CHAPTER 2

Basic Theory of Fourier and Fourier-Stieltjes Algebras

In this chapter the Fourier and Fourier-Stieltjes algebras, $A(G)$ and $B(G)$, associated to a locally compact group G, are introduced and studied almost to the extent of Eymard's fundamental paper [**73**]. In particular, $B(G)$ is identified as the Banach space dual of the group C^*-algebra $C^*(G)$ and a fair number of basic functorial properties are presented. Similarly, for the Fourier algebra $A(G)$, the elements of which are shown to be precisely the convolution products of L^2-functions on G.

Given a commutative Banach algebra A, immediate problems arising are to determine the spectrum (or Gelfand space) of A and to check whether the range of the Gelfand transform is a regular function algebra. As we show in Section 2.3, the spectrum $\sigma(A(G))$ turns out to be homeomorphic to G and the Gelfand homomorphism is then nothing but the identity mapping. Moreover, $A(G)$ is regular.

We next identify, following Eymard's seminal paper [**73**], the Banach space dual of $A(G)$ as the von Neumann subalgebra $VN(G)$ of $\mathcal{B}(L^2(G))$ generated by the left regular representation of G. The fact that $A(G)$ is the predual of a von Neumann algebra will prove to be of great importance. For instance, it allows us to equip $A(G)$ with a natural operator space structure and employing the theory of operator spaces has led to significant progress, as will be shown in Chapters 4 and 6.

In Section 2.5 the very important notion of support of an operator in $VN(G)$ is introduced and several properties of these supports, which are extremely useful later on, are shown. An immediate consequence of one of the results about the support is that singletons in G are sets of synthesis for $A(G)$.

Let H be a closed subgroup of the locally compact group G. A challenging problem is whether functions in $A(H)$ and $B(H)$ extend to functions in $A(G)$ and $B(G)$, respectively. For the Fourier algebras there is a very satisfactory solution to the effect that every function in $A(H)$ extends to a function in $A(G)$ of the same norm (Section 2.6). For Fourier-Stieltjes algebras, however, the problem is considerably more difficult and its investigation will cover a major portion of Chapter 7.

If A is a nonunital Banach algebra, then often the existence of a bounded approximate identity in A proves useful. In Section 2.7 we present Leptin's theorem [**191**] saying that $A(G)$ has a bounded approximate identity precisely when the group G is amenable. The proof uses several different characterizations of amenability of a locally compact group.

The notion of Fourier algebra has been generalized by Arsac [**5**]. He associated to any unitary representation π of G a closed subspace $A_\pi(G)$ of $B(G)$ and studied these spaces extensively. When π is the left regular representation of G, then

$A_\pi(G)$ equals $A(G)$. We present in Section 2.8 those results from [**5**] which either will be needed in Chapter 3 or used in Section 2.9 to show that for certain examples of locally compact groups the Fourier-Stieltjes algebra $B(G)$ decomposes into the direct sum of $A(G)$ and $B(G/N)$ for a large normal subgroup N of G.

2.1. The Fourier-Stieltjes algebra $B(G)$

Let G be a locally compact group. In this section we introduce the Fourier-Stieltjes algebra $B(G)$ and its subspaces $B_{\mathcal{S}}(G)$, where $\mathcal{S}$ is a collection of (equivalence classes of) unitary representations of G, and prove a number of basic results on these spaces.

Let Σ denote the equivalence classes of continuous unitary representations of G. For $\mathcal{S} \subseteq \Sigma$ and $\mu \in M(G)$, let $\|\mu\|_{\mathcal{S}} = \sup\{\|\pi(\mu)\| : \pi \in \mathcal{S}\}$. Then the assignment $\mu \to \|\mu\|_{\mathcal{S}}$ is a semi-norm on $M(G)$, and for $\mu, \nu \in M(G)$, $f \in L^1(G)$ and $x, y \in G$, we have

(1) $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|$ and $\|\mu * \nu\|_{\mathcal{S}} \leqslant \|\mu\|_{\mathcal{S}}\|\nu\|_{\mathcal{S}}$
(2) $\|\mu^*\|_{\mathcal{S}} = \|\mu\|_{\mathcal{S}}$ and $\|\mu * \mu^*\|_{\mathcal{S}} = \|\mu^*\|_{\mathcal{S}}^2$
(3) $\|L_x f\|_{\mathcal{S}} = \|f\|_{\mathcal{S}}$ and $\|R_y f\|_{\mathcal{S}} = \Delta(y^{-1})\|f\|_{\mathcal{S}}$.

Let $N_{\mathcal{S}} = \{f \in L^1(G) : \pi(f) = 0 \text{ for all } \pi \in \mathcal{S}\}$. Then $N_{\mathcal{S}}$ is a closed $*$-ideal of $L^1(G)$, and if $\dot{f} = f + N_{\mathcal{S}} \in L^1(G)/N_{\mathcal{S}}$, then $\|\dot{f}\|_{\mathcal{S}} = \|f\|_{\mathcal{S}}$ defines a norm on $L^1(G)/N_{\mathcal{S}}$. It is clear that $L^1(G)/N_{\mathcal{S}}$ becomes a normed $*$-algebra, and the norm satisfies $\|\dot{f} * \dot{f}^*\|_{\mathcal{S}} = \|\dot{f}\|_{\mathcal{S}}^2$. Let $C^*_{\mathcal{S}}(G)$ denote the completion of $L^1(G)/N_{\mathcal{S}}$. The group G acts on $L^1(G)/N_{\mathcal{S}}$, and we have

$$\|L_x \dot{f}\|_{\mathcal{S}} = \|\dot{f}\|_{\mathcal{S}} = \Delta(y)\|R_y \dot{f}\|_{\mathcal{S}}$$

for all $x, y \in G$. These actions extend uniquely to the C^*-algebra $C^*_{\mathcal{S}}(G)$.

LEMMA 2.1.1. *The mapping $f \to \dot{f} = f + k(\mathcal{S})$ from $L^1(G)$ onto $(L^1(G) + k(\mathcal{S}))/k(\mathcal{S})$ extends to a homomorphism from $C^*(G)$ onto $C^*_{\mathcal{S}}(G)$ with kernel $k(\mathcal{S})$, where $k(\mathcal{S})$ denotes for all $g \in \mathcal{S}$ such that $\pi(g) = 0$.*

PROOF. Since $\|\dot{f}\|_{\mathcal{S}} = \|f\|_{\mathcal{S}} \leqslant \|f\|_{C^*}$ for $f \in L^1(G)$, the map $f \to \dot{f}$ extends uniquely to a $*$-homomorphism $\phi : C^*(G) \to C^*_{\mathcal{S}}(G)$, and $\|\phi\| \leqslant 1$. Since $\phi(C^*(G)) \supseteq \phi(L^1(G))$ and a homomorphism between C^*-algebras with dense range is surjective, $\phi(C^*(G)) = C^*_{\mathcal{S}}(G)$. It remains to show that $\ker \phi = k(\mathcal{S})$.

Let $g \in C^*(G)$ such that $\phi(g) = 0$. Then there exist $f_n \in L^1(G)$, $n \in \mathbb{N}$, such that

$$\sup_{\pi \in \mathcal{S}} \|\pi(g) - \pi(f_n)\| \leqslant \|g - f_n\|_{C^*} \to 0$$

and $\|\dot{f}_n\|_{\mathcal{S}} = \|f_n\|_{\mathcal{S}} \to 0$. This implies that $\sup_{\pi\in\mathcal{S}} \|\pi(g)\| = 0$, i.e. $g \in k(\mathcal{S})$. Conversely, let $g \in k(\mathcal{S})$ and let $f_n \in L^1(G)$ such that $\|g - f_n\|_{C^*} \to 0$. Then $\lim_{n\to\infty} \|f_n\|_{\mathcal{S}} = 0$, and hence $\phi(g) = \lim_{n\to\infty} \phi(f_n) = 0$. □

Let G be a locally compact abelian group with dual group $\widehat{G}$ and let $\mathcal{S} \subseteq \widehat{G}$. Then, for $f \in L^1(G)$, $f \in k(\mathcal{S})$ if and only if $\widehat{f}$ vanishes on $\overline{\mathcal{S}} \subseteq \widehat{G}$. Thus $N_{\mathcal{S}} = \{0\}$ if and only if $\mathcal{S}$ is dense in $\widehat{G}$, and in this case $\|f\|_{\mathcal{S}} = \|\widehat{f}\|_\infty$, and the Fourier transform is an isometric isomorphism between $C^*_{\mathcal{S}}(G) = C^*(G)$ and $C_0(\widehat{G})$. If $\mathcal{S}$ is not dense in $\widehat{G}$, then $L^1(G)/N_{\mathcal{S}}$ is isometrically isomorphic to the subalgebra of $C_0(\overline{\mathcal{S}})$ consisting of all $\widehat{f}|_{\mathcal{S}}$, $f \in L^1(G)$, equipped with the uniform norm, and $C^*_{\mathcal{S}}(G)$ identifies with $C_0(\overline{\mathcal{S}})$.

LEMMA 2.1.2. *For $\mathcal{S} \subseteq \Sigma$ and $u \in P(G)$, the following are equivalent.*

(i) $\pi_u \prec \mathcal{S}$.

