CHAPTER 4

Casimir elements for $\mathfrak{gl}_N$

In the following two chapters we construct 'quantum analogues' of the invariants in the symmetric algebras considered in Chapter 2. Those invariants will be 'lifted' to the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ thus providing its central elements. As in Chapter 2, we will focus on such elements produced with the use of idempotents for the symmetric group and Brauer algebra. However, keeping in mind affine counterparts of these constructions, we will depart from our reliance on the adjoint action of the associated group. Instead, we will develop matrix techniques by writing the commutation relations of Lie algebras in a matrix form. This will allow us to work within the Lie algebra settings. The same approach will then be applied to affine Kac–Moody algebras in Chapters 7 and 8.

We will rely on some standard facts about simple Lie algebras and their representations which can be found in the books by Dixmier [**29**], Goodman and Wallach [**59**] and Humphreys [**70**]. We start by recalling matrix presentations of Lie algebras.

4.1. Matrix presentations of simple Lie algebras

Suppose that $\mathfrak{g}$ is a finite-dimensional Lie algebra over $\mathbb{C}$ equipped with a nondegenerate symmetric invariant bilinear form $\langle\ ,\ \rangle$. Choose a basis $J^1, \dots, J^d$ of $\mathfrak{g}$ and let $J_1, \dots, J_d$ be its dual with respect to the form so that $\langle J_i, J^k\rangle = \delta_{ik}$. Let π be a faithful representation of $\mathfrak{g}$ afforded by a finite-dimensional vector space V,

$$\pi : \mathfrak{g} \to \operatorname{End} V.$$

Introduce the elements

$$G = \sum_{i=1}^{d} \pi(J^i) \otimes J_i \in \operatorname{End} V \otimes \mathrm{U}(\mathfrak{g})$$

and

$$\Omega = \sum_{i=1}^{d} \pi(J^i) \otimes \pi(J_i) \in \operatorname{End} V \otimes \operatorname{End} V.$$

It is easy to verify that G and Ω are independent of the choice of the basis J^i. In particular,

$$\Omega = \sum_{i=1}^{d} \pi(J_i) \otimes \pi(J^i). \tag{4.1}$$

Consider the tensor product algebra $\operatorname{End} V \otimes \operatorname{End} V \otimes \mathrm{U}(\mathfrak{g})$ and identify Ω with the element $\Omega \otimes 1$. Introduce elements of this algebra by

$$G_1 = \sum_{i=1}^{d} \pi(J^i) \otimes 1 \otimes J_i \qquad \text{and} \qquad G_2 = \sum_{i=1}^{d} 1 \otimes \pi(J^i) \otimes J_i.$$

Write the commutation relations for $\mathfrak{g}$,

$$[J_i, J_j] = \sum_{k=1}^{d} c_{ij}^k \, J_k \tag{4.2}$$

with structure coefficients c_{ij}^k. We will regard the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ as the associative algebra with generators J_i subject to the defining relations (4.2), where the left hand side is understood as the commutator $J_i J_j - J_j J_i$.

Proposition 4.1.1. *The defining relations of* $\mathrm{U}(\mathfrak{g})$ *are equivalent to the matrix relation*

$$G_1 G_2 - G_2 G_1 = -\Omega \, G_2 + G_2 \, \Omega. \tag{4.3}$$

Proof. The left hand side of (4.3) reads

$$\sum_{i,j=1}^{d} \pi(J^i) \otimes \pi(J^j) \otimes (J_i J_j - J_j J_i).$$

For the right hand side we have

$$-\sum_{i,k=1}^{d} \pi(J^i) \otimes \pi\big([J_i, J^k]\big) \otimes J_k. \tag{4.4}$$

By the invariance of the form, we find

$$\langle [J_i, J^k], J_j \rangle = -\langle J^k, [J_i, J_j] \rangle = -c_{ij}^k.$$

Hence (4.4) equals

$$\sum_{i,j,k=1}^{d} c_{ij}^k \, \pi(J^i) \otimes \pi(J^j) \otimes J_k.$$

Since the representation π is faithful, we conclude that (4.3) is equivalent to the defining relations (4.2) of $\mathrm{U}(\mathfrak{g})$. □

The defining relations (4.3) can be written in an equivalent form

$$G_1 G_2 - G_2 G_1 = \Omega \, G_1 - G_1 \, \Omega, \tag{4.5}$$

which is easily verified with the use of (4.1). The element G can be regarded as an $n \times n$ matrix ($n = \dim V$) with entries in $\mathrm{U}(\mathfrak{g})$ so that Proposition 4.1.1 gives a *matrix presentation* of $\mathrm{U}(\mathfrak{g})$. As we will see below in Proposition 4.2.1, single matrix relations (4.3) or (4.5), encoding the commutation relations of $\mathfrak{g}$, are convenient for constructing central elements of $\mathrm{U}(\mathfrak{g})$.

4.2. Harish-Chandra isomorphism

The *center* $\mathrm{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is defined by

$$\mathrm{Z}(\mathfrak{g}) = \{z \in \mathrm{U}(\mathfrak{g}) \mid z\,u = u\,z \quad \text{for all} \ \ u \in \mathrm{U}(\mathfrak{g})\}.$$

Any element of the center is called a *Casimir element* for $\mathfrak{g}$. Since $\mathrm{U}(\mathfrak{g})$ is generated by basis elements of $\mathfrak{g}$, for any $z \in \mathrm{U}(\mathfrak{g})$ we have

$$z \in \mathrm{Z}(\mathfrak{g}) \qquad \text{if and only if} \qquad z\,x = x\,z \quad \text{for all} \quad x \in \mathfrak{g}.$$

This condition can be restricted further to a subset of elements $x \in \mathfrak{g}$ which generate $\mathfrak{g}$ as a Lie algebra. A family of Casimir elements (often called *Gelfand invariants* following [**55**]) can be produced with the use of the matrix presentations introduced in Section 4.1.

PROPOSITION 4.2.1. *All elements* $\operatorname{tr} G^k$ *with* $k \geqslant 1$ *belong to the center of* $\mathrm{U}(\mathfrak{g})$.

PROOF. Relation (4.3) implies

$$G_1\, G_2^k - G_2^k\, G_1 = \sum_{r=1}^{k} G_2^{r-1}(-\Omega\, G_2 + G_2\,\Omega) G_2^{k-r} = -\Omega\, G_2^k + G_2^k\,\Omega.$$

By taking trace over the second copy of $\operatorname{End} V$ and using its cyclic property (Lemma 1.4.1), we get $[G_1, \operatorname{tr}_2 G_2^k] = 0$ as required. □

The Casimir elements $\operatorname{tr} G^k$ are widely used in representation theory, especially for the Lie algebras $\mathfrak{g}$ of classical types. In those cases one usually takes V to be the first fundamental (or vector) representation; see Section 4.8 and Example 5.3.3 below.

