Contents

Preface xi
Origin of the book and structure of the chapters xi
Some notes on using the book as a course text xi
Acknowledgments xii

Chapter 1. Background: Simple groups and their properties 1
Introduction: Statement of the CFSG—the list of simple groups 1
1.1. Alternating groups 2
1.2. Sporadic groups 3
1.3. Groups of Lie type 5
Some easy applications of the CFSG-list 19
1.4. Structure of $\mathcal{K}$-groups: Via components in $F^*(G)$ 20
1.5. Outer automorphisms of simple groups 23
1.6. Further CFSG-consequences: e.g. doubly-transitive groups 25

Chapter 2. Outline of the proof of the CFSG: Some main ideas 29
2.0. A start: Proving the Odd/Even Dichotomy Theorem 29
2.1. Treating the Odd Case: Via standard form 36
2.2. Treating the Even Case: Via trichotomy and standard type 38
2.3. Afterword: Comparison with later CFSG approaches 44
Applying the CFSG toward Quillen's Conjecture on $\mathcal{S}_p(G)$ 45
2.4. Introduction: The poset $\mathcal{S}_p(G)$ and the contractibility conjecture 45
2.5. Quillen-dimension and the solvable case 47
2.6. The reduction of the p-solvable case to the solvable case 49
2.7. Other uses of the CFSG in the Aschbacher-Smith proof 52

Chapter 3. Thompson Factorization—and its failure: FF-methods 55
Introduction: Some forms of the Frattini factorization 55
3.1. Thompson Factorization: Using $J(T)$ as weakly-closed "W" 57
3.2. Failure of Thompson Factorization: FF-methods 59
3.3. Pushing-up: FF-modules in Aschbacher blocks 61
3.4. Weak-closure factorizations: Using other weakly-closed "W" 66
Applications related to the Martino-Priddy Conjecture 70
3.5. The conjecture on classifying spaces and fusion systems 70
3.6. Oliver's proof of Martino-Priddy using the CFSG 72
3.7. Oliver's conjecture on $J(T)$ for p odd 74

Chapter 4. Recognition theorems for simple groups 77
Introduction: Finishing classification problems 77
4.1. Recognizing alternating groups 80

4.2. Recognizing Lie-type groups 80
4.3. Recognizing sporadic groups 82
Applications to recognizing some quasithin groups 84
4.4. Background: 2-local structure in the quasithin analysis 84
4.5. Recognizing rank-2 Lie-type groups 86
4.6. Recognizing the Rudvalis group Ru 87

Chapter 5. Representation theory of simple groups 89
Introduction: Some standard general facts about representations 89
5.1. Representations for alternating and symmetric groups 91
5.2. Representations for Lie-type groups 92
5.3. Representations for sporadic groups 97
Applications to Alperin's conjecture 98
5.4. Introduction: The Alperin Weight Conjecture (AWC) 98
5.5. Reductions of the AWC to simple groups 99
5.6. A closer look at verification for the Lie-type case 100
A glimpse of some other applications of representations 102

Chapter 6. Maximal subgroups and primitive representations 105
Introduction: Maximal subgroups and primitive actions 105
6.1. Maximal subgroups of symmetric and alternating groups 106
6.2. Maximal subgroups of Lie-type groups 110
6.3. Maximal subgroups of sporadic groups 113
Some applications of maximal subgroups 114
6.4. Background: Broader areas of applications 114
6.5. Random walks on S_n and minimal generating sets 115
6.6. Applications to p-exceptional linear groups 117
6.7. The probability of 2-generating a simple group 119

Chapter 7. Geometries for simple groups 121
Introduction: The influence of Tits's theory of buildings 121
7.1. The simplex for S_n; later giving an apartment for $GL_n(q)$ 122
7.2. The building for a Lie-type group 125
7.3. Geometries for sporadic groups 129
Some applications of geometric methods 131
7.4. Geometry in classification problems 131
7.5. Geometry in representation theory 133
7.6. Geometry applied for local decompositions 136

Chapter 8. Some fusion techniques for classification problems 139
8.1. Glauberman's Z^*-theorem 139
8.2. The Thompson Transfer Theorem 143
8.3. The Bender-Suzuki Strongly Embedded Theorem 145
Analogous p-fusion results for odd primes p 149
8.4. The Z_p^*-theorem for odd p 149
8.5. Thompson-style transfer for odd p 150
8.6. Strongly p-embedded subgroups for odd p 150

Chapter 9. Some applications close to finite group theory 153
9.1. Distance-transitive graphs 153

9.2. The proportion of p-singular elements 154
9.3. Root subgroups of maximal tori in Lie-type groups 156
Some applications more briefly treated 157
9.4. Frobenius' conjecture on solutions of $x^n = 1$ 157
9.5. Subgroups of prime-power index in simple groups 158
9.6. Application to 2-generation and module cohomology 159
9.7. Minimal nilpotent covers and solvability 160
9.8. Computing composition factors of permutation groups 160

Chapter 10. Some applications farther afield from finite groups 161
10.1. Polynomial subgroup-growth in finitely-generated groups 161
10.2. Relative Brauer groups of field extensions 162
10.3. Monodromy groups of coverings of Riemann surfaces 163
Some exotic applications more briefly treated 165
10.4. Locally finite simple groups and Moufang loops 165
10.5. Waring's problem for simple groups 167
10.6. Expander graphs and approximate groups 167

Appendix 169

Appendix A. Some supplementary notes to the text 171
A.1. Notes for 6.1.1: Deducing the structures-list for S_n 171
A.2. Notes for 8.2.1: The cohomological view of the transfer map 172
A.3. Notes for (8.3.4): Some details of proofs in Holt's paper 174

Appendix B. Further remarks on certain exercises 183
B.1. Some exercises from Chapter 1 183
B.2. Some exercises from Chapter 4 184
B.3. Some exercises from Chapter 5 191
B.4. Some exercises from Chapter 6 193

Bibliography 199

Index 213