(ii) *There exists a positive linear functional φ on $C^*_{\mathcal{S}}(G)$ such that, for each $f \in L^1(G)$,*

$$\varphi(f + k(\mathcal{S})) = \int_G f(x)u(x)dx.$$

PROOF. Let $u(x) = \langle \pi_u(x)\xi, \xi\rangle$, where ξ is a cyclic vector for π_u. Then $\pi_u \prec \mathcal{S}$ is equivalent to $\langle \pi_u(g)\xi, \xi\rangle = 0$ for all $g \in k(\mathcal{S})$. In fact, the sufficiency of this latter condition is immediate from the facts that $k(\mathcal{S})$ is a two-sided ideal of $C^*(G)$ and that ξ is a cyclic vector. Now, by Lemma 2.1.1, the positive linear functionals on $C^*_{\mathcal{S}}(G)$ are exactly the positive linear functionals on $C^*(G)$ which are zero on $k(\mathcal{S})$. It follows that (i) and (ii) are equivalent. □

From now on, $P_{\mathcal{S}}(G)$ will denote the set of all $u \in P(G)$ which satisfy any of the equivalent conditions in Lemma 2.1.2.

LEMMA 2.1.3. *For collections $\mathcal{S}$ and $\mathcal{T}$ of representations of G, the following conditions are equivalent.*

(i) $\mathcal{S} \prec \mathcal{T}$.

(ii) *For every $\mu \in M(G)$, $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|_{\mathcal{T}}$.*

(iii) *For every $f \in L^1(G)$, $\|f\|_{\mathcal{S}} \leqslant \|f\|_{\mathcal{T}}$.*

PROOF. First, assume (iii) and let $u \in P_{\mathcal{S}}(G)$. Then, for any $f \in L^1(G)$,

$$\left|\int_G f(x)u(x)dx\right| \leqslant \|f\|_{\mathcal{S}}u(e) \leqslant \|f\|_{\mathcal{T}}u(e).$$

Thus u defines a positive linear functional on $L^1G)/N_{\mathcal{T}}$, which extends to a positive linear functional on $C^*_{\mathcal{T}}(G) = C^*(G)/k(\mathcal{T})$. This implies that $\pi_u \prec \mathcal{T}$. Since this holds for every $u \in P_{\mathcal{S}}(G)$, (i) follows.

Now suppose that (i) holds. Since $k(\mathcal{T}) \subseteq k(\mathcal{S})$, we get

$$\|f\|_{\mathcal{S}} = \inf\{\|f + g\|_{C^*(G)} : g \in k(\mathcal{S})\} \leqslant \inf\{\|f + g\|_{C^*(G)} : g \in k(\mathcal{T})\} = \|f\|_{\mathcal{T}}$$

for every $f \in L^1(G)$. From this inequality we are going to deduce $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|_{\mathcal{T}}$ for $\mu \in M(G)$. Let $\mathcal{V}$ be a neighbourhood basis of the identity, and for each $V \in \mathcal{V}$, choose a nonnegative continuous function g_V with $\operatorname{supp} g_V \subseteq V$, $\|g_V\|_1 = 1$ and let $f_V = \mu * g_V$. Then the bounded net $(f_V)_V$ in $L^1(G)$ converges to μ in the weak topology $\sigma(M(G), C^b(G))$ of G. Let π be an arbitrary unitary representation and $\xi, \eta \in \mathcal{H}(\pi)$. Then

$$\begin{aligned}\langle \pi(\mu)\xi, \eta\rangle &= \int_G \langle \pi(x)\xi, \eta\rangle d\mu(x) = \lim_V \int_G \langle \pi(x)\xi, \eta\rangle f_V(x)dx \\ &= \lim_V \langle \pi(f_V)\xi, \eta\rangle,\end{aligned}$$

and hence the net of operators $\pi(f_V)$ converges to $\pi(\mu)$ in the weak operator topology of $\mathcal{B}(\mathcal{H}(\pi))$ of G. Since $\|\pi(f_V)\| \leqslant \|\pi(\mu)\| \cdot \|\pi(g_V)\| \leqslant \|\pi(\mu)\|$ and the ball of radius $\|\pi(\mu)\|$ in $\mathcal{B}(\mathcal{H}(\pi))$ is weakly closed and the weak topology agrees with the ultraweak topology, it follows that $\|\pi(\mu)\| = \sup_{V \in \mathcal{V}} \|\pi(f_V)\|$. Since $\mathcal{S} \prec \mathcal{T}$, for

$\pi \in \mathcal{S}$ we get

$$\begin{aligned}\|\pi(\mu)\| &= \sup_{V\in\mathcal{V}} \|\pi(f_V)\| \leqslant \sup_{V\in\mathcal{V}} \|\pi(f_V)\|_{\mathcal{S}} \\ &\leqslant \sup_{V\in\mathcal{V}} \|\pi(f_V)\|_{\mathcal{T}} = \sup_{\sigma\in\mathcal{T}} \left(\sup_{V\in\mathcal{V}} \|\sigma(f_V)\|\right) = \|\mu\|_{\mathcal{T}}.\end{aligned}$$

Since $\pi \in \mathcal{S}$ was arbitrary, we conclude that $\|\mu\|_{\mathcal{S}} \leqslant \|\mu\|_{\mathcal{T}}$. This completes the proof. □

LEMMA 2.1.4. *For a function u on G and a collection $\mathcal{S}$ of unitary representations of G the following assertions are equivalent.*

(i) *u is a finite linear combination of functions in $P_{\mathcal{S}}(G)$.*

(ii) *There exist a unitary representation π of G, which is weakly contained in the $\mathcal{S}$, and vectors $\xi, \eta \in \mathcal{H}(\pi)$ such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for all $x \in G$.*

(iii) *u is a bounded continuous function and*

$$\sup_{f\in L^1(G), \|f\|_{\mathcal{S}}\leqslant 1} \left|\int_G f(x)u(x)dx\right| < \infty.$$

PROOF. (i) $\Rightarrow$ (ii) Since every $u \in P_{\mathcal{S}}(G)$ can be represented as in (ii), it suffices to observe that the functions in (ii) form a linear space. So let π_1 and π_2 be representations that are both weakly contained in $\mathcal{S}$ and let $\xi_j, \eta_j \in \mathcal{H}(\pi_j)$, $j = 1, 2$. Then $\pi_1 \oplus \pi_2$ is weakly contained in $\mathcal{S}$ and

$$\langle(\pi_1 \oplus \pi_2)(\xi_1 \oplus \xi_2), \eta_1 \oplus \eta_2\rangle = \sum_{j=1}^{2} \langle \pi_j(x)\xi_j, \eta_j\rangle.$$

(ii) $\Rightarrow$ (iii) Since $\pi \prec \mathcal{S}$, for each $f \in L^1(G)$,

$$\left|\int_G f(x)u(x)dx\right| = |\langle \pi(f)\xi, \eta\rangle| \leqslant \|\pi(f)\| \cdot \|\xi\| \cdot \|\eta\| \leqslant \|f\|_S\|\xi\| \cdot \|\eta\|.$$

(iii) $\Rightarrow$ (i) Condition (iii) implies that u defines a bounded linear functional on $C^*_{\mathcal{S}}(G)$, which then can be written as a linear combination of positive functionals on $C^*_{\mathcal{S}}(G)$. By Lemma 2.1.2(iii), each of the latter functionals is given by a function in $P_{\mathcal{S}}(G)$. □

DEFINITION 2.1.5. Let $B_{\mathcal{S}}(G)$ denote the set of functions satisfying the equivalent conditions of Lemma 2.1.4. Of course $B_{\mathcal{S}}(G)$ is translation invariant. We simply write $B(G)$ for $B_{\Sigma(G)}(G)$. Thus $B(G)$ consists of all finite linear combinations of continuous positive definite functions and hence equals the collection of all coefficient functions $\langle \pi(\cdot)\xi, \eta\rangle$, where $\pi \in \Sigma(G)$ and $\xi, \eta \in \mathcal{H}(\pi)$.

It follows from Lemma 2.1.4 that $B(G)$ identifies with the Banach space dual of $C^*(G)$ through the pairing

$$\langle f, u\rangle = \int_G f(x)u(x)\, dx, \quad f \in L^1(G), u \in B(G). \tag{2.1}$$

The norm on $B(G)$ is then given by

$$\|u\| = \sup\left\{\left|\int_G f(x)u(x)dx\right| : f \in L^1(G), \|f\|_{C^*} \leqslant 1\right\}. \tag{2.2}$$

Note that if $u(\cdot) = \langle \pi(\cdot)\xi, \eta\rangle$, then $\langle g, u\rangle = \langle \pi(g)\xi, \eta\rangle$ for all $g \in C^*(G)$.