Now suppose that $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ as in Chapter 2. Recall that the *Harish-Chandra homomorphism*

$$\mathrm{U}(\mathfrak{g})^{\mathfrak{h}} \to \mathrm{U}(\mathfrak{h}) \tag{4.6}$$

is the projection of the $\mathfrak{h}$-centralizer $\mathrm{U}(\mathfrak{g})^{\mathfrak{h}}$ in the universal enveloping algebra to $\mathrm{U}(\mathfrak{h})$ whose kernel is the two-sided ideal $\mathrm{U}(\mathfrak{g})^{\mathfrak{h}} \cap \mathrm{U}(\mathfrak{g})\mathfrak{n}_+$. This ideal coincides with $\mathrm{U}(\mathfrak{g})^{\mathfrak{h}} \cap \mathfrak{n}_-\, \mathrm{U}(\mathfrak{g})$. The restriction of the homomorphism (4.6) to the center $\mathrm{Z}(\mathfrak{g})$ of $\mathrm{U}(\mathfrak{g})$ yields an isomorphism

$$\chi : \mathrm{Z}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{h})^W \tag{4.7}$$

called the *Harish-Chandra isomorphism*, where $\mathrm{U}(\mathfrak{h})^W$ denotes the subalgebra of invariants in $\mathrm{U}(\mathfrak{h})$ with respect to a certain (shifted) action of the Weyl group W of $\mathfrak{g}$. This leads to a description of the center $\mathrm{Z}(\mathfrak{g})$ as an algebra of polynomials,

$$\mathrm{Z}(\mathfrak{g}) = \mathbb{C}[P_1^\circ, \dots, P_n^\circ],$$

for certain algebraically independent central elements $P_1^\circ, \dots, P_n^\circ$, where n is the rank of $\mathfrak{g}$. The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is equipped with a canonical filtration so that the associated graded algebra gr $\mathrm{U}(\mathfrak{g})$ is isomorphic to the symmetric algebra $\mathrm{S}(\mathfrak{g})$. Moreover, the associated graded algebra gr $\mathrm{Z}(\mathfrak{g})$ is isomorphic to the subalgebra of $\mathfrak{g}$-invariants in $\mathrm{S}(\mathfrak{g})$ as defined in (2.1). For each i denote by P_i the *symbol* of the element P_i°, that is, the image of P_i in the corresponding graded component of $\mathrm{S}(\mathfrak{g})$. Then relation (2.2) holds, and the respective degrees $d_1, \dots, d_n$ of the elements $P_1^\circ, \dots, P_n^\circ$ coincide with the exponents of $\mathfrak{g}$ increased by 1.

Now recall the general linear Lie algebra $\mathfrak{gl}_N$ defined by the commutation relations (2.4). We will regard the basis elements E_{ij} of $\mathfrak{gl}_N$ as generators of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. So we will think of $\mathrm{U}(\mathfrak{gl}_N)$ as the associative algebra with these generators subject to the defining relations

$$E_{ij}\,E_{kl} - E_{kl}\,E_{ij} = \delta_{kj}\,E_{il} - \delta_{il}\,E_{kj}, \qquad i,j,k,l \in \{1,\dots,N\}. \tag{4.8}$$

Taking $\mathcal{A} = \mathrm{U}(\mathfrak{gl}_N)$ in (1.59), combine the elements E_{ij} into the matrix E so that

$$E = \sum_{i,j=1}^{N} e_{ij} \otimes E_{ij} \in \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{gl}_N).$$

Consider the tensor product algebra

$$\operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N \otimes \mathrm{U}(\mathfrak{gl}_N) \tag{4.9}$$

and use the notation of Section 1.4. We identify the permutation operator (1.64) with the element $P \otimes 1$ of the algebra (4.9).

Proposition 4.2.2. *The defining relations of the algebra* $\mathrm{U}(\mathfrak{gl}_N)$ *can be written in the form*

$$E_1\,E_2 - E_2\,E_1 = (E_1 - E_2)\,P. \tag{4.10}$$

Proof. This can be derived from Proposition 4.1.1 and it is also easy to check directly. Namely, compare the coefficients of the basis vectors $e_{ij} \otimes e_{kl} \otimes 1$ on both sides of (4.10). This is equivalent to applying the operators on both sides to the basis vector $e_j \otimes e_l$ of $\mathbb{C}^N \otimes \mathbb{C}^N$ and then comparing the coefficients of the vector $e_i \otimes e_k$. For the left hand side we have

$$\sum_{i,k=1}^{N} e_i \otimes e_k \otimes \big(E_{ij}\,E_{kl} - E_{kl}\,E_{ij}\big),$$

while for the right hand side we get

$$(E_1 - E_2)\,P(e_j \otimes e_l) = (E_1 - E_2)\,(e_l \otimes e_j) = \sum_{i=1}^{N} e_i \otimes e_j \otimes E_{il} - \sum_{k=1}^{N} e_l \otimes e_k \otimes E_{kj}.$$

Equating the coefficients of $e_i \otimes e_k$ we recover the defining relations (4.8). $\square$

The advantage of having the defining relations for $\mathrm{U}(\mathfrak{gl}_N)$ written as a single relation (4.10) for the matrix E will be apparent in Theorem 4.5.1 below, where we produce a family of Casimir elements for the Lie algebra $\mathfrak{gl}_N$.

Remark 4.2.3. The matrix form of the defining relations given in Proposition 4.2.2 leads to the definition of the *degenerate affine Hecke algebra* $\mathcal{H}_m$ as follows; see [**31**]. First note the relations

$$P_{a\,b}\,E_c = E_c\,P_{a\,b} \qquad \text{and} \qquad P_{a\,b}\,E_a = E_b\,P_{a\,b}, \tag{4.11}$$

which hold in the algebra (1.60) with $\mathcal{A} = \mathrm{U}(\mathfrak{gl}_N)$ for $1 \leqslant a < b \leqslant m$ and $c \neq a, b$. Moreover, by (4.10) for $a < b$ we have

$$E_a\,E_b - E_b\,E_a = E_a\,P_{a\,b} - P_{a\,b}\,E_a. \tag{4.12}$$

It was pointed out in [**3**] that $\mathcal{H}_m$ is isomorphic to the algebra generated by (abstract) elements $E_1,\dots,E_m$ and the group algebra $\mathbb{C}[\mathfrak{S}_m]$ subject to the relations (4.11) and (4.12), where $P_{a\,b}$ is understood as the transposition $s_{a\,b} \in \mathfrak{S}_m$. To see the connection with the definition of [**31**], set $u_a = E_a - x_a$ for $a = 1, \dots, m$, where x_a is the Jucys–Murphy element defined in (1.10). The elements u_a pairwise commute, $u_a u_b = u_b u_a$, while

$$s_a\, u_a = u_{a+1}\, s_a + 1 \qquad \text{and} \qquad s_b\, u_a = u_a\, s_b \quad \text{if} \quad b \neq a-1, a.$$

The map taking E_a to $u_a + x_a$ and identical on $\mathbb{C}[\mathfrak{S}_m]$ provides an isomorphism between the two presentations of $\mathcal{H}_m$. □

Remark 4.2.4. The *adjoint action* of the group GL_N of all invertible $N \times N$ matrices over $\mathbb{C}$ on the Lie algebra $\mathfrak{gl}_N$ is defined by (2.7). This extends to a unique action of the group GL_N on $\mathrm{U}(\mathfrak{gl}_N)$ so that each element of the group acts as an automorphism. The center coincides with the subalgebra of invariants under this action, $\mathrm{Z}(\mathfrak{gl}_N) = \mathrm{U}(\mathfrak{gl}_N)^{\mathrm{GL}_N}$. □

Example 4.2.5. It is an easy calculation to verify directly or with the use of Remark 4.2.4 that

$$\sum_{i=1}^{N} E_{ii} \qquad \text{and} \qquad \sum_{i,j=1}^{N} E_{ij}\, E_{ji}$$

are Casimir elements for $\mathfrak{gl}_N$. □

Given an N-tuple of complex numbers $\lambda = (\lambda_1, \dots, \lambda_N)$, the corresponding *irreducible highest weight representation* $L(\lambda)$ of the Lie algebra $\mathfrak{gl}_N$ is generated by a nonzero vector $\xi \in L(\lambda)$ (the *highest vector*) such that

$$E_{ij}\, \xi = 0 \qquad \text{for} \quad 1 \leqslant i < j \leqslant N, \qquad \text{and} \tag{4.13}$$

$$E_{ii}\, \xi = \lambda_i\, \xi \qquad \text{for} \quad 1 \leqslant i \leqslant N. \tag{4.14}$$

