to the paper by H. Sugawara [**144**] and an unpublished work of Graeme Segal; see e.g. I. Frenkel [**52**]. We chose to reserve the longer name, *Segal–Sugawara vectors*, for elements of $\mathfrak{z}(\widehat{\mathfrak{g}})$ to make a clearer distinction between the vectors and operators.

More recently, new families of Segal–Sugawara vectors were constructed by A. Chervov and D. Talalaev [**24**] for type A, by the author [**110**] in types B, C and D, and in joint work with E. Ragoucy and N. Rozhkovskaya [**116**] in type G_2. Furthermore, these constructions lead to a direct proof of the Feigin–Frenkel theorem in those cases relying on an affine analogue of the Chevalley isomorphism (0.2). This analogue provides an isomorphism

$$\mathrm{S}\left(t^{-1}\mathfrak{g}[t^{-1}]\right)^{\mathfrak{g}[t]} \cong \mathbb{C}\big[T^r M_1, \dots, T^r M_n \mid r = 0, 1, \dots\big],$$

for the 'classical limit' of $\mathfrak{z}(\widehat{\mathfrak{g}})$, and is due to M. Raïs and P. Tauvel [**134**] and to A. Beilinson and V. Drinfeld; see [**46**, Theorem 3.4.2].

Our goal in the book is to review these constructions of Segal–Sugawara vectors and to give an introduction to the subject. We hope that together with the general results explained in the book [**46**], they would bring more content to make the beautiful theory more accessible via concrete examples. The explicit Segal–Sugawara vectors will also be used in the applications of the theory as envisaged by the seminal work of B. Feigin, E. Frenkel and N. Reshetikhin [**40**]. Elements $S \in \mathfrak{z}(\widehat{\mathfrak{g}})$ give rise to Hamiltonians of the Gaudin model describing quantum spin chain. Their eigenvalues on the Bethe vectors can be calculated by using an affine version of the Harish-Chandra isomorphism for the algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$. The role of the invariant polynomials occurring in (0.1) will now be played by elements of the *classical* $\mathcal{W}$*-algebra* $\mathcal{W}({}^L\mathfrak{g})$ associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$. In parallel to the finite-dimensional theory, the affine Harish-Chandra isomorphism can be understood via the action of elements of the center $\mathrm{Z}(\widehat{\mathfrak{g}})$ in the *Wakimoto modules* over $\widehat{\mathfrak{g}}$: central elements act by scalar multiplication with the scalars interpreted as the Harish-Chandra images.

As another application of the constructions of Segal–Sugawara vectors, an explicit solution of E. Vinberg's quantization problem [**149**] will be given. It is based on the general results of L. Rybnikov [**139**] and B. Feigin, E. Frenkel and V. Toledano Laredo [**42**] which provide algebraically independent families of generators of commutative subalgebras of $\mathrm{U}(\mathfrak{g})$ from generators of the algebra $\mathfrak{z}(\widehat{\mathfrak{g}})$.

All constructions of the Segal–Sugawara vectors which we discuss in the book can be explained in a uniform way with the use of the *fusion procedure* allowing one to represent primitive idempotents for the centralizer algebras associated with representations of $\mathfrak{g}$, as products of rational *R-matrices*. This approach is therefore applicable, in principle, to all simple Lie algebras, depending on the availability of such a procedure. Its development for the exceptional types would give a uniform description of the Feigin–Frenkel center.

An R-matrix is a solution of the *Yang–Baxter equation*. Given such a solution, one can define the corresponding algebra by an *RTT-relation*, where the generators of the algebra are combined into a matrix. This general approach originated in the work of L. Faddeev and the St. Petersburg (Leningrad) school on the *quantum inverse scattering method* in the early 1980s. Motivated by this work, V. Drinfeld [**30**] and M. Jimbo [**81**] came to the discovery of *quantum groups*. Deformations of universal enveloping algebras in the class of Hopf algebras form one of the most important families of quantum groups. The presentations of these Hopf algebras involving R-matrices give rise to special algebraic methods often referred to as the *R-matrix techniques*, to investigate their structure and representations; see e.g. [**32**], [**96**], [**136**] and references therein for more details on the origins of the methods. Moreover, these techniques can also be used to study the underlying Lie algebras themselves to bring new insights into their properties. It is these techniques which will underpin our approach. As a starting point, we will consider their applications to the simple Lie algebras $\mathfrak{g}$ of classical types. Then we apply the R-matrix techniques to the corresponding affine Kac–Moody algebras $\widehat{\mathfrak{g}}$ and a class of quantum groups $\mathrm{Y}(\mathfrak{g})$ known as *Yangians*. In both cases, the defining relations of the algebras will be written in terms of certain generator matrices which can be understood as 'operators' on the space of tensors $(\mathbb{C}^N)^{\otimes m}$ with coefficients in the respective algebras. Therefore, as essential role will be played by the *Schur–Weyl duality* involving natural actions of the classical Lie algebras on the space $(\mathbb{C}^N)^{\otimes m}$ and the commuting actions of the symmetric group in type A or the Brauer algebra in types B, C and D.

We will begin by reviewing constructions of primitive idempotents for the symmetric group and the Brauer algebra based on the respective fusion procedures which provide multiplicative R-matrix formulas for these idempotents (Chapter 1). We apply them to construct invariants in symmetric algebras $\mathrm{S}(\mathfrak{g})$ in Chapter 2. Then we use the R-matrix techniques to derive some basic algebraic properties of *Manin matrices* (Chapter 3). They will be applied for constructions of Casimir elements for the general linear Lie algebras (Chapter 4). Similar constructions based on symmetrizers and anti-symmetrizers for the Brauer algebra will be used for the orthogonal and symplectic Lie algebras (Chapter 5).

In Chapter 6 we introduce the center of the affine vertex algebra at the critical level associated with the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$. We will produce explicit generators of the center in the classical types in Chapters 7 and 8 and show how this leads to a proof of the Feigin–Frenkel theorem. In Chapter 9 the generators are used to construct commutative subalgebras of the classical universal enveloping algebras which 'quantize' the shift of argument subalgebras of the symmetric algebras.

Our calculation of the Harish-Chandra images of the Segal–Sugawara vectors will be based on explicit formulas for the characters of some finite-dimensional representations of the Yangian $\mathrm{Y}(\mathfrak{g})$. The R-matrix techniques will play a key role in the derivation of the character formulas which we review in Chapters 10 and 11. In Chapter 12 we discuss the classical $\mathcal{W}$-algebras and construct their generators. The images of the Segal–Sugawara vectors with respect to an affine version of the Harish-Chandra isomorphism will be calculated in Chapter 13. This will produce special families of generators of the classical $\mathcal{W}$-algebra $\mathcal{W}({}^L\mathfrak{g})$ associated with the Langlands dual Lie algebra ${}^L\mathfrak{g}$. Applications to the Gaudin model will be discussed in Chapter 14. In the final Chapter 15 we will give a construction of the Wakimoto modules over $\widehat{\mathfrak{g}}$ for all classical types and calculate the eigenvalues of the Sugawara operators in these modules.

Bibliographical notes at the end of each chapter contain some comments on the origins of the results and references.

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Alexander Molev
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