REMARK 2.1.6. (1) For any $\mathcal{S} \subseteq \Sigma(G)$, $C^*_{\mathcal{S}}(G)$ is isometrically isomorphic to a quotient of $C^*(G)$ and hence $B_{\mathcal{S}}(G) = C^*_{\mathcal{S}}(G)^*$ is a closed subspace of $B(G)$. Thus the norms of u, considered as an element of $C^*_{\mathcal{S}}(G)$ and of $C^*(G)$, respectively, are equal. In particular, if $u \in B_\lambda(G)$, then

$$\|u\| = \sup \left\{ \left| \int_G f(x)u(x)dx \right| : f \in C_c(G), \|\lambda_G(f)\| \leqslant 1 \right\}.$$

(2) If $u \in B_{\mathcal{S}}(G)$, then u admits Jordan decompositions

$$u = u^+ - u^-, \ u^+, u^- \in P(G), \ \|u\| = u^+(e) + u^-(e) \tag{2.3}$$

as an element of $B(G)$, and

$$u = u^+_{\mathcal{S}} - u^-_{\mathcal{S}}, \ u^+_{\mathcal{S}}, u^-_{\mathcal{S}} \in P_{\mathcal{S}}(G), \ \|u\| = u^+_{\mathcal{S}}(e) + u^-_{\mathcal{S}}(e)$$

as an element of $B_{\mathcal{S}}(G)$. It follows from the uniqueness of the decomposition (2.3) that $u^+_{\mathcal{S}} = u^+$ and $u^-_{\mathcal{S}} = u^-$. Similarly, for $u \in B_{\mathcal{S}}(G) = C^*_{\mathcal{S}}(G)^*$, the absolute value $|u|$ does not depend on S. In fact, this follows from the uniqueness of the polar decomposition. This in particular shows that if $u \in B_{\mathcal{S}}(G)$, then $|u| \in P_{\mathcal{S}}(G)$ and if $u = \widetilde{u}$, then also u^+ and u^- belong to $P_{\mathcal{S}}(G)$.

(3) Let ω denote the universal representation of G (see [**270**, page 122]). Then $C^*(G)^{**}$ is the von Neumann subalgebra of $\mathcal{B}(\mathcal{H}(\omega))$ generated by either the operators $\omega(f)$, $f \in L^1(G)$, or the operators $\omega(x)$, $x \in G$, since the sets $\omega(L^1(G))$ and $\omega(G)$ have the same commutant in $\mathcal{B}(\mathcal{H}(\omega))$. Let $\mu \in M(G)$ and let $\mathcal{V}$ be a neighbourhood basis of e. For each $V \in \mathcal{V}$, let g_V and $f_V = \mu * g_V$ be as in the proof of Lemma 2.1.3. Then $\omega(\mu)$ is the ultraweak limit of the net $(\omega(f_V))_V$ and hence $\omega(\mu) \in C^*(G)^{**}$. Since $B(G) = C^*(G)^*$ and

$$\langle \omega(f_V), u \rangle = \int_G f_V(x)u(x)dx,$$

for all V, passing to the limit, by duality of $B(G)$ and $C^*(G)^{**}$, we conclude that

$$\langle \omega(\mu), u \rangle = \int_G u(x)d\mu(x), \tag{2.4}$$

and in particular

$$\langle \omega(x), u \rangle = u(x) \tag{2.5}$$

for $u \in B(G)$ and $x \in G$.

(4) Let $u \in B(G)$ and let π be a representation of G and $\xi, \eta \in \mathcal{H}(\pi)$ such that $u(x) = \langle \pi(x)\xi, \eta \rangle$ for all $x \in G$. Then, for any $g \in C^*(G)$,

$$\langle \omega(g)\xi, \eta \rangle = \langle \pi(g)\xi, \eta \rangle = \langle \pi''(\omega(g))\xi, \eta \rangle.$$

Again, passing to the ultraweak limit, we get

$$\langle T, u \rangle = \langle \pi''(T)\xi, \eta \rangle \tag{2.6}$$

for all $T \in C^*(G)^{**}$.

LEMMA 2.1.7. *Let $u = \widetilde{u} \in B(G)$. Then the functions u^+, u^- and $|u|$ are uniform limits on G of finite linear combinations of right translates of u.*

PROOF. Let $u = V|u|$ be the polar decomposition of u. Then, since $u^+ = \frac{1}{2}(|u| + u)$ and $u^- = \frac{1}{2}(|u| - u)$, it suffices to prove the assertion for $|u|$. Let A denote the $*$-subalgebra of $C^*(G)^{**}$ generated by the operators $\omega(x)$, $x \in G$. Since V^* is a partial isometry, V^* is contained in the unit ball of $C^*(G)^{**}$. Then, by

Kaplansky's density theorem, V^* is the limit in the strong operator topology of a net $(S_\alpha)_\alpha$ in A such that $\|S_\alpha\| \leqslant 1$ for all α. Each of the functions

$$x \to S_\alpha u(x) = \langle \omega(x), S_\alpha u\rangle = \langle \omega(x) S_\alpha, u\rangle$$

is a linear combination of right translates of u. Now, using that $u = \widetilde{u}$, formula (2.5) and the estimate $|\langle T^*, u\rangle|^2 \leqslant \|u\| \langle T^*T, |u|\rangle$ for every $T \in C^*(G)^{**}$, we obtain for $x \in G$

$$\begin{aligned} ||u|(x) - S_\alpha u(x)|^2 &= |\langle \omega(x), |u|\rangle - \langle \omega(x) S_\alpha, u\rangle|^2 \\ &= |\langle \omega(x), V^* u\rangle - \langle \omega(x) S_\alpha, u\rangle|^2 \\ &= |\langle \omega(x)(V^* - S_\alpha), u\rangle|^2 \\ &= |\langle (V^* - S_\alpha)^* \omega(x^{-1}), u\rangle|^2 \\ &\leqslant \|u\| \langle (V^* - S_\alpha)^* \omega(x^{-1}) \omega(x)(V^* - S_\alpha), |u|\rangle \\ &= \|u\| \langle (V^* - S_\alpha)^*(V^* - S_\alpha), |u|\rangle. \end{aligned}$$

Now $S_\alpha \to V^*$ in the strong operator topology, and therefore the bounded net $[(V^* - S_\alpha)^*(V^* - S_\alpha)]_\alpha$ in $C^*(G)^{**}$ converges in the weak operator topology and hence ultraweakly to zero. Since the weak operator topology on $C^*(G)^{**}$ coincides with the topology $\sigma(C^*(G)^{**}, B(G))$ on bounded sets, the above estimate implies that the net $(S_\alpha u)_\alpha$ converges uniformly on G to $|u|$. □

The next two lemmas provide additional expressions for the norms of elements in $B_{\mathcal{S}}(G)$.

LEMMA 2.1.8. *Let $\mathcal{S} \subseteq \Sigma(G)$ and $u \in B_{\mathcal{S}}(G)$. Then*

$$\|u\| = \sup\left\{ \left| \sum_{j=1}^n c_j u(x_j) \right| : x_i \in G, c_i \in \mathbb{C}, 1 \leqslant i \leqslant n, n \in \mathbb{N}, \left\| \sum_{j=1}^n c_j \delta_{x_j} \right\|_{\mathcal{S}} \leqslant 1 \right\}.$$

PROOF. Suppose first that $\mathcal{S} = \Sigma(G)$ and let A denote the C^*-subalgebra of $C^*(G)^{**}$ generated by $\omega(G)$. Then, by Kaplansky's density theorem, the unit ball of A is ultraweakly dense in the unit ball of $C^*(G)^{**}$. By (2.5) this implies

$$\begin{aligned} \|u\| &= \sup\{|\langle T, u\rangle| : T \in C^*(G)^{**}, \|T\| \leqslant 1\} \\ &= \sup\{|\langle S, u\rangle| : S \in A, \|S\| \leqslant 1\} \\ &= \sup\left\{ \left| \sum_{j=1}^n c_j u(x_j) \right| \right\}, \end{aligned}$$

where the supremum extends over all $x_1, \ldots, x_n \in G, c_1, \ldots, c_n \in \mathbb{C}$ such that $\left\| \sum_{j=1}^n c_j \delta_j \right\|_{\mathcal{S}} \leqslant 1$.

Now let $\mathcal{S}$ be arbitrary and consider G_d, the group G made discrete. Since $P_{\mathcal{S}}(G) \subseteq P_{\mathcal{S}}(G_d)$, $u \in B_{\mathcal{S}}(G_d)$ and Remark 2.1.6(1), applied to G_d, shows that

$$\sup\left\{ \left| \sum_{j=1}^n c_j u(x_j) \right| : \left\| \sum_{j=1}^n c_j \delta_j \right\|_{S} \leqslant 1 \right\} = \sup\left\{ \left| \sum_{j=1}^n c_j u(x_j) \right| : \left\| \sum_{j=1}^n c_j \delta_j \right\|_{\Sigma(G)} \leqslant 1 \right\}.$$

The statement of the lemma now follows from the first part of the proof. □

LEMMA 2.1.9. *Let $u \in B(G)$ and $u(x) = \langle \pi(x)\xi, \eta\rangle$, $x \in G$. Then $\|u\| \leqslant \|\xi\| \cdot \|\eta\|$. Conversely, if $u \in B_{\mathcal{S}}(G)$ then there exist a representation π which is weakly contained in $\mathcal{S}$ and $\xi, \eta \in \mathcal{H}(\pi)$ such that*

$$u(x) = \langle \pi(x)\xi, \eta\rangle,\ x \in G, \quad \text{and} \quad \|u\| = \|\xi\| \cdot \|\eta\|.$$

More precisely, if $u = V|u|$, $V \in C^*(G)^{**}$, denotes the polar decomposition of u, it suffices to take π and η such that $|u|(x) = \langle \pi(x)\eta, \eta\rangle$, where η is a cyclic vector in $\mathcal{H}(\pi)$, and then put $\xi = \pi''(V)\eta$.