Any element $z \in \mathrm{Z}(\mathfrak{gl}_N)$ acts in $L(\lambda)$ by multiplying each vector by a scalar $\chi(z)$. When regarded as a function of the highest weight, $\chi(z)$ is a symmetric polynomial in the variables $l_1, \dots, l_N$, where $l_i = \lambda_i - i + 1$. This provides an equivalent interpretation of the Harish-Chandra isomorphism (4.7) as the mapping $z \mapsto \chi(z)$ defines an algebra isomorphism

$$\chi : \mathrm{Z}(\mathfrak{gl}_N) \to \mathbb{C}[l_1, \dots, l_N]^{\mathfrak{S}_N}, \tag{4.15}$$

where $\mathbb{C}[l_1, \dots, l_N]^{\mathfrak{S}_N}$ denotes the algebra of $\mathfrak{S}_N$-invariant (symmetric) polynomials in $l_1, \dots, l_N$ and l_i is identified with the element $E_{ii} - i + 1 \in \mathrm{U}(\mathfrak{h})$. Thus, the commutative algebra $\mathrm{Z}(\mathfrak{gl}_N)$ can be regarded as an algebra of polynomials in N variables. The preimages of any family of algebraically independent generators of the algebra of symmetric polynomials are algebraically independent generators of $\mathrm{Z}(\mathfrak{gl}_N)$.

Remark 4.2.6. The shifts in the definition of the variables l_i are determined by the half-sum of the positive roots for the simple Lie algebra $\mathfrak{sl}_N$. Clearly, given any constant $a \in \mathbb{C}$, one could set $l_i = \lambda_i - i + a$ for $i = 1, \dots, N$ to get an alternative definition of the isomorphism (4.15). This extra freedom is explained by the fact that the reductive Lie algebra $\mathfrak{gl}_N$ is isomorphic to the direct sum of $\mathfrak{sl}_N$ and the one-dimensional center spanned by the scalar matrices. Apart from the value $a = 1$ which we use in (4.15), some other common choices of a in the literature include $(N+1)/2$ and N. □

EXAMPLE 4.2.7. For the Harish-Chandra images of the Casimir elements of Example 4.2.5 we have

$$\chi : \sum_{i=1}^{N} E_{ii} \mapsto \sum_{i=1}^{N} l_i + \binom{N}{2},$$

$$\chi : \sum_{i,j=1}^{N} E_{ij} E_{ji} \mapsto \sum_{i=1}^{N} l_i^2 + (N-1) \sum_{i=1}^{N} l_i + \binom{N}{3}.$$

They are found by the application of the Casimir elements to the highest vector ξ in the representation $L(\lambda)$ of $\mathfrak{gl}_N$. □

4.3. Factorial Schur polynomials

Certain particular families of symmetric polynomials will occur as Harish-Chandra images of central elements for the simple Lie algebras of all classical types. To describe them, consider the algebra of symmetric polynomials in the independent variables $x_1, \dots, x_n$ over $\mathbb{C}$ and fix a sequence $a = (a_1, a_2, \dots)$ of complex numbers. The *factorial elementary* and *complete symmetric polynomials* are defined by the respective formulas

$$e_k(x_1, \dots, x_n \,|\, a) = \sum_{1 \leqslant p_1 < \dots < p_k \leqslant n} (x_{p_1} - a_{p_1})(x_{p_2} - a_{p_2 - 1}) \dots (x_{p_k} - a_{p_k - k + 1}), \tag{4.16}$$

and

$$h_k(x_1, \dots, x_n \,|\, a) = \sum_{1 \leqslant p_1 \leqslant \dots \leqslant p_k \leqslant n} (x_{p_1} - a_{p_1})(x_{p_2} - a_{p_2 + 1}) \dots (x_{p_k} - a_{p_k + k - 1}), \tag{4.17}$$

so that $e_k(x_1, \dots, x_n \,|\, a) = 0$ for $k > n$. These polynomials are particular cases of the *factorial* (or *double*) *Schur polynomials* $s_\mu(x_1, \dots, x_n \,|\, a)$ defined as follows. Suppose that μ is a diagram with at most n rows; see Section 1.1. Then

$$s_\mu(x_1, \dots, x_n \,|\, a) = \sum_{\mathrm{sh}(\mathcal{T}) = \mu} \prod_{\alpha \in \mu} \big(x_{\mathcal{T}(\alpha)} - a_{\mathcal{T}(\alpha) + c(\alpha)} \big), \tag{4.18}$$

summed over semistandard tableaux $\mathcal{T}$ of shape μ with entries in $\{1, \dots, n\}$, where $\mathcal{T}(\alpha)$ denotes the entry of the tableau $\mathcal{T}$ at the box $\alpha = (i, j)$ of μ and $c(\alpha) = j - i$ is the content of this box. In the particular cases $\mu = (1^k)$ and $\mu = (k)$ the polynomial (4.18) coincides with (4.16) and (4.17), respectively.

When a is specialized to the sequence of zeros, then (4.16) and (4.17) become the respective elementary and complete symmetric polynomials $e_k(x_1, \dots, x_n)$ and $h_k(x_1, \dots, x_n)$, and $s_\mu(x_1, \dots, x_n \,|\, a)$ becomes the Schur polynomial $s_\mu(x_1, \dots, x_n)$. These homogeneous polynomials also coincide with the top degree components of the respective factorial counterparts. This implies, in particular, that the factorial Schur polynomials with μ running over all diagrams with at most n rows form a basis of the algebra of symmetric polynomials $\mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n}$. Some other equivalent definitions of the factorial Schur polynomials (4.18) and their basic properties are discussed in the book by Macdonald [**104**, Section I.3]. Here we derive their vanishing and characterization properties to be used below.

We will suppose that the sequence a is multiplicity-free, that is $a_k \neq a_l$ for all $k \neq l$. Set $x = (x_1, \dots, x_n)$ and for any partition $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\ell(\lambda) \leqslant n$ introduce the n-tuple of elements of a by

$$a_\lambda = (a_{\lambda_1+n}, \dots, a_{\lambda_n+1}). \tag{4.19}$$

As before we let λ'_j denote the number of boxes in the column j of the diagram λ.

PROPOSITION 4.3.1. *If $\mu \not\subset \lambda$ then $s_\mu(a_\lambda \,|\, a) = 0$. Moreover,*

$$s_\mu(a_\mu \,|\, a) = \prod_{(i,j)\in\mu} \big(a_{\mu_i+n-i+1} - a_{n-\mu'_j+j}\big). \tag{4.20}$$

PROOF. The polynomials $s_\mu(x \,|\, a)$ are symmetric in x, so replacing x with $(x_n, \dots, x_1)$, we may rewrite (4.18) in the form

$$s_\mu(x \,|\, a) = \sum_{\mathrm{sh}(\mathcal{T})=\mu} \prod_{\alpha\in\mu} \big(x_{n-\mathcal{T}(\alpha)+1} - a_{\mathcal{T}(\alpha)+c(\alpha)}\big), \tag{4.21}$$

with the summation over semistandard μ-tableau $\mathcal{T}$ with entries from $\{1, \dots, n\}$. Suppose that $s_\mu(a_\lambda \,|\, a) \neq 0$. Then at least one summand in (4.21) does not vanish for $x = a_\lambda$,

$$\prod_{\alpha\in\mu} \big(a_{\lambda_{n-\mathcal{T}(\alpha)+1}+\mathcal{T}(\alpha)} - a_{\mathcal{T}(\alpha)+c(\alpha)}\big) \neq 0. \tag{4.22}$$

