Preface

Let G be a locally compact group. Let $C^b(G)$ be the C^*-algebra of bounded continuous complex-valued functions on G with the supremum norm, and let $C_0(G)$ be the closed $*$-subalgebra of $C^b(G)$ that consists of functions vanishing at infinity. If G is abelian, let $\widehat{G}$ be the dual group of G, and let $A(G)$ be all $\widehat{f}$ (Fourier transform of f), $f \in L^1(\widehat{G})$ (the group algebra of the dual group $\widehat{G}$); and let $B(G)$ be all $\widehat{\mu}$ (the Fourier-Stieltjes transform of μ), $\mu \in M(\widehat{G})$ (the measure algebra of $\widehat{G}$). Then $A(G)$ is a subalgebra of $C_0(G)$, and $B(G)$ is a subalgebra of $C^b(G)$. Furthermore, $A(G)$ (respectively, $B(G)$) with norm from $L^1(\widehat{G})$ (respectively, $M(\widehat{G})$) is a commutative Banach algebra called the Fourier (respectively, Fourier-Stieltjes) algebra of G.

In Chapter 2, we shall introduce and study some basic properties of Fourier and Fourier-Stieltjes algebras, $A(G)$ and $B(G)$, associated to a locally compact group G based on the fundamental paper of Eymard [**73**]. $B(G)$ will be identified as the Banach space dual of the group C^*-algebra $C^*(G)$ and a fair number of basic functorial properties will be presented. Similarly, for the Fourier algebra $A(G)$, the elements will be shown to be precisely the convolution products of L^2-functions on G.

In Chapter 3, we shall study some further topics of $A(G)$ and $B(G)$. Generalizing the classical description of idempotents in the measure algebra of a locally compact abelian group, Host [**129**] has identified the integer-valued functions in $B(G)$. Host's idempotent theorem, which has numerous applications, will be shown in this chapter. A natural question is whether either of the Banach algebras $A(G)$ and $B(G)$ determines G as a topological group. This question has been affirmatively answered by Walter [**280**]. If G_1 and G_2 are locally compact groups and $B(G_1)$ and $B(G_2)$ (respectively, $A(G_1)$ and $A(G_2)$) are isometrically isomorphic, then G_1 and G_2 are topologically isomorphic or anti-isomorphic.

Amenable Banach algebras were introduced by B. E. Johnson. He showed the fundamental result that a locally compact group is amenable if and only if the group algebra $L^1(G)$ is amenable. We present a proof of the "only if" part of Johnson's result in Chapter 4. In particular, if G is abelian, then $A(G)$, being isometrically isomorphic to the L^1-algebra of the dual group $\widehat{G}$, is amenable. However, when G is nonabelian, then $A(G)$ need not be weakly amenable, even when G is compact.

In Chapter 4, we will also consider the completely bounded cohomology theory of the Fourier algebra $A(G)$ and of the Fourier-Stieltjes algebra $B(G)$. We will show that $A(G)$, equipped with the operator space structure inherited from being embedded into $VN(G)^*$, is a completely contractive Banach algebra. Using this, we establish in this chapter the fundamental result, due to Ruan [**245**], that a locally compact group G is amenable precisely when $A(G)$ is operator amenable.

An important object associated to any (nonunital) commutative Banach algebra A is the multiplier algebra $M(A)$ of A; that is, the algebra of all bounded linear maps $T : A \to A$ satisfying the equation $T(ab) = aT(b)$ for all $a, b \in A$. When A is faithful, then the map $a \to T_a$, where $T_a(b) = ab$ for $b \in A$, is a continuous embedding of A into $M(A)$.

Let G be a locally compact group. Then $M(A(G))$ consists of all bounded continuous functions u on G such that $uA(G) \subseteq A(G)$, and since $A(G)$ is an ideal in $B(G)$, $B(G)$ embeds continuously into $M(A(G))$. If G is abelian, then as shown by Wendel [**288**], $M(L^1(G)) = M(G)$, and hence $M(A(G)) = B(G)$. It is not difficult to see that this holds true, more generally, when G is amenable. One of the profound achievements in abstract harmonic analysis has been that the converse holds; that is, $M(A(G)) = B(G)$ forces G to be amenable. This was shown by Nebbia [**219**] for discrete groups G and by Losert [**201**] for nondiscrete G. We will present these results in Chapter 5.

In Chapter 6, we study spectral synthesis and ideal theory for $A(G)$. A famous theorem of Malliavin [**207**] states that spectral synthesis fails for $A(G)$ whenever G is any nondiscrete abelian locally compact group. Using this and a deep theorem of Zel′manov [**293**] ensuring the existence of infinite abelian subgroups of infinite compact groups, we prove that for an arbitrary locally compact group G, under a mild additional hypothesis, spectral synthesis holds for $A(G)$ if and only if G is discrete.

One of the most interesting problems in the ideal theory of a commutative Banach algebra is to identify the closed ideals with bounded approximate identities. For Fourier algebras this problem is also treated in Chapter 6.

The Hahn-Banach extension theorem asserts that if E is a normed linear space and F is a closed linear subspace of E, then each continuous linear functional on F extends to a continuous linear functional on E. From this it follows that given $x \in E \backslash F$, there exists a continuous linear functional ϕ on E such that $\phi = 0$ on F and $\phi(x) \neq 0$ (the Hahn-Banach separation theorem). In Chapter 7, we address the analogous properties for positive definite functions on locally compact groups.

Let G be an arbitrary locally compact group, and let H be a closed subgroup of G. We show in Chapter 2 that the restriction map $u \to u|_H$ from $A(G)$ into $A(H)$ is surjective. The corresponding problem for Fourier-Stieltjes algebras is much more delicate. We say that G has the extension property if for every closed subgroup H, each $\varphi \in P(H)$ admits an extension $\phi \in P(G)$ (equivalently, $B(H) = B(G)|_H$). The largest class of locally compact groups sharing this extension property is formed by the groups with small conjugation invariant neighbourhoods of the identity, the SIN-groups. The converse implication is true for connected Lie groups and for compactly generated nilpotent groups. More precisely, a connected Lie group has the extension property only if it is a direct product of a vector group and a compact group. On the other hand, there exists a compactly generated 2-step solvable group which has the extension property, but fails to be a SIN-group.