PROOF. The first statement follows from

$$\begin{aligned} \|u\| &= \sup\left\{\left|\int_G f(x)u(x)dx\right| : f \in L^1(G), \|f\|_{C^*} \leqslant 1\right\} \\ &= \sup\left\{|\langle \pi(f)\xi, \eta\rangle| : f \in L^1(G), \|f\|_{C^*} \leqslant 1\right\} \\ &\leqslant \|\xi\| \cdot \|\eta\|. \end{aligned}$$

Now choose π, η and ξ as announced. Then, for every $x \in G$,

$$\begin{aligned} u(x) &= \langle \omega(x), u\rangle = \langle \omega(x), V|u|\rangle = \langle \omega(x)V, |u|\rangle \\ &= \langle \pi''(\omega(x)V)\eta, \eta\rangle = \langle \pi(x)\pi''(V)\eta, \eta\rangle \\ &= \langle \pi(x)\xi, \eta\rangle. \end{aligned}$$

Since $|u| \in P_{\mathcal{S}}(G)$, π is weakly contained in $\mathcal{S}$ (compare Lemma 2.1.2). So it only remains to show that $\|u\| \geqslant \|\xi\| \cdot \|\eta\|$. Now, since V is a partial isometry, $\|V\| = 1$ and hence $\|\pi''(V)\| \leqslant 1$ and $\|\xi\| = \|\pi''(V)\eta\| \leqslant \|\eta\|$. Consequently,

$$\|u\| = \||u|\| = |u|(e) = \|\eta\|^2 \geqslant \|\xi\| \cdot \|\eta\|,$$

as required. □

REMARK 2.1.10. (1) For each $u \in B(G)$, we have $\|u\|_\infty \leqslant \|u\|$. In fact, since $\|f\|_{C^*} \leqslant \|f\|_1$ for every $f \in L^1(G)$, (2.2) implies

$$\begin{aligned} \|u\| &= \sup\left\{\left|\int_G f(x)u(x)dx\right| : f \in L^1(G), \|f\|_{C^*} \leqslant 1\right\} \\ &\geqslant \sup\left\{\left|\int_G f(x)u(x)dx\right| : f \in L^1(G), \|f\|_1 \leqslant 1\right\} = \|u\|_\infty. \end{aligned}$$

(2) Let $\mu \in M(G)$. Then, for any $\mathcal{S} \subseteq \sum(G)$,

$$\|\mu\|_{\mathcal{S}} = \sup\left\{\left|\int_G u(x)d\mu(x)\right| : u \in B_{\mathcal{S}}(G), \|u\| \leqslant 1\right\}.$$

To see this, let ϕ denote the continuous linear functional on $B_{\mathcal{S}}(G)$ defined by $\langle \phi, u\rangle = \int_G u(x)d\mu(x)$ and let $u(x) = \langle \pi(x)\xi, \eta\rangle$, where π is weakly contained in $\mathcal{S}$ and $\xi, \eta \in \mathcal{H}(\pi)$ satisfy $\|u\| = \|\xi\| \cdot \|\eta\|$ (Lemma 2.1.9). Then

$$|\langle \phi, u\rangle| = |\langle \pi(\mu)\xi, \eta\rangle| \leqslant \|\pi(\mu)\| \cdot \|\xi\| \cdot \|\eta\| \leqslant \|\mu\|_S \|u\|$$

and hence $\|\mu\|_S \geqslant \|\phi\|$. Conversely, consider any $\pi \in S$ and $\xi, \eta \in \mathcal{H}(\pi)$ with $\|\xi\| \leqslant 1$ and $\|\eta\| \leqslant 1$ and let $v(x) = \langle \pi(x)\xi, \eta\rangle$. Then $v \in B_{\mathcal{S}}(G)$ and

$$|\langle \pi(\mu)\xi, \eta\rangle| = \left|\int_G v(x)d\mu(x)\right| \leqslant \|\phi\| \cdot \|v\| \leqslant \|\phi\| \cdot \|\xi\| \cdot \|\ \eta\| \leqslant \|\phi\|.$$

Since $\|\mu\|_S$ is the supremum of all such values $|\langle \pi(\mu)\xi, \eta\rangle|$, it follows that $\|\mu\|_S \leqslant \|\phi\|$.

(3) Let $u \in B(G)$, (respectively, $u \in B_\lambda(G)$). Then the functions $\widetilde{u}$, $\overline{u}$ and $\check{u}$ all belong to $B(G)$ (respectively, $B_\lambda(G)$) and

$$\|u\| = \|\widetilde{u}\| = \|\overline{u}\| = \|\check{u}\|.$$

The first statement follows from the fact that it holds for $P(G)$ (respectively, $P_\lambda(G)$). For the equality of the norms, let $u(x) = \langle \pi(x)\xi, \eta\rangle$ with $\|u\| = \|\xi\| \cdot \|\eta\|$. Then

$$\widetilde{u}(x) = \langle \pi(x)\eta, \xi\rangle \quad \text{and} \quad \overline{u}(x) = \langle \eta, \pi(x)\xi\rangle$$

and therefore $\|\widetilde{u}\| \leqslant \|\eta\| \cdot \|\xi\| = \|u\|$ and $\|\overline{u}\| \leqslant \|u\|$. Since $(\widetilde{u})^\sim = u$ and $(\overline{u})^- = u$, the reverse norm inequalities follow. Finally, as $\check{u} = (\overline{u})^\sim$, all four norms have to be equal.

Theorem 2.1.11. *Let G be a locally compact group. Then $B(G)$, equipped with pointwise multiplication and the norm*

$$\|u\| = \sup\left\{\left|\int_G f(x)u(x)dx\right| : f \in L^1(G), \|f\|_{C^*} \leqslant 1\right\},$$

is a unital commutative Banach algebra, called the Fourier-Stieltjes algebra of G, and $B(G)$ contains $B_\lambda(G)$ as a closed ideal.

Proof. Let u and v be elements of $B(G)$. By Lemma 2.1.9 u and v admit representations

$$u(x) = \langle \pi(x)\xi, \eta\rangle \quad \text{and} \quad v(x) = \langle \pi'(x)\xi', \eta'\rangle,$$

where $\|u\| = \|\xi\| \cdot \|\eta\|$ and $\|v\| = \|\xi'\| \cdot \|\eta'\|$. Thus

$$u(x)v(x) = \langle (\pi \otimes \pi')(x)(\xi \otimes \xi'), \eta \otimes \eta'\rangle$$

and hence $uv \in B(G)$. Also,

$$\|uv\| \leqslant \|\xi \otimes \xi'\| \cdot \|\eta \otimes \eta'\| = \|\xi\| \cdot \|\eta\| \cdot \|\xi'\| \cdot \|\eta'\|.$$

So the norm $\|\cdot\|$ on $B(G)$ is submultiplicative.

Finally, $B_\lambda(G)$ is a closed linear subspace of $B(G)$. To see that $B_\lambda(G)$ is an ideal in $B(G)$, it suffices to verify that $P(G)P_\lambda(G) \subseteq P_\lambda(G)$ because $B(G)$ and $B_\lambda(G)$ are spanned by $P(G)$ and $P_\lambda(G)$, respectively. Now, $P(G)P_\lambda(G) \subseteq P_\lambda(G)$ follows from the fact that for any representation π of G, $\pi \otimes \lambda \sim \lambda$ (see Section 1.6). □

Proposition 2.1.12. *Let $\mathcal{T}$ and $\mathcal{S}$ be collections of unitary representations of G such that $P_\mathcal{T}(G)P_\mathcal{S}(G) \subseteq P_\mathcal{S}(G)$. If $u \in B_\mathcal{T}(G)$ and $\mu \in M(G)$, then $u\mu \in B_\mathcal{S}(G)$ and*

$$\|u\mu\|_\mathcal{T} \leqslant \|u\| \cdot \|\mu\|_\mathcal{S}.$$

In particular, $\|u\mu\|_{\Sigma(G)} \leqslant \|u\| \cdot \|\mu\|_{\Sigma(G)}$ for any $u \in B(G)$, and if $u \in B_\lambda(G)$, then $\|u\mu\|_{\Sigma(G)} \leqslant \|u\| \cdot \|\lambda(\mu)\|$.

Proof. The hypothesis that $P_\mathcal{T}(G)P_\mathcal{S}(G) \subseteq P_\mathcal{S}(G)$ implies $B_\mathcal{T}(G)B_\mathcal{S}(G) \subseteq B_\mathcal{S}(G)$. Then, by Remark 2.1.10(2),

$$\begin{aligned}\|u\mu\|_\mathcal{T} &= \sup\left\{\left|\int_G u(x)v(x)d\mu(x)\right| : v \in B_\mathcal{T}(G), \|v\| \leqslant 1\right\} \\ &\leqslant \sup\{\|uv\| \cdot \|\mu\|_S : v \in B_\mathcal{T}(G), \|v\| \leqslant 1\} \\ &\leqslant \|u\| \cdot \|\mu\|_\mathcal{S}.\end{aligned}$$

The remaining statements follow since $B_\lambda(G)$ is an ideal in $B(G)$. □

LEMMA 2.1.13. *Let $\mathcal{S}$ be a collection of equivalence classes of unitary representations of G and let $u \in B_{\mathcal{S}}(G)$.*

(i) *If $\mu \in M(G)$, then $\mu * u \in B_{\mathcal{S}}(G)$ and*

$$\|\mu * u\| \leqslant \|u\| \cdot \sup\{\|\sigma(\mu)\| : \sigma \in S\}.$$

(ii) *If $\Delta^{-1}\mu \in M(G)$, then $u * \mu \in B_{\mathcal{S}}(G)$ and*

$$\|u * \mu\| \leqslant \|u\| \cdot \sup\{\|\sigma(\Delta^{-1}\mu)\| : \sigma \in S\}.$$

PROOF. By Lemma 2.1.9, there exist a unitary representation π of G which is weakly contained in $\mathcal{S}$ and $\xi, \eta \in \mathcal{H}(\pi)$ such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ for all $x \in G$ and $\|u\| = \|\xi\| \cdot \|\eta\|$. Then

$$\begin{aligned} \mu * u(x) &= \int_G u(y^{-1}x)d\mu(y) = \int_G \langle \pi(x)\xi, \pi(y)\eta\rangle d\mu(y) \\ &= \langle \pi(x)\xi, \pi(\mu)\eta\rangle. \end{aligned}$$

This shows that $\mu * u \in B_{\mathcal{S}}(G)$ and $\|\mu * u\| \leqslant \|\xi\| \cdot \|\eta\| \cdot \|\pi(\mu)\|$. Since π is subordinate to S, (i) follows.