Since the sequence a is multiplicity free, this implies that

$$\lambda_{n-\mathcal{T}(\alpha)+1} \neq c(\alpha) \tag{4.23}$$

for all $\alpha \in \mu$. For the entries of the first row of the tableau $\mathcal{T}$ we have

$$\mathcal{T}(1,1) \leqslant \dots \leqslant \mathcal{T}(1,\mu_1).$$

Note that $c(1,1) = 0$ so that using (4.23) with $\alpha = (1,1)$ we obtain $\lambda_{n-\mathcal{T}(1,1)+1} \geqslant 1$. Furthermore, $c(1,2) = 1$ and since

$$\lambda_{n-\mathcal{T}(1,2)+1} \geqslant \lambda_{n-\mathcal{T}(1,1)+1} \geqslant 1,$$

by using (4.23) with $\alpha = (1,2)$ we get $\lambda_{n-\mathcal{T}(1,2)+1} \geqslant 2$. Continuing in the same manner, we conclude that $\lambda_{n-\mathcal{T}(1,i)+1} \geqslant i$ for all $i = 1, \dots, \mu_1$. On the other hand, for the entries of the i-th column of $\mathcal{T}$ we have

$$\mathcal{T}(1,i) < \dots < \mathcal{T}(\mu'_i, i).$$

Therefore,

$$\lambda_{n-\mathcal{T}(\mu'_i,i)+1} \geqslant \dots \geqslant \lambda_{n-\mathcal{T}(1,i)+1} \geqslant i. \tag{4.24}$$

This means that the diagram λ has at least μ'_i rows of length at least i so that $\lambda'_i \geqslant \mu'_i$. Hence, $\mu \subset \lambda$ thus proving the first part of the proposition.

Now take $\lambda = \mu$ and suppose that (4.22) holds for a certain tableau $\mathcal{T}$. By the above argument, the inequalities (4.24) hold for $\lambda = \mu$ which implies that such tableau $\mathcal{T}$ is determined uniquely by $\mathcal{T}(k,i) = n - \mu'_i + k$ for $k = 1, \dots, \mu'_i$. Thus, there is a unique nonzero summand in (4.21) with $x = a_\mu$. By using the values of the tableau, we come to (4.20). □

The vanishing properties of the factorial Schur polynomials given by Proposition 4.3.1 are characteristic in the sense of the next proposition. We suppose that μ is a diagram with at most n rows.

PROPOSITION 4.3.2. *Let $f(x)$ be a symmetric polynomial of degree not exceeding $|\mu|$. If the top degree component of $f(x)$ coincides with $s_\mu(x)$ and $f(a_\lambda) = 0$ for all λ such that $|\lambda| < |\mu|$, then $f(x) = s_\mu(x\,|\,a)$.*

PROOF. Represent the difference $f(x) - s_\mu(x\,|\,a)$ as a linear combination of the basis polynomials,

$$f(x) - s_\mu(x\,|\,a) = \sum_{\nu\in S} c_\nu\, s_\nu(x\,|\,a), \tag{4.25}$$

where $c_\nu \in \mathbb{C}$ and the sum is taken over the set S of the diagrams ν with $|\nu| < |\mu|$. Let λ run over the same set S. Then $\mu \not\subset \lambda$ and $s_\mu(a_\lambda\,|\,a) = 0$ by Proposition 4.3.1. Hence putting $x = a_\lambda$ in (4.25) for all such λ, we obtain a system of linear equations on the coefficients c_ν of the form

$$\sum_{\nu\in S} c_\nu\, s_\nu(a_\lambda\,|\,a) = 0.$$

Equip the set S with any linear ordering $\prec$ such that $|\lambda| < |\nu|$ implies $\lambda \prec \nu$. By arranging the system of equations according to the ordering $\prec$ we find that the corresponding matrix is triangular due to the vanishing properties of Proposition 4.3.1. The diagonal entries of the matrix are the values $s_\nu(a_\nu\,|\,a)$. All the numbers $\nu_i + n - i + 1$ and $n - \nu'_j + j$ are distinct and so the product in (4.20) is nonzero since the sequence a is multiplicity-free. Thus $s_\nu(a_\nu\,|\,a) \neq 0$ and the system has only the trivial solution $c_\nu = 0$ which means that $f(x) = s_\mu(x\,|\,a)$. □

4.4. Schur–Weyl duality

Recall the action of the symmetric group $\mathfrak{S}_m$ on the tensor product space $(\mathbb{C}^N)^{\otimes m}$ as defined in Section 1.5. The general linear group GL_N acts on this space diagonally; see (2.13). The corresponding action of the Lie algebra $\mathfrak{gl}_N$ is given by

$$X : v_1 \otimes \ldots \otimes v_m \mapsto \sum_{a=1}^{m} v_1 \otimes \ldots \otimes X\, v_a \otimes \ldots \otimes v_m, \qquad v_i \in \mathbb{C}^N, \qquad X \in \mathfrak{gl}_N.$$

In line with our tensor notation, these actions are written as

$$\mathbf{h} \mapsto \mathbf{h}_1 \ldots \mathbf{h}_m \qquad \text{and} \qquad X \mapsto X_1 + \cdots + X_m,$$

where

$$\mathbf{h}_a = 1^{\otimes(a-1)} \otimes \mathbf{h} \otimes 1^{\otimes(m-a)} \qquad \text{and} \qquad X_a = 1^{\otimes(a-1)} \otimes X \otimes 1^{\otimes(m-a)}.$$

As we verified in Section 2.1, the action of any element $s \in \mathfrak{S}_m$ on the vector space $(\mathbb{C}^N)^{\otimes m}$ commutes with that of any element $\mathbf{h} \in \mathrm{GL}_N$ (and hence with the action of any element $X \in \mathfrak{gl}_N$). By the classical *Schur–Weyl duality*, these actions of $\mathfrak{S}_m$ and GL_N centralize each other. This leads to the multiplicity-free decomposition as a representation of the group $\mathfrak{S}_m \times \mathrm{GL}_N$,

$$(\mathbb{C}^N)^{\otimes m} \cong \bigoplus_{\lambda\vdash m,\ \ell(\lambda)\leqslant N} V_\lambda \otimes L(\lambda), \tag{4.26}$$

where V_λ and $L(\lambda)$ are the respective irreducible representations of $\mathfrak{S}_m$ and GL_N associated with a Young diagram λ which contains $|\lambda| = m$ boxes, and the number of nonzero rows $\ell(\lambda)$ does not exceed N. As a $\mathfrak{gl}_N$-module, $L(\lambda)$ is the highest weight representation with the highest weight $\lambda = (\lambda_1, \dots, \lambda_N)$, where one sets $\lambda_i = 0$ for all $i = \ell(\lambda) + 1, \dots, N$.

Let $\mathcal{U}$ be a standard tableau of shape $\lambda \vdash m$ and let $e_{\mathcal{U}} \in \mathbb{C}[\mathfrak{S}_m]$ be the associated primitive idempotent; see Section 1.1. Denote by $\mathcal{E}_{\mathcal{U}}$ the image of $e_{\mathcal{U}}$ under the action of the symmetric group $\mathfrak{S}_m$ given by (1.65). The space $\mathcal{E}_{\mathcal{U}}(\mathbb{C}^N)^{\otimes m}$ is an irreducible representation of GL_N isomorphic to $L(\lambda)$. Therefore, the trace $\mathrm{tr}_{1,\dots,m}\, \mathcal{E}_{\mathcal{U}}\, \mathbf{h}_1 \dots \mathbf{h}_m$ coincides with the character of the representation $L(\lambda)$ evaluated at the element $\mathbf{h}$. This value is given by the Weyl character formula so that the trace equals the Schur polynomial s_μ evaluated at the eigenvalues $h_1, \dots, h_N$ of the matrix $\mathbf{h}$,

$$\mathrm{tr}_{1,\dots,m}\, \mathcal{E}_{\mathcal{U}}\, \mathbf{h}_1 \dots \mathbf{h}_m = s_\mu(h_1, \dots, h_N); \tag{4.27}$$

see also (2.19). In particular, the dimension $\dim L(\lambda)$ coincides with the trace

$$\mathrm{tr}_{1,\dots,m}\, \mathcal{E}_{\mathcal{U}} = s_\mu(1, \dots, 1).$$

By the correspondence (1.72), the trace formula for the idempotents given in Proposition 1.3.5 implies the *Robinson hook dimension formula*

$$\dim L(\lambda) = \frac{1}{h(\lambda)} \prod_{(i,j)\in\lambda} (N + j - i). \tag{4.28}$$

4.5. A general construction of central elements

Now suppose that $s \in \mathbb{C}[\mathfrak{S}_m]$ is an arbitrary element and let S denote its image under the map (1.65). We regard S as an element of the algebra

$$\underbrace{\mathrm{End}\, \mathbb{C}^N \otimes \dots \otimes \mathrm{End}\, \mathbb{C}^N}_{m} \otimes \mathrm{U}(\mathfrak{gl}_N) \tag{4.29}$$

by identifying it with $S \otimes 1$.

