

Preface

Most mathematicians are at least aware of the Classification of Finite Simple Groups (CFSG), a major project involving work by hundreds of researchers. The work was largely completed by about 1983, though final publication of the “quasithin” part was delayed until 2004. Since the 1980s, the result has had a huge influence on work in finite group theory and in many adjacent fields of mathematics. This book attempts to survey and sample a number of topics from the very large and increasingly active research area of *applications* of the CFSG.

The book cannot hope to systematically cover all applications. Indeed, the particular applications chosen were mainly provided by contacting colleagues who are experts in many related areas. The resulting collection of material might seem somewhat scattered; however, I have tried to present the choices within contexts of wider areas of applications, and I am grateful to the referee for helpful suggestions in that direction.

Origin of the book and structure of the chapters

This book began life as a series of 10 two-hour lectures that I was invited to give during the September 2015 Venice Summer School on Finite Groups.

The material in the book has been adapted only fairly lightly from that lecture format, mainly in order to try to preserve the introductory, and comparatively informal, tone of the original. The primary difference in the book form has been to try to bring the cross-references and the literature citations up to the more formal level expected of a published reference book.

I will mention briefly one particular feature of the chapters in the book, arising from the lectures. Each original lecture was divided into two parts: typically, the first part introduced some basic theory from a particular aspect of the CFSG, and the second (“applications”) part then demonstrated ways in which that theory can be put to use. The 10 lectures have now become the 10 chapters in this book format. The reader will notice, even from the Table of Contents, that the later sections of each chapter usually reflect the applications focus of the second part of the original lecture.

Some notes on using the book as a course text

The material is intended to be accessible to an audience with basic mathematical training: for example, to beginning graduate students with at least an undergraduate course in abstract algebra. Preferably, the background should also include first-year graduate algebra, especially some experience with examples of the most familiar types of groups and, ideally, at least the basics of Lie algebras. It will also be helpful, at times, to have some exposure to various adjacent areas of

mathematics: for example, the fundamentals of algebraic topology and homological algebra, and possibly also a little combinatorics.

Of course, the original audience for the lectures varied from early graduate students through postdoctoral researchers; this variation led to the inclusion, during the lectures, of some additional explicit background material and references at various levels.

The style of presentation is deliberately fairly informal: for example, statements of some results (and even some definitions) are given with a warning such as “roughly”. The purpose is to communicate mainly just the overall flavor of the original. These approximations are normally accompanied by fuller references to the precise statements, since one main motivation for the survey is to get the reader interested enough, for at least some of the topics, to pursue the details—and maybe even to look for research problems.

The book has retained the exercises from the original lectures, typically without providing solutions. Very often, these exercises come directly after a more fully worked-out example; the exercise is then to mimic that work for some other groups so that the example in effect provides a “hint” for the exercise. Furthermore, some of the exercises also provide explicit hints, including reference to sources where similar material is worked out. Finally, some exercises are detailed further in Appendix B.

The Appendix also provides some supplementary material to the text. Much of this was generated during the lectures, in response to student questions, and was originally provided to the students via pdf files on the Web, as the lectures progressed.

The Index is intended to be substantial enough to help indicate where the main ideas (and relevant papers) are used in practice. In particular, the most substantial applications of many entries are indicated in **boldface**.

Boldface is also often used in the Index to indicate definitions, but there is some variation in the level of these definitions: Fairly standard background concepts, likely to be familiar to most readers, may be indicated in passing within the text or recalled via a footnote. Definitions which are fairly brief may be indicated in a LaTeX equation environment, while longer definitions appear in a more formal definition environment. Finally, as noted earlier, definitions which are only roughly approximated in the text should also be accompanied by a reference to an appropriate source for the full details.

Acknowledgments

I am deeply indebted to the organizers of the Summer School: Mario Mainardis, Clara Franchi, John van Bon, and Rebecca Waldecker. In particular, they made many helpful suggestions during the preparation and delivery of the lectures, as well as further urging me to publish the material afterwards. I emphasize that this post-course-updated version of the material also owes very much to the contributions of the students during the course.

During the preparation of the original lectures, I consulted many colleagues with deep expertise in applications of the CFSG (since by contrast I had personally been more active in the classification itself, rather than applications). They

were most generous in directing me to their favorite applications and in improving my outsider's description of their work. I particularly mention Michael Aschbacher, Russell Blyth, Persi Diaconis, David Green, Jesper Grodal, Bob Guralnick, Jon Hall, Derek Holt, Bill Kantor, Bob Oliver, Cheryl Praeger, Gary Seitz, Ron Solomon, Gernot Stroth, and P. H. Tiep, though many others also provided recommendations and assistance.

In an area as broad as applications of the CFSG, I had to select a comparatively small number of topics. I apologize to readers in advance if I have omitted their favorite applications.

Finally, I am grateful to the referees for very constructive recommendations.

A note on certain references. As a primary reference on the CFSG, I am mainly using the recent "CFSG outline" book:

Aschbacher–Lyons–Smith–Solomon [ALSS11].

For convenience of reference, within the lecture-course format, I tended to refer to this central source for many results, rather than to the corresponding original papers in the literature. (If the reader in fact requires those original references, they can in turn be found in the corresponding areas of [ALSS11].)

While many of the applications described in this book were suggested to me by experts in other areas, I also chose some from areas more well known to me; in particular, it was convenient for me to refer at various times to some of my other publications, notably:

- the quasithin classification: Aschbacher-Smith [AS04a, AS04b];
- the Quillen Conjecture: Aschbacher-Smith [AS93];
- a survey of subgroup complexes: Smith [Smi11].

I am grateful to Sergei Gelfand of the AMS for allowing me to provide the students with temporary online access to these during the 2015 Venice Summer School.

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