The proof of (ii) is of course similar. In fact,

$$\begin{aligned} u * \mu(x) &= \int_G u(xy^{-1})\Delta(y^{-1})d\mu(y) \\ &= \int_G \langle \pi(xy^{-1})\xi, \eta\rangle \Delta^{-1}(y)d\mu(y) \\ &= \langle \pi(x)\pi(\Delta^{-1}\mu)^*\xi, \eta\rangle. \end{aligned}$$

This implies that $u * \mu \in B_{\mathcal{S}}(G)$ and

$$\|u * \mu\| \leqslant \|\xi\| \cdot \|\eta\| \cdot \|\pi(\Delta^{-1}\mu)\|,$$

so that (ii) follows. □

Taking $\mu = \delta_x$ in (i) and $\mu = \Delta\delta_{x^{-1}}$ in (ii) of Lemma 2.1.13, respectively, we get

COROLLARY 2.1.14. *Let $u \in B_{\mathcal{S}}(G)$ and $x \in G$. Then $L_x u \in B_{\mathcal{S}}(G)$ and $R_x u \in B_{\mathcal{S}}(G)$ and $\|L_x u\| = \|u\| = \|R_x u\|$.*

REMARK 2.1.15. Let G be a locally compact abelian group and $\widehat{G}$ its dual group. We claim that $B(\widehat{G})$ is isometrically isomorphic to the measure algebra $M(G)$ via the Fourier-Stieltjes transform $\mu \to \widehat{\mu}$, $\mu \in M(G)$. In fact, this can be seen as follows.

Given $u \in B(\widehat{G})$, by Bochner's theorem there exists $\mu \in M(G)$ such that

$$u(\chi) = \widehat{\mu}(\chi) = \int_G \chi(x)d\mu(x)$$

for all $\chi \in \widehat{G}$. Then, using the inversion formula,

$$\begin{aligned}\|u\| &= \sup\left\{\left|\int_{\widehat{G}} u(\chi)f(\chi)d\chi\right| : f \in C_c(\widehat{G}), \|f\|_{C^*(\widehat{G})} \leqslant 1\right\}\\ &= \sup\left\{\left|\int_{\widehat{G}} u(\chi)\widehat{g}(\chi)d\chi\right| : g \in C_c(G), \|g\|_\infty \leqslant 1\right\}\\ &= \sup\left\{\left|\int_G \left(\int_{\widehat{G}} \widehat{g}(\chi)\chi(x)\right) d\mu(x)\right| : g \in C_c(G), \|g\|_\infty \leqslant 1\right\}\\ &= \sup\left\{\left|\int_G g(x)d\mu(x)\right| : g \in C_c(G), \|g\|_\infty = 1\right\}\\ &= \|\mu\|.\end{aligned}$$

Recall that by the Pontryagin duality theorem, a locally compact abelian group G is topologically isomorphic to the dual group of $\widehat{G}$. Therefore, the above can be restated as follows. For any locally compact abelian group G, $B(G)$ is isometrically isomorphic to the measure algebra $M(\widehat{G})$, the isomorphism being performed by the inverse Fourier-Stieltjes transform $\mu \to \check{\mu}$, $\mu \in M(\widehat{G})$.

2.2. Functorial properties of $B(G)$

Let G and H be locally compact groups and $\phi : H \to G$ a continuous homomorphism. If π is a unitary representation of G, then $\pi \circ \phi$ is a unitary representation of H in the same Hilbert space. If u is a positive definite function associated with π, that is, $u(x) = \langle \pi(x)\xi, \xi\rangle$ for some $\xi \in \mathcal{H}(\pi)$ and all $x \in G$, then $u \circ \phi(y) = \langle \pi \circ \phi(y)\xi, \xi\rangle$ defines a positive definite function of H associated with $\pi \circ \phi$. Moreover, if $\mathcal{S}$ is a set of equivalence classes of representations of G, then we denote by $\mathcal{S} \circ \phi$ the set of equivalence classes of representations $\pi \circ \phi$, $\pi \in \mathcal{S}$. Note that if π and π' are equivalent representations of G then $\pi \circ \phi$ and $\pi' \circ \phi$ are equivalent.

THEOREM 2.2.1. *Let ϕ be a continuous homomorphism from H into G.*

(i) *The map $j : u \to u \circ \phi$ is a norm decreasing homomorphism from $B(G)$ into $B(H)$, and for any $\mathcal{S}$,*

$$j(P_{\mathcal{S}}(G)) \subseteq P_{\mathcal{S}\circ\phi}(H) \quad \text{and} \quad j(B_{\mathcal{S}}(G)) \subseteq B_{\mathcal{S}\circ\phi}(H).$$

(ii) *Suppose that $\phi(H)$ is dense in G. Then j is isometric and*

$$j(B(G)) = B(H) \cap j(C(G)) = B_{\Sigma(H)\circ\phi}(H) \cap j(C(G)). \tag{2.7}$$

Moreover, if $u = \widetilde{u} \in B(G)$, then

$$(u \circ \phi)^+ = u^+ \circ \phi \quad \text{and} \quad (u \circ \phi)^- = u^- \circ \phi.$$

(iii) *Suppose that ϕ is surjective and that given any compact subset K of G, there exists a compact subset C of H such that $\phi(C) = K$. Then*

$$j(B_{\mathcal{S}}(G)) = B_{\mathcal{S}\circ\phi}(H) \cap j(C(G)) \tag{2.8}$$

for any subset $\mathcal{S}$ of $\Sigma(G)$.

PROOF. (i) It is clear that if u is positive definite, then so is $u \circ \phi$. Now assume that $u \in P_{\mathcal{S}}(G)$. To show that $u \circ \phi \in P_{\mathcal{S}\circ\phi}(H)$, let a compact subset C of H and $\epsilon > 0$ be given. Since $K = \phi(C)$ is compact, there exist elements $u_1, \ldots, u_n$

of $P(G)$ associated with representations $\pi_1, \ldots, \pi_n$ in S, respectively, such that $|u(x) - \sum_{j=1}^n u_j(x)| \leqslant \epsilon$ for all $x \in K$. Thus

$$\left| u \circ \phi(y) - \sum_{j=1}^{n} u_j \circ \phi(y) \right| \leqslant \epsilon$$

for all $y \in C$. Since $u_j \circ \phi$ is associated to $\pi_j \circ \phi$, it follows that $u \circ \phi \in P_{S \circ \phi}(H)$. By linearity, we obtain that $j(B_{\mathcal{S}}(G)) \subseteq B_{S\circ\phi}(H)$.

Taking $\mathcal{S} = \Sigma(G)$, we conclude that j is a homomorphism from $B(G)$ into $B(H)$. To show that j is norm decreasing, let $u \in B(G)$ such that $u(x) = \langle \pi(x)\xi, \eta\rangle$ with $\|u\| = \|\xi\| \cdot \|\eta\|$. Then $u \circ \phi(y) = \langle \pi \circ \phi(y)\xi, \eta\rangle$ and hence $\|u \circ \phi\| \leqslant \|\xi\| \cdot \|\eta\| = \|u\|$.

(ii) Suppose that $\phi(H)$ is dense in G. Let ω denote the universal representation of G and A the subalgebra of $C^*(G)^{**}$ consisting of all finite linear combinations of operators $\omega(x)$, $x \in \phi(H)$. Since the mapping $x \to \omega(x)$ from G into $\mathcal{B}(\mathcal{H}(\omega))$ is continuous with respect to the strong operator topology, A is strongly dense in $C^*(G)^{**}$. Then the Kaplansky density theorem assures that the unit ball of A is ultra-weakly dense in the unit ball of $C^*(G)^{**}$.

Now, let $u \in B(G) = B_\omega(G)$. Then $u \circ \phi \in B_{\omega\circ\phi}(H)$ by (i), and hence, by Lemma 2.1.8,

$$\begin{aligned}
\|u \circ \phi\| &= \sup\left\{ \left| \sum_{j=1}^{n} c_j (u \circ \phi)(y_j) \right| : y_j \in H, c_j \in \mathbb{C}, \left\| \sum_{j=1}^{n} c_j (\omega \circ \phi)(y_j) \right\| \leqslant 1 \right\} \\
&= \sup\left\{ \left| \sum_{j=1}^{n} c_j u(x_j) \right| : x_j \in \phi(H), c_j \in \mathbb{C}, \left\| \sum_{j=1}^{n} c_j \omega(x_j) \right\| \leqslant 1 \right\} \\
&= \sup\{ |\langle T, u\rangle| : T \in A, \|T\| \leqslant 1\} \\
&= \sup\{ |\langle T, u\rangle| : T \in C^*(G)^{**}, \|T\| \leqslant 1\} \\
&= \|u\|.
\end{aligned}$$

This shows that j is isometric.