THEOREM 4.5.1. *For any $s \in \mathbb{C}[\mathfrak{S}_m]$ and $u_1, \dots, u_m \in \mathbb{C}$ the element*

$$\mathrm{tr}_{1,\dots,m}\, S\, (u_1 + E_1) \dots (u_m + E_m) \tag{4.30}$$

belongs to the center $\mathrm{Z}(\mathfrak{gl}_N)$.

PROOF. Consider the tensor product

$$\mathrm{End}\, \mathbb{C}^N \otimes \mathrm{End}\, (\mathbb{C}^N)^{\otimes m} \otimes \mathrm{U}(\mathfrak{gl}_N) \tag{4.31}$$

with an additional copy of the endomorphism algebra $\mathrm{End}\, \mathbb{C}^N$. We will label the copies of this algebra respectively by $0, 1, \dots, m$. It will be sufficient to show that the following commutator in the algebra (4.31) is zero,

$$\big[E_0, \mathrm{tr}_{1,\dots,m}\, S\, (u_1 + E_1) \dots (u_m + E_m)\big] = 0. \tag{4.32}$$

To this end, note that by (1.67) and Proposition 4.2.2 we can write

$$[E_0, u_a + E_a] = P_{0\,a}(u_a + E_a) - (u_a + E_a)P_{0\,a}$$

so that

$$\begin{aligned}&\big[E_0, S\,(u_1+E_1)\dots(u_m+E_m)\big]\\ &\qquad=\sum_{a=1}^{m} S\,(u_1+E_1)\dots\big(P_{0\,a}(u_a+E_a)-(u_a+E_a)P_{0\,a}\big)\dots(u_m+E_m)\\ &\qquad=S\sum_{a=1}^{m} P_{0\,a}\,(u_1+E_1)\dots(u_m+E_m)-S\,(u_1+E_1)\dots(u_m+E_m)\sum_{a=1}^{m} P_{0\,a},\end{aligned}$$

where we used the observation that $E_0\,S = S\,E_0$ and that $P_{0\,a}$ commutes with E_b for $b \neq a$. Furthermore, the sum of the permutation operators $P_{0\,a}$ commutes with the image of any element of $\mathfrak{S}_m$ under the map (1.65) so that

$$\begin{aligned}\mathrm{tr}_{1,\dots,m}\, S\sum_{a=1}^{m} P_{0\,a}\,(u_1+E_1)\dots(u_m+E_m)&\\ =\mathrm{tr}_{1,\dots,m}\sum_{a=1}^{m} P_{0\,a}\, S\,(u_1+E_1)\dots(u_m+E_m)&\\ =\mathrm{tr}_{1,\dots,m}\, S\,(u_1+E_1)\dots(u_m+E_m)\sum_{a=1}^{m} P_{0\,a},&\end{aligned}$$

where the last equality holds by the cyclic property of trace; see Lemma 1.4.1. Its application relies on the fact that two elements of the algebra (4.31) of the form

$$X\otimes 1^{\otimes m}\otimes 1 \qquad \text{and} \qquad 1\otimes 1^{\otimes m}\otimes y$$

commute for any $X \in \mathrm{End}\,\mathbb{C}^N$ and $y \in \mathrm{U}(\mathfrak{gl}_N)$. This proves (4.32). □

Remark 4.5.2. A slight modification of the above argument shows that all elements of the form

$$\mathrm{tr}_{1,\dots,m}\, S\, E_{a_1}\dots E_{a_k}$$

with arbitrary parameters $a_i \in \{1,\dots,m\}$ also belong to $\mathrm{Z}(\mathfrak{gl}_N)$. □

In what follows we consider particular choices of the parameters $u_1,\dots,u_m$ and the element $s \in \mathbb{C}[\mathfrak{S}_m]$ in Theorem 4.5.1. The next lemma will allow us to rely on the properties of Manin matrices discussed in Chapter 3.

Lemma 4.5.3. *Suppose that α and β are elements of a certain unital associative algebra $\mathcal{D}$ which satisfy the relation*

$$\alpha\,\beta-\beta\,\alpha=\beta^2. \tag{4.33}$$

Then the matrix $\alpha + E\,\beta$ with entries in the algebra $\mathrm{U}(\mathfrak{gl}_N)\otimes\mathcal{D}$ is a Manin matrix.

Proof. Set $M = \alpha + E\,\beta$ (we abbreviate notation by omitting the tensor product signs). Using Proposition 4.2.2 we find

$$\begin{aligned}M_1\,M_2-M_2\,M_1&=\big(\alpha+E_1\,\beta\big)\big(\alpha+E_2\,\beta\big)-\big(\alpha+E_2\,\beta\big)\big(\alpha+E_1\,\beta\big)\\ &=(E_1\,E_2-E_2\,E_1)\,\beta^2-(E_1-E_2)\,(\alpha\,\beta-\beta\,\alpha)=(E_1-E_2)(P-1)\,\beta^2.\end{aligned}$$

Hence, on multiplying this element from the right by $1+P$ we get 0, and the claim follows by using the equivalent definition (3.5) of Manin matrices. □

Examples of elements α and β satisfying (4.33) are provided by

$$\alpha = -\partial_t, \quad \beta = t^{-1} \qquad \text{and} \qquad \alpha = u\, e^{-\partial_u}, \quad \beta = e^{-\partial_u}.$$

In the first example we can take $\mathcal{D}$ to be the algebra of polynomial differential operators of the form

$$a_0 + a_1\, \partial_t + \dots + a_k\, \partial_t^k, \qquad k \geqslant 0,$$

where each coefficient a_i is a Laurent polynomial in t. In the second example we take $\mathcal{D}$ to be generated by all polynomials in u and an additional element $e^{-\partial_u}$, subject to the relations

$$e^{-\partial_u} f(u) = f(u-1)\, e^{-\partial_u} \tag{4.34}$$

for any polynomial $f(u)$.