Let $u = \widetilde{u} \in B(G)$. Then

$$u \circ \phi = (u \circ \phi)^{\sim} = u^+ \circ \phi - u^- \circ \phi.$$

On the other hand, since j is isometric,

$$\|u \circ \phi\| = \|u\| = u^+(e_G) + u^-(e_G) = u^+ \circ \phi(e_H) - u^- \circ \phi(e_H).$$

These two equations imply that $(u \circ \phi)^+ = u^+ \circ \phi$ and $(u \circ \phi)^- = u^- \circ \phi$. Since, by (i),

$$j(B(G)) \subseteq B_{\Sigma(H)\circ\phi}(H) \cap j(C(G)) \subseteq B(H) \cap j(C(G)),$$

to finish the proof of (ii), it remains to show that if $v = u \circ \phi$, where $u \in C(G)$ and $v \in B(H)$, then $u \in B(G)$. Obviously, we can assume that $v = \widetilde{v}$. By Lemma 2.1.7, v^+ and v^- are uniform limits on H of linear combinations of right translates of $v = u \circ \phi$. Therefore, given $m \in \mathbb{N}$, there exist $y_1, \ldots, y_n \in H$ and $c_1, \ldots, c_n \in \mathbb{C}$ such that, for every $y \in H$,

$$\left| v^+(y) - \sum_{j=1}^{n} c_j R_{y_j}(u \circ \phi)(y) \right| \leqslant 1/m.$$

Setting $v_m = \sum_{j=1}^n c_j R_{\phi(y_j)} u$ and observing that

$$\sum_{j=1}^n c_j R_{y_j}(u \circ \phi) = \sum_{j=1}^n c_j (R_{\phi(y_j)} u) \circ \phi = \left(\sum_{j=1}^n c_j R_{\phi(y_j)} u \right) \circ \phi = v_m \circ \phi,$$

we see that for any $m \in \mathbb{N}$, there exists v_m, a linear combination of translates of u, such that

$$|v^+(y) - v_m \circ \phi(y)| \leqslant 1/m, \quad y \in H. \tag{2.9}$$

By (2.9) the sequence of functions v_m, $m \in \mathbb{N}$, converges uniformly on $\phi(H)$ and hence, by density, uniformly on G, to a function w^+ on G. Being a linear combination of translates of u, v_m is a continuous function on G and therefore w^+ is continuous. It follows from (2.9) that w^+ satisfies $v^+ = w^+ \circ \phi$. Moreover, since v^+ is positive definite on H, w^+ is positive definite on $\phi(H)$ and hence positive definite on G by continuity. So we have shown that $v^+ = w^+ \circ \phi$ for some $w^+ \in P(G)$.

Similarly, it is shown that there exists $w^- \in P(G)$ such that $v^- = w^- \circ \phi$. Since

$$u \circ \phi = v = v^+ - v^- = w^+ \circ \phi - w^- \circ \phi = (w^+ - w^-) \circ \phi,$$

we get $u = w^+ - w^- \in B(G)$. This shows (2.7).

(iii) Suppose that ϕ is onto and satisfies the condition on compact sets in (iii). Let $v \in B_{\mathcal{S}\circ\phi}(H)$, so that v^+ and v^- belong to $P_{\mathcal{S}\circ\phi}(H)$. Let K be a compact subset of G and $\epsilon > 0$, and choose a compact subset C of H such that $\phi(C) = K$. Since v^+ and v^- can be uniformly approximated on C, up to ϵ, by sums of continuous positive definite functions associated to $\pi\circ\phi$, where $\pi \in \mathcal{S}$, w^+ and w^- (see (ii)) can be uniformly approximated on K, up to ϵ, by continuous positive definite functions associated with representation π, $\pi \in \mathcal{S}$. Since K and $\epsilon > 0$ are arbitrary, it follows that w^+, $w^- \in P_{\mathcal{S}}(G)$ and hence $u = w^+ - w^- \in B_{\mathcal{S}}(G)$. This shows that

$$B_{\mathcal{S}\circ\phi} \cap j(C(H)) \subseteq j(B_{\mathcal{S}}(H)),$$

and hence (2.8) follows. □

As an immediate consequence of the preceding theorem we obtain the following corollary. It extends, to arbitrary locally compact groups, a theorem which is due to Bochner and Schoenberg for $\mathbb{R}$ and to Eberlein for general locally compact abelian groups G and characterizes the Fourier-Stieltjes transforms of measures in $M(G)$.

Corollary 2.2.2. *Let G be a locally compact group and G_d the same group equipped with the discrete topology. A function u on G belongs to $B(G)$ if and only if u is continuous and $u \in B(G_d)$. In that case, the norms of u in $B(G)$ and in $B(G_d)$ are equal.*

Corollary 2.2.3. *The unit ball of $B(G)$ is closed in $C(G)$ with respect to the topology of pointwise convergence.*

Proof. If $v \in B(G)$ is such that $\|v\| \leqslant 1$, then by Lemma 2.1.8

$$\left| \sum_{j=1}^n c_j v(x_j) \right| \leqslant \left\| \sum_{j=1}^n c_j \delta_{x_j} \right\|_{\Sigma(G)}$$

for any finitely many $x_1, \ldots, x_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$.

Now, let $u \in C(G)$ be a pointwise limit of such functions v. Then the same inequality holds for u. It then follows for any $f \in l^1(G_d)$ that

$$\left|\sum_{x\in G} f(x)u(x)\right| \leqslant \|f\|_{\Sigma(G)}$$

and hence $u \in B(G_d)$. Since u is continuous, Corollary 2.2.2 shows that $u \in B(G)$. □

COROLLARY 2.2.4. *Let G be a locally compact group, N a closed normal subgroup of G and $q : G \to G/N$ the quotient homomorphism. Then the map $u \to u \circ q$ is an isometric isomorphism from $B(G/N)$ onto the subspace of $B(G)$ consisting of all functions in $B(G)$ which are constant on cosets of N. For each $\mathcal{S} \subseteq \Sigma(G/N)$, the image of $B_{\mathcal{S}}(G/N)$ under this map is $B_{\mathcal{S}\circ q}(G)$.*

PROOF. The homomorphism q satisfies the hypotheses of Theorem 2.2.1(iii). Moreover, a function u on G/N is continuous if and only if $u \circ q$ is continuous on G. The statement now follows from Theorem 2.2.1(iii). □

Alternatively, Corollary 2.2.4 can be obtained more directly by using duality arguments for Banach spaces as follows. Let T_N denote the canonical $*$-homomorphism from $C^*(G)$ onto $C^*(G/N)$. Then the dual map, T_N^*, from $B(G/N)$ into $B(G)$ is given by

$$\begin{aligned} \langle T_N^*(u), f\rangle &= \langle u, T_N f\rangle \\ &= \int_{G/N} u(xN)\left(\int_N f(xn)dn\right) d(xN) \\ &= \int_G u(q(x))f(x)dx \\ &= \langle u \circ q, f\rangle, \end{aligned}$$

so that T_N^* equals j, where j denotes the map $u \to u \circ q$.

COROLLARY 2.2.5. *Let G be a locally compact group, bG its Bohr compactification and $\phi : G \to bG$ the canonical homomorphism. Let $AP(G)$ denote the space of almost periodic functions on G. Then the map $j : u \to u \circ \phi$ is an isometry from $B(bG)$ onto $B(G) \cap AP(G)$.*

PROOF. Since $\phi(G)$ is dense in bG, by Theorem 2.2.1(ii) j is an isometry and $j(B(bG)) = B(G) \cap j(C(bG))$. Now, just note that $j(C(bG)) = AP(G)$. □

LEMMA 2.2.6. *Let H be an open subgroup of the locally compact group G, and for a function f on H, let $\overset{\circ}{f}$ denote the trivial extension of f to all of G. Then the map $f \to \overset{\circ}{f}$ from $C_c(H)$ into $C_c(G)$ extends uniquely to an isometric $*$-homomorphism of $C_\lambda^*(H)$ into $C_\lambda^*(G)$. Its adjoint map, which is the restriction map $u \to u|_H$, is a norm decreasing algebra homomorphism from $B_\lambda(G)$ onto $B_\lambda(H)$ and it maps $P_{\lambda_G}(G)$ onto $P_{\lambda_H}(H)$.*

PROOF. The map $f \to \overset{\circ}{f}$ is clearly an injective $*$-homomorphism of $C_c(H)$ into $C_c(G)$, and

$$\begin{aligned}\|\lambda_H(f)\| &= \sup\{\|f * g\|_2 : g \in C_c(H), \|g\|_2 \leqslant 1\}\\ &= \sup\{\|\overset{\circ}{f} * \overset{\circ}{g}\|_2 : g \in C_c(H), \|g\|_2 \leqslant 1\}\\ &\leqslant \sup\{\|\overset{\circ}{f} * h\|_2 : h \in C_c(G), \|h\|_2 \leqslant 1\}\\ &= \|\lambda_G(\overset{\circ}{f})\|.\end{aligned}$$

Conversely, since $B_\lambda(G)|_H \subseteq B_\lambda(H)$ and $\|v|_H\|_{B(H)} \leqslant \|v\|_{B(G)}$ for any $v \in B(G)$,

$$\begin{aligned}\|\lambda_G(\overset{\circ}{f})\| &= \sup\left\{\left|\int_G \overset{\circ}{f}(x)v(x)dx\right| : v \in B_\lambda(G), \|v\|_{B(G)} \leqslant 1\right\}\\ &= \sup\left\{\left|\int_H f(x)v|_H(x)dx\right| : v \in B_\lambda(G), \|v\|_{B(G)} \leqslant 1\right\}\\ &\leqslant \sup\left\{\left|\int_H f(x)u(x)dx\right| : u \in B_\lambda(H), \|u\|_{B(H)} \leqslant 1\right\}\\ &= \|\lambda_H(f)\|.\end{aligned}$$

So $\|\lambda_H(f)\| = \|\lambda_G(\overset{\circ}{f})\|$ for $f \in C_c(H)$, and hence the map $f \to \overset{\circ}{f}$ extends uniquely to an isometric $*$-homomorphism ϕ from $C^*_\lambda(H)$ into $C^*_\lambda(G)$. Since

$$\int_G \overset{\circ}{f}(x)v(x)dx = \int_H f(x)v|_H(x)dx$$

for $f \in C_c(H)$ and $v \in B_\lambda(G)$, the adjoint map of ϕ is simply the map $v \to v|_H$. Finally, ϕ^* is surjective since ϕ is an isometry. □

2.3. The Fourier algebra $A(G)$, its spectrum and its dual space

Let G be a locally compact group. In this section we first introduce the Fourier algebra of G and then identify its Gelfand spectrum and its Banach space dual.

LEMMA 2.3.1. *Let $f, g \in L^2(G)$. Then $f * \widetilde{g} \in B_\lambda(G)$ [Section 2.1] and $\|f * \widetilde{g}\| \leqslant \|f\|_2\|g\|_2$.*

PROOF. For $x \in G$, we have

$$(f * \widetilde{g})(x) = \int_G f(xy)\overline{g(y)}dy = \langle \lambda_G(x^{-1})f, g\rangle.$$

The statement now follows from Lemma 2.1.9 and Remark 2.1.10. □

PROPOSITION 2.3.2. *Let G be a locally compact group, C a compact subset of G and U an open subset of G such that $C \subseteq U$. Then there exists a function u on G which is a finite linear combination of functions in $P(G) \cap C_c(G)$ and satisfies*

$$0 \leqslant u \leqslant 1,\ u|_C = 1 \text{ and } u|_{G\setminus U} = 0.$$

PROOF. Since C is compact there exists a compact symmetric neighbourhood V of the identity such that $CV^2 \subseteq U$. Let

$$u(x) = |V|^{-1}(1_{CV} * \check{1}_V)(x) = |V|^{-1} \cdot |xV \cap CV|,$$

$x \in G$. Then $0 \leqslant u(x) \leqslant 1$ for all $x \in G$. If $x \in C$ then $|xV \cap CV| = |xV| = |V|$ and hence $u(x) = 1$, whereas if $x \notin CV^2$, then $xV \cap CV = \varnothing$ and so $u(x) = 0$. In particular, supp $u \subseteq CV^2$, which is compact, and $u(x) = 0$ for $x \in G \backslash U$.

Finally, the identity

$$\begin{aligned} 4(f * g^*) &= (f+g)*(f+g)^* - (f-g)*(f-g)^* \\ &\quad + i(f+ig)*(f+ig)^* - i(f-ig)*(f-ig)^*, \end{aligned}$$

$f, g \in L^1(G)$, ensures that u is a finite linear combination of functions of the form $h * h^*$, where $h \in L^\infty(G)$ and h has compact support and hence $h * h^* \in P(G) \cap C_c(G)$. □

Proposition 2.3.3. *For $1 \leqslant j \leqslant 10$, define a subset M_j of $B(G)$ as follows:*
$M_1 = \{f * \widetilde{g} : f, g \in C_c(G)\}$;
$M_2 = \{h * \widetilde{h} : h \in C_c(G)\}$;
$M_3 = \{f * \widetilde{g} : f, g \in L^\infty(G)$ *with compact support*$\}$;
$M_4 = \{h * \widetilde{h} : h \in L^\infty(G)$ *with compact support*$\}$;
$M_5 = B(G) \cap C_c(G)$;
$M_6 = P(G) \cap C_c(G)$;
$M_7 = \{u \in P(G) : \Delta^{-1/2} u \in L^1(G)\}$;
$M_8 = P(G) \cap L^2(G)$;
$M_9 = \{h * \widetilde{h} : h \in L^2(G)\}$;
$M_{10} = \{f * \widetilde{g} : f, g \in L^2(G)\}$.
Let E_j denote the linear span of M_j, $1 \leqslant j \leqslant 10$. Then

$$E_1 = E_2 \subseteq E_3 = E_4 \subseteq E_5 = E_6 \subseteq E_7 \subseteq E_8 \subseteq E_9 = E_{10} \subseteq B_\lambda(G),$$

and all these subspaces of $B_\lambda(G)$ have the same closure, denoted $A(G)$, in $B_\lambda(G)$. Moreover, $A(G)$ is an ideal in $B(G)$.

Proof. The equalities $E_1 = E_2$, $E_3 = E_4$ and $E_9 = E_{10}$ all follow from the polar identity. The inclusions $E_2 \subseteq E_3$, $E_4 \subseteq E_5$ and $E_6 \subseteq E_7$ are evident.

To see that $E_5 \subseteq E_6$, let $v \in B(G) \cap C_c(G)$ and $v = v_1 - v_2 + i(v_3 - v_4)$, where $v_k \in P(G)$, $1 \leqslant k \leqslant 4$. By Proposition 2.3.2 there exists a function u of the form $u = \sum_{j=1}^n c_j u_j$, where $c_j \in \mathbb{C}$ and $u_j \in P(G) \cap C_c(G)$, such that $u = 1$ on supp v. Then

$$v = uv = \sum_{j=1}^n c_j u_j [v_1 - v_2 + i(v_3 - v_4)],$$

which shows that $v \in E_6$. The inclusion $E_7 \subseteq E_8$ is a consequence of [**100**, Proposition 12], and $E_8 \subseteq E_9$ follows from the fact that every $u \in P(G) \cap L^2(G)$ can be written in the form $u = f * \widetilde{f}$ with $f \in L^2(G)$ [**60**, Theorem 13.8.6].

Finally, all these subspaces will have the same closure once we have shown that E_1 is dense in E_{10}. To this end, let $f, g \in L^2(G)$ and $\epsilon > 0$ be given. Then there exist functions $h, k \in C_c(G)$ such that $\|f - h\|_2 \leqslant \epsilon$ and $\|g - k\|_2 \leqslant \epsilon$. Now, by Lemma 2.3.1,

$$\begin{aligned} \|f * \widetilde{g} - h * \widetilde{k}\|_{B(G)} &= \|(f-h) * \widetilde{g} + h * (\widetilde{g} - \widetilde{k})\|_{B(G)} \\ &\leqslant \|(f-h) * \widetilde{g}\|_{B(G)} + \|h * (\widetilde{g} - \widetilde{k})\|_{B(G)} \\ &\leqslant \|f-h\|_2 \|g\|_2 + \|h\|_2 \|g-k\|_2 \\ &\leqslant \epsilon(\|g\|_2 + \|f\|_2 + \epsilon). \end{aligned}$$

This shows that $M_{10} \subseteq \overline{M_1}$ and hence $E_{10} \subseteq \overline{E_1}$. □

DEFINITION 2.3.4. The algebra $A(G)$, defined in Proposition 2.3.3, is called the *Fourier algebra* of the locally compact group G.

COROLLARY 2.3.5. *Let G be a locally compact group.*

(i) *Every $u \in A(G)$ vanishes at infinity.*
(ii) *$A(G)$ is uniformly dense in $C_0(G)$.*

PROOF. (i) follows from $\|u\|_\infty \leqslant \|u\|$ and the density of the subspace $B(G) \cap C_c(G) = A(G) \cap C_c(G)$ in $A(G)$ (Proposition 2.3.3).

(ii) $A(G) \cap C_c(G)$ is a self-adjoint subalgebra of $C_0(G)$ which, by Proposition 2.3.2, strongly separates the points of G. Thus $A(G)$ is uniformly dense in $C_0(G)$ by the Stone-Weierstrass theorem. □

LEMMA 2.3.6. *Let $u \in A(G)$ and $x, y \in G$. Then the functions $L_x u$, $R_y u$, $\overline{u}$, $\check{u}$ and $\widetilde{u}$ all belong to $A(G)$.*

PROOF. By Remark 2.1.10(3) and Corollary 2.1.14, all the linear maps of $B(G)$ into itself in question are continuous. Since they clearly map $B(G) \cap C_c(G)$ into itself and $B(G) \cap C_c(G)$ is dense in $A(G)$, the statements of the lemma follow. □

Our next purpose in this section is to show that the spectrum of $A(G)$ can be canonically identified with G and that $A(G)$ is a regular algebra of functions on G. To that end, we need the following lemma.