Lemma 4.5.4. *Let $u_1, \dots, u_m$ be complex parameters and let S be the image of an element s of the center of the group algebra $\mathbb{C}[\mathfrak{S}_m]$ under the homomorphism* (1.65). *For any permutations $\sigma, \tau \in \mathfrak{S}_m$ we have the identity*

$$\operatorname{tr}_{1,\dots,m} S\,(E_{\sigma(1)} + u_{\tau(1)}) \dots (E_{\sigma(m)} + u_{\tau(m)}) = \operatorname{tr}_{1,\dots,m} S\,(E_1 + u_1) \dots (E_m + u_m).$$

Proof. By the cyclic property of trace, the right hand side equals

$$\begin{aligned} \operatorname{tr}_{1,\dots,m} P_\sigma S\,(E_1 + u_1) \dots (E_m + u_m) P_\sigma^{-1} & \\ &= \operatorname{tr}_{1,\dots,m} S\, P_\sigma\,(E_1 + u_1) \dots (E_m + u_m) P_\sigma^{-1} \\ &= \operatorname{tr}_{1,\dots,m} S\,(E_{\sigma(1)} + u_1) \dots (E_{\sigma(m)} + u_m), \end{aligned}$$

where we also used (1.67). Thus, it suffices to verify the identity in the case where σ is the identity permutation. We may assume that τ is an adjacent transposition $s_a = (a\ a+1)$. By Proposition 4.2.2,

$$(E_a + u_{a+1})(E_{a+1} + u_a) - (E_{a+1} + u_a)(E_a + u_{a+1}) = P_{a\,a+1}\, E_{a+1} - E_{a+1}\, P_{a\,a+1}.$$

Since $S\, P_{a\,a+1} = P_{a\,a+1}\, S$, the claim follows from the cyclic property of trace and the first part of the proof. □

In particular, Lemma 4.5.4 holds for the symmetrizer and anti-symmetrizer $S = H^{(m)}$ and $S = A^{(m)}$.

4.6. Capelli determinant

Take $S = A^{(m)}$ with $1 \leqslant m \leqslant N$ in Theorem 4.5.1 and specialize the parameters by

$$u_a = u - a + 1, \qquad a = 1, \dots, m,$$

for a variable u. Then the element (4.30) becomes a polynomial in u of degree m,

$$\operatorname{tr}_{1,\dots,m} A^{(m)}\,(u + E_1) \dots (u + E_m - m + 1), \tag{4.35}$$

whose coefficients are Casimir elements for $\mathfrak{gl}_N$.

Proposition 4.6.1. *The Harish-Chandra images of the coefficients of the polynomial* (4.35) *are found by*

$$\begin{aligned} \chi : \operatorname{tr}_{1,\dots,m} A^{(m)}\,(u + E_1) \dots (u + E_m - m + 1) & \\ \mapsto \sum_{1 \leqslant i_1 < \dots < i_m \leqslant N} & (u + \lambda_{i_1}) \cdots (u + \lambda_{i_m} - m + 1). \end{aligned}$$

Proof. Use Lemma 4.5.3 to apply Proposition 3.2.2 to the Manin matrix $M = (u+E)\,e^{-\partial_u}$. By (4.34) the left hand side of (3.17) can be written as

$$\operatorname{tr}_{1,\dots,m} A^{(m)}\big(u+E_1\big)\,e^{-\partial_u}\dots\big(u+E_m\big)\,e^{-\partial_u} = \operatorname{tr}_{1,\dots,m} A^{(m)}\big(u+E_1\big)\dots\big(u+E_m-m+1\big)\,e^{-m\,\partial_u}.$$

Now consider the product $M_{i_{\sigma(1)}\,i_1}\dots M_{i_{\sigma(m)}\,i_m}$ which occurs as a summand in the right hand side of (3.17). It is clear from (4.13) that its application to the highest vector ξ of a highest weight representation $L(\lambda)$ yields zero unless σ is the identity permutation in $\mathfrak{S}_m$. In that case, by (4.14)

$$M_{i_1\,i_1}\dots M_{i_m\,i_m}\,\xi = \big(u+E_{i_1i_1}\big)\,e^{-\partial_u}\dots\big(u+E_{i_mi_m}\big)\,e^{-\partial_u}\,\xi = \big(u+\lambda_{i_1}\big)\dots\big(u+\lambda_{i_m}-m+1\big)\,e^{-m\,\partial_u}\,\xi.$$

Taking the sum over the indices $1 \leqslant i_1 < \dots < i_m \leqslant N$ we get the desired formula for the Harish-Chandra image of the polynomial (4.35). □

Due to Proposition 4.6.1 all coefficients of the polynomial in u given by

$$\sum_{1\leqslant i_1<\dots<i_m\leqslant N} (u+\lambda_{i_1})\cdots(u+\lambda_{i_m}-m+1) \tag{4.36}$$

are symmetric polynomials in the variables $l_i = \lambda_i - i + 1$ with $i = 1,\dots,N$. In particular, taking $u = 0$ in (4.36) we get the factorial elementary symmetric polynomial

$$e_m(l_1,\dots,l_N\,|\,a) = \sum_{1\leqslant i_1<\dots<i_m\leqslant N} \big(l_{i_1}+i_1-1\big)\cdots\big(l_{i_m}+i_m-m\big),$$

associated with the parameter sequence $a = (a_i)$, $a_i = -i+1$; see (4.16).

We use the column-determinant (3.23) to define the *Capelli determinant* by

$$C(u) = \operatorname{cdet}\begin{bmatrix} u+E_{11} & E_{12} & \dots & E_{1N}\\ E_{21} & u+E_{22}-1 & \dots & E_{2N}\\ \vdots & \vdots & & \vdots\\ E_{N1} & E_{N2} & \dots & u+E_{NN}-N+1 \end{bmatrix}. \tag{4.37}$$

This is a polynomial in u,

$$C(u) = u^N + C_1^\circ\,u^{N-1} + \dots + C_N^\circ, \qquad C_k^\circ \in \mathrm{U}(\mathfrak{gl}_N). \tag{4.38}$$

Note that the symbols $C_k \in \mathrm{S}(\mathfrak{gl}_N)$ of the elements C_k° coincide with the coefficients of the characteristic polynomial $\det(u+E)$; see (2.6). Relation (3.28) and the proof of Proposition 4.6.1 imply the properties of $C(u)$ given by the next corollary.

Corollary 4.6.2. *The Capelli determinant coincides with the polynomial in u defined in* (4.35) *with $m = N$,*

$$C(u) = \operatorname{tr}_{1,\dots,N} A^{(N)}\,(u+E_1)\dots(u+E_N-N+1)$$

so that all coefficients C_k° are Casimir elements for $\mathfrak{gl}_N$. Moreover, under the Harish-Chandra isomorphism (4.15) *we have*

$$\chi : C(u) \mapsto (u+l_1)\dots(u+l_N).$$

In particular, the elements $C_1^\circ,\dots,C_N^\circ$ are algebraically independent generators of the center $\mathrm{Z}(\mathfrak{gl}_N)$ of the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_N)$. □

The last property holds since the elementary symmetric polynomials are algebraically independent generators of the algebra $\mathbb{C}[l_1,\dots,l_N]^{\mathfrak{S}_N}$.

4.7. Permanent-type elements

Now take $S = H^{(m)}$ with $m \geqslant 1$ in Theorem 4.5.1 and specialize the parameters by

$$u_a = u + a - 1, \qquad a = 1,\dots,m,$$

for a variable u. The element (4.30) becomes a polynomial in u of degree m:

$$\operatorname{tr}_{1,\dots,m} H^{(m)}\,(u+E_1)\dots(u+E_m+m-1). \tag{4.39}$$

Its coefficients are Casimir elements for $\mathfrak{gl}_N$.

Proposition 4.7.1. *The Harish-Chandra images of the coefficients of the polynomial* (4.39) *are found by*

$$\chi : \operatorname{tr}_{1,\dots,m} H^{(m)}\,(u+E_1)\dots(u+E_m+m-1) \\ \mapsto \sum_{1\leqslant i_1\leqslant\dots\leqslant i_m\leqslant N} (u+\lambda_{i_1})\cdots(u+\lambda_{i_m}+m-1).$$

Proof. We use again Proposition 3.2.2 and Lemma 4.5.3. The left hand side of (3.18) can be written for the Manin matrix $M = (u+E+m-1)\,e^{-\partial_u}$ as

$$\operatorname{tr}_{1,\dots,m} H^{(m)}\big(u+E_1+m-1\big)\,e^{-\partial_u}\dots\big(u+E_m+m-1\big)\,e^{-\partial_u} \\ = \operatorname{tr}_{1,\dots,m} H^{(m)}\big(u+E_1+m-1\big)\dots\big(u+E_m\big)\,e^{-m\,\partial_u} \\ = \operatorname{tr}_{1,\dots,m} H^{(m)}\big(u+E_1\big)\dots\big(u+E_m+m-1\big)\,e^{-m\,\partial_u},$$

where the second equality holds due to Lemma 4.5.4. For $\sigma \in \mathfrak{S}_m$ consider the product $M_{i_m\, i_{\sigma(m)}}\dots M_{i_1\, i_{\sigma(1)}}$ which occurs as a summand on the right hand side of (3.18). By (4.13) its application to the highest vector ξ of a highest weight representation $L(\lambda)$ yields zero unless σ belongs to the stabilizer of the multiset $\{i_1,\dots,i_m\}$. In this case, by (4.14) we find

$$M_{i_m\, i_m}\dots M_{i_1\, i_1}\,\xi = \big(u+E_{i_m i_m}+m-1\big)\,e^{-\partial_u}\dots\big(u+E_{i_1 i_1}+m-1\big)\,e^{-\partial_u}\,\xi \\ = \big(u+\lambda_{i_m}+m-1\big)\dots\big(u+\lambda_{i_1}\big)\,e^{-m\,\partial_u}\,\xi.$$

Taking into account the number $\alpha_1!\dots\alpha_N!$ of permutations which stabilize the multiset $\{i_1,\dots,i_m\}$, we obtain the desired formula. □

Due to Proposition 4.7.1 all coefficients of the polynomial in u given by

$$\sum_{1\leqslant i_1\leqslant\dots\leqslant i_m\leqslant N} (u+\lambda_{i_1})\cdots(u+\lambda_{i_m}+m-1)$$

are symmetric polynomials in the variables $l_i = \lambda_i - i + 1$ with $i = 1,\dots,N$. In particular, taking $u = 0$ we get the factorial complete symmetric polynomial

$$h_m(l_1,\dots,l_N\,|\,a) = \sum_{1\leqslant i_1\leqslant\dots\leqslant i_m\leqslant N} \big(l_{i_1}+i_1-1\big)\cdots\big(l_{i_m}+i_m+m-2\big),$$

associated with the parameter sequence $a = (a_i)$, $a_i = -i+1$; see (4.17).

4.8. Gelfand invariants

Take $S = P_\sigma$ in Theorem 4.5.1, where $\sigma = (m, m-1, \dots, 1)$ is a long cycle in the symmetric group $\mathfrak{S}_m$. We have

$$P_\sigma = P_{m-1\,m} \dots P_{2\,3} P_{1\,2}. \tag{4.40}$$

We also set $u_a = 0$ for all $a = 1, \dots, m$. Then by (1.68) and the cyclic property of trace, (4.30) becomes the Casimir element $\operatorname{tr} E^m$ known as the *Gelfand invariant*. Indeed, note that by (1.67)

$$\operatorname{tr}_{1,\dots,m} E_1 \dots E_m P_{m-1\,m} \dots P_{2\,3} P_{1\,2}$$
$$= \operatorname{tr}_{1,\dots,m} E_1 \dots E_{m-1} P_{m-1\,m} E_{m-1} P_{m-2\,m-1} \dots P_{2\,3} P_{1\,2}.$$

Since $\operatorname{tr}_m P_{m-1\,m} = 1$, applying the partial trace tr_m we bring the expression to the form

$$\operatorname{tr}_{1,\dots,m-1} E_1 \dots E_{m-1}^2 P_{m-2\,m-1} \dots P_{2\,3} P_{1\,2}$$

and then continue by induction to get

$$\operatorname{tr}_{1,\dots,m} P_\sigma E_1 \dots E_m = \operatorname{tr} E^m \tag{4.41}$$

which is therefore a Casimir element for $\mathfrak{gl}_N$ for any $m \geqslant 1$.

A more direct way to come to this conclusion is to use the following straightforward consequence of Proposition 4.2.2:

$$E_1 E_2^m - E_2^m E_1 = P_{12} E_2^m - E_2^m P_{12},$$

then take the partial trace tr_2 and apply Lemma 1.4.1; cf. Proposition 4.2.1.

The next theorem is the *Newton identity* for the Gelfand invariants. The argument will be similar to the proof of Theorem 3.2.10 but we need to use a difference operator instead of the derivative. We use the Capelli determinant $C(u)$; see (4.38).

THEOREM 4.8.1. *We have the identity*

$$1 + \sum_{m=0}^{\infty} \frac{(-1)^m \operatorname{tr} E^m}{(u - N + 1)^{m+1}} = \frac{C(u+1)}{C(u)}.$$

PROOF. By Corollary 4.6.2,

$$\operatorname{tr}_{1,\dots,N} A^{(N)} (u + E_1) \dots (u + E_N - N + 1) = C(u).$$

Hence, by Lemma 4.5.4 we also have

$$\operatorname{tr}_{1,\dots,N} A^{(N)} (u + E_1) \dots (u + E_{N-1} - N + 2)(u + E_N + 1) = C(u + 1).$$

Therefore,

$$C(u+1) - C(u) = N \operatorname{tr}_{1,\dots,N} A^{(N)} (u + E_1) \dots (u + E_{N-1} - N + 2)$$
$$= N \operatorname{tr}_{1,\dots,N} A^{(N)} C(u) (u + E_N - N + 1)^{-1}.$$

Applying the conjugation by P_{1N} we can write the right hand side as

$$N \operatorname{tr}_{1,\dots,N} A^{(N)} C(u) (u + E_1 - N + 1)^{-1}.$$

Finally, calculate the partial trace $\operatorname{tr}_{2,\dots,N}$ by using (3.26) to get

$$C(u+1) - C(u) = C(u) \sum_{m=0}^{\infty} \frac{(-1)^m \operatorname{tr} E^m}{(u - N + 1)^{m+1}},$$

as required. □

Corollary 4.8.2. *The Harish-Chandra images of the Gelfand invariants are found by*

$$1+\sum_{m=0}^{\infty}\frac{(-1)^m\,\chi(\operatorname{tr}E^m)}{(u-N+1)^{m+1}}=\prod_{i=1}^{N}\frac{u+l_i+1}{u+l_i}. \tag{4.42}$$

Equivalently, setting $\bar{l}_i=\lambda_i-i+N$ *we have*

$$\chi(\operatorname{tr}E^m)=\sum_{k=1}^{N}\bar{l}_k^{\,m}\,\frac{(\bar{l}_1-\bar{l}_k+1)\dots(\bar{l}_N-\bar{l}_k+1)}{(\bar{l}_1-\bar{l}_k)\dots\wedge\dots(\bar{l}_N-\bar{l}_k)}, \tag{4.43}$$

where the symbol $\wedge$ *indicates that the zero factor is skipped.*

Proof. Relation (4.42) follows from Corollary 4.6.2 and Theorem 4.8.1. Setting $v=u-N+1$ we can write this relation in the form

$$1+\sum_{m=0}^{\infty}(-1)^m\,\chi(\operatorname{tr}E^m)\,v^{-m-1}=\prod_{i=1}^{N}\frac{v+\bar{l}_i+1}{v+\bar{l}_i}.$$

Now use a partial fraction decomposition for the right hand side,

$$\prod_{i=1}^{N}\frac{v+\bar{l}_i+1}{v+\bar{l}_i}=1+\frac{a_1}{v+\bar{l}_1}+\dots+\frac{a_N}{v+\bar{l}_N}.$$

The constants a_k are found by multiplying both sides by $v+\bar{l}_k$ and setting $v=-\bar{l}_k$. This gives (4.43) by expanding the rational functions into power series in v^{-1}. □

The Harish-Chandra images of certain modified Gelfand invariants will be calculated in Corollary 13.4.3 below.

4.9. Quantum immanants

As in Section 1.1, suppose that $\mathcal{U}$ is a standard tableau of shape $\mu\vdash m$ and let $e_{\mathcal{U}}\in\mathbb{C}[\mathfrak{S}_m]$ be the associated primitive idempotent. By $c_a=c_a(\mathcal{U})$ we denote the content $c(\alpha)=j-i$ of the box $\alpha=(i,j)$ in $\mathcal{U}$ occupied by $a\in\{1,\dots,m\}$. As before, we let $\mathcal{E}_{\mathcal{U}}$ denote the image of $e_{\mathcal{U}}$ under the action of the symmetric group $\mathfrak{S}_m$ given by (1.65). The operator $\mathcal{E}_{\mathcal{U}}$ is zero if the length $\ell(\mu)$ of μ exceeds N. So we will assume that $\ell(\mu)\leqslant N$. Now specialize the parameters in Theorem 4.5.1 by

$$u_a=u+c_a,\quad a=1,\dots,m$$

for a variable u and take $S=\mathcal{E}_{\mathcal{U}}$. Denote the corresponding Casimir element by

$$\mathbb{S}_\mu(u)=\operatorname{tr}_{1,\dots,m}\mathcal{E}_{\mathcal{U}}\,(u+E_1+c_1)\dots(u+E_m+c_m). \tag{4.44}$$

Proposition 4.9.1. *The polynomial* $\mathbb{S}_\mu(u)$ *is independent of the choice of the standard tableau* $\mathcal{U}$ *of shape* μ. *Moreover, the Harish-Chandra images of its coefficients are found by*

$$\chi:\mathbb{S}_\mu(u)\mapsto\sum_{\operatorname{sh}(\mathcal{T})=\mu}\prod_{\alpha\in\mu}\big(u+\lambda_{\mathcal{T}(\alpha)}+c(\alpha)\big),$$

summed over semistandard tableau $\mathcal{T}$ *of shape* μ *with entries in* $\{1,\dots,N\}$.

We postpone the proof till Chapter 10 as Proposition 4.9.1 will follow from a more general result on Yangian characters; see Remark 10.1.5(i). In the particular cases where μ is the column or row-diagram, the statement reduces to Propositions 4.6.1 and 4.7.1, as $\mathbb{S}_\mu(u)$ coincides with the polynomials (4.35) and (4.39), respectively.

The Harish-Chandra image of the polynomial $\mathbb{S}_\mu(u)$ is a symmetric polynomial in the variables $l_i = \lambda_i - i + 1$ with $i = 1, \dots, N$. The Casimir elements $\mathbb{S}_\mu = \mathbb{S}_\mu(0)$ given by

$$\mathbb{S}_\mu = \operatorname{tr}_{1,\dots,m} \mathcal{E}_{\mathcal{U}} \, (E_1 + c_1) \dots (E_m + c_m) \tag{4.45}$$

are called the *quantum immanants*. Their images are the factorial Schur polynomials

$$s_\mu(l_1, \dots, l_N \,|\, a) = \sum_{\operatorname{sh}(\mathcal{T})=\mu} \prod_{\alpha \in \mu} \big(l_{\mathcal{T}(\alpha)} + \mathcal{T}(\alpha) + c(\alpha) - 1 \big),$$

associated with the parameter sequence $a = (a_i)$, $a_i = -i + 1$; see (4.18). The following is a quantum version of Proposition 2.1.3.

Corollary 4.9.2. *The quantum immanants $\mathbb{S}_\mu$ with μ running over all diagrams with at most N rows form a basis of $\mathrm{Z}(\mathfrak{gl}_N)$.*

Proof. The factorial Schur polynomial $s_\mu(l_1, \dots, l_N \,|\, a)$ is a non-homogeneous symmetric polynomial in $l_1, \dots, l_N$ whose top degree component coincides with the Schur polynomial $s_\mu(l_1, \dots, l_N)$. Since the Schur polynomials form a basis of the algebra of symmetric polynomials so do their factorial counterparts. The claim follows due to the Harish-Chandra isomorphism (4.15). □

Remark 4.9.3. Our definition of the quantum immanants (4.45) is slightly different from the original one due to Okounkov [**128**], where they are given by

$$\operatorname{tr}_{1,\dots,m} \mathcal{E}_{\mathcal{U}} \, (E_1 - c_1) \dots (E_m - c_m). \tag{4.46}$$

These elements possess certain stability properties which motivated their definition. To make a connection between the two families, consider the automorphism of the algebra $\mathrm{U}(\mathfrak{gl}_N)$ defined by

$$\phi : E_{ij} \mapsto -E_{ji}, \tag{4.47}$$

so that ϕ takes E to the negative transpose matrix $-E^t$. Then

$$\phi : \mathbb{S}_\mu \mapsto \operatorname{tr}_{1,\dots,m} \mathcal{E}_{\mathcal{U}} \, (-E_1^t + c_1) \dots (-E_m^t + c_m).$$

It follows by induction from (1.14) that the element $\mathcal{E}_{\mathcal{U}}$ is stable under the transposition applied simultaneously to all m copies of $\operatorname{End} \mathbb{C}^N$ in (4.29). Since this operation does not affect the trace, we obtain

$$\phi : \mathbb{S}_\mu \mapsto (-1)^m \operatorname{tr}_{1,\dots,m} \mathcal{E}_{\mathcal{U}} \, (E_1 - c_1) \dots (E_m - c_m).$$

On the other hand, by twisting the action of $\mathfrak{gl}_N$ on the finite-dimensional irreducible representation $L(\lambda_1, \dots, \lambda_N)$ by the automorphism (4.47) we obtain a representation isomorphic to $L(-\lambda_N, \dots, -\lambda_1)$. Hence, the shifted variables $l_i = \lambda_i - i + 1$ get mapped by

$$l_i \mapsto -l_{N-i+1} - N + 1, \qquad i = 1, \dots, N.$$

The conclusion is that the Harish-Chandra image of the Casimir element (4.46) coincides with the polynomial $s_\mu(l_1 + N - 1, \dots, l_N + N - 1 \,|\, a)$; see also [**109**, Theorem 7.4.6]. □

4.10. Bibliographical notes

Matrix presentations of simple Lie algebras were used by Gould [**60**] in relation with characteristic identities. In a different way, they also emerge in the work of Drinfeld [**32**] in the context of quantum groups and can be obtained by restricting the R-matrix presentations of the Yangians $\mathrm{Y}(\mathfrak{g})$ to the subalgebras $\mathrm{U}(\mathfrak{g})$. A detailed exposition of properties of the factorial Schur functions can be found e.g. in the paper by Macdonald [**103**]. Propositions 4.3.1 and 4.3.2 are due to Okounkov [**128**]. More details on the use of the idempotents for the derivation of the Robinson formula (4.28) can be found in [**117**]; see also [**34**] and [**104**, Section I.3, Example 4] for other proofs. The first direct proof that the coefficients of the Capelli determinant $C(u)$ in (4.38) are Casimir elements was given by Howe and Umeda [**68**]. The images of the Gelfand invariants under the Harish-Chandra isomorphism (Corollary 4.8.2) were first found by Perelomov and Popov [**133**]. This corollary implies the Newton identity of Theorem 4.8.1. Direct proofs of this theorem were given by Umeda [**148**] and Itoh [**76**]; see also [**109**, Chapter 7] for its derivation with the use of the Yangians and more references. The matrices of the form $\alpha + E\beta$ as in Lemma 4.5.3 are among principle examples of Manin matrices; see [**21**]. The quantum immanants and related higher Capelli identities are due to Okounkov [**128**]; see also Nazarov [**124**] and Okounkov and Olshanski [**129**].