LEMMA 2.3.7. *Let $a \in G$ and $f \in A(G)$ such that $f(a) = 0$. Then, given $\epsilon > 0$, there exists $h \in A(G) \cap C_c(G)$ vanishing in a neighbourhood of a such that $\|h - f\|_{A(G)} \leqslant \epsilon$.*

PROOF. Notice first that, since $A(G) \cap C_c(G)$ is dense in $A(G)$, without loss of generality we can assume that $f \neq 0$, f has compact support and $\epsilon \leqslant \|f\|_\infty$ and $\epsilon < 1$. Let

$$W = \{y \in G : \|f - R_y f\|_{A(G)} \leqslant \epsilon\}.$$

Then W is a compact neighbourhood of e in G. Choose an open neighbourhood V of e such that $V \subseteq W$ and $\sup\{|f(ay)| : y \in V\} \leqslant \epsilon$. By regularity of Haar measure, there exists a compact neighbourhood U of e such that $U \subseteq V$ and $|U| \geqslant |V|(1-\epsilon)$. Now, define functions u, g and h on G by setting $u = |U|^{-1} 1_U$, $g = 1_{aV} f$ and

$$h = (f - g) * \check{u} \in A(G).$$

Then h has compact support since W is compact and f has compact support. For any $x \in G$,

$$h(x) = |U|^{-1} \int_U f(xy)[1 - 1_{aV}(xy)]dy.$$

It follows that, if $x \in G$ satisfies $a^{-1}xU \subseteq V$, then $h(x) = 0$. Thus h vanishes in a neighbourhood of a. Moreover,

$$\|u\|_2 = |U|^{-1/2} \leqslant |V|^{-1/2} \left(\frac{1}{1-\epsilon}\right)^{1/2},$$

$$\|g\|_2 = \left(\int_{aV} |f(y)|^2 dy\right)^{1/2} \leqslant \epsilon |V|^{1/2},$$

and

$$\begin{aligned}\|f - f * \check{u}\|_{A(G)} &= \left\| f - |U|^{-1} \int_U (R_y f) dy \right\|_{A(G)} \\ &\leqslant \sup_{y \in U} \|f - R_y f\|_{A(G)} \leqslant \epsilon.\end{aligned}$$

Combining all these estimates, we obtain

$$\|f - h\|_{A(G)} \leqslant \|f - f * \check{u}\|_{A(G)} + \|g\|_2 \|\check{u}\|_2 \leqslant \epsilon + \epsilon \left(\frac{1}{1-\epsilon} \right)^{1/2}.$$

This finishes the proof. □

THEOREM 2.3.8. *Let G be a locally compact group. For each $x \in G$, let*

$$\varphi_x : A(G) \to \mathbb{C}, \ u \to u(x).$$

Then the map $x \to \varphi_x$ is a homeomorphism from G onto $\sigma(A(G))$. Moreover, $A(G)$ is regular.

PROOF. It is obvious that $\varphi_x \in \sigma(A(G))$ and that the map $x \to \varphi_x$ is injective. Now let $\varphi \in \sigma(A(G))$ be given and suppose that $\varphi \neq \varphi_x$ for all $x \in G$. Then, for each $x \in G$ there exists $f_x \in A(G)$ such that $\varphi(f_x) = 1$, but $\varphi_x(f_x) = 0$. By Lemma 2.3.6, every $g \in A(G)$ vanishing at x is the limit of a sequence $(g_n)_n$ in $A(G)$ with the property that each g_n vanishes in a neighbourhood of x. Therefore we can assume that f_x vanishes in a neighbourhood V_x of x.

Since $A(G) \cap C_c(G)$ is dense in $A(G)$, there exists $f_0 \in C_c(G) \cap A(G)$ such that $\varphi(f_0) = 1$. Choose $x_1, \ldots, x_n \in \operatorname{supp} f_0$ such that

$$\operatorname{supp} f_0 \subseteq \textstyle\bigcup_{j=1}^n V_{x_j}$$

and let

$$f = f_0 f_{x_1} \ldots f_{x_n} \in A(G).$$

Then $f(x) = 0$ for every $x \in G$, whereas

$$\varphi(f) = \varphi(f_0) \textstyle\prod_{j=1}^n \varphi(f_{x_j}) = 1.$$

This contradiction shows that $\varphi = \varphi_x$ for some $x \in G$.

Now, since the subalgebra $A(G)$ of $C_0(G)$ is uniformly dense in $C_0(G)$, the topology on G coincides with the weak topology defined by the set of functions $x \to f(x) = \varphi_x(f), f \in A(G)$. Thus the map $x \to \varphi_x$ from G to $\sigma(A(G))$ is a homeomorphism.

Finally, Proposition 2.3.2 implies that $A(G)$ is a regular algebra of functions on G. □

Of course, after identifying $\sigma(A(G))$ with G, the Gelfand homomorphism of $A(G)$ is nothing but the identity mapping. In particular, $A(G)$ is a semisimple commutative Banach algebra.

We remind the reader that if G is a locally compact abelian group with dual group $\widehat{G}$, then $A(G)^* = L^1(\widehat{G})^* = L^\infty(\widehat{G})$ and that $VN(G) = L^\infty(\widehat{G})$ (see Remark 1.8.21). The following theorem shows that $VN(G)$ is isometrically isomorphic to the dual space of $A(G)$ for general locally compact groups G.

THEOREM 2.3.9. *Let G be a locally compact group. For any $\varphi \in A(G)^*$ there exists a unique operator $T_\varphi \in VN(G)$ such that*

$$\langle T_\varphi(f), g\rangle_2 = \langle \varphi, \overline{g} * \check{f}\,\rangle = \langle \varphi, (f * \widetilde{g})\check{}\,\rangle$$

for all $f, g \in L^2(G)$. The mapping $\varphi \to T_\varphi$ from $A(G)^*$ to $VN(G)$ is a surjective linear isometry and has the following additional properties.

(i) *If $u = \sum_{j=1}^\infty (g_j * \check{f}_j)$, $f_j, g_j \in L^2(G)$, with $\sum_{j=1}^\infty \|f_j\|_2\|g_j\|_2 < \infty$, then*

$$\langle \varphi, u\rangle = \sum_{j=1}^{\infty} \langle T_\varphi(f_j), \overline{g}_j\rangle_2.$$

(ii) *If $\mu \in M(G)$ and φ_μ is the element of $A(G)^*$ defined by*

$$\langle \varphi_\mu, u\rangle = \int_G u(x)d\mu(x), \quad u \in A(G),$$

then $T_{\varphi_\mu} = \lambda_G(\mu)$.

(iii) *$\varphi \to T_\varphi$ is a homeomorphism for the w^*-topology on $A(G)^*$ and the ultraweak topology on $VN(G)$.*

PROOF. If $\varphi \in A(G)^*$, then for any $f, g \in L^2(G)$,

$$|\langle \varphi, \overline{g} * \check{f}\rangle| \leqslant \|\varphi\| \cdot \|\overline{g} * \check{f}\|_{A(G)} \leqslant \|\varphi\| \cdot \|g\|_2\|f\|_2.$$

Thus, for each $f \in L^2(G)$, the assignment $g \to \langle \varphi, \overline{g} * \check{f}\rangle$ defines a conjugate linear functional on $L^2(G)$. Hence there exists a unique $f_\varphi \in L^2(G)$ such that $\langle \varphi, \overline{g} * \check{f}\rangle = \langle f_\varphi, g\rangle_2$ for all $g \in L^2(G)$. Define $T_\varphi : L^2(G) \to L^2(G)$ by $T_\varphi(f) = f_\varphi$. Then T_φ is linear and $\|T_\varphi(f)\|_2 \leqslant \|\varphi\| \cdot \|f\|_2$ for all $f \in L^2(G)$, so that $\|T_\varphi\| \leqslant \|\varphi\|$. For any $f, g \in L^2(G)$ and $h \in L^1(G)$, we have

$$\begin{aligned}
\langle T_\varphi(\lambda_G(h)f), g\rangle_2 &= \langle T_\varphi(f * h), g\rangle_2 = \langle \varphi, \overline{g} * (f * h)\check{}\,\rangle \\
&= \langle \varphi, \overline{g} * \check{h} * \check{f}\rangle = \langle \varphi, \overline{g * h^*} * \check{f}\rangle \\
&= \langle T_\varphi(f), g * h^*\rangle_2 = \langle T_\varphi(f), \lambda_G(h)^* g\rangle_2 \\
&= \langle \lambda_G(h) T_\varphi(f), g\rangle_2.
\end{aligned}$$

So T_φ commutes with the right regular representation operators and therefore $T_\varphi \in VN(G)$. If u is as in (i), the series $\sum_{j=1}^\infty (g_j * \check{f}_j)$ is absolutely convergent and hence

$$\langle \varphi, u\rangle = \sum_{j=1}^{\infty} \langle \varphi, g_j * \check{f}_j\rangle = \sum_{j=1}^{\infty} \langle T_\varphi(f_j), \overline{g}_j\rangle_2$$

and $|\langle \varphi, u\rangle| \leqslant \|T_\varphi\| \cdot \sum_{j=1}^\infty \|f_j\|_2\|g_j\|_2$. Since this holds for all such representations of $u \in A(G)$, we conclude that $|\langle \varphi, u\rangle| \leqslant \|T_\varphi\| \cdot \|u\|_{A(G)}$. It follows that $\varphi \to T_\varphi$ is an isometry, and the above equation yields statement (iii). If $\mu \in M(G)$, then for all $f, g \in L^2(G)$,

$$\begin{aligned}
\langle T_{\varphi_\mu}(f), g\rangle_2 &= \langle \varphi_\mu, \overline{g} * \check{f}\rangle = \int_G \int_G \overline{g(xy)} f(y) dy d\mu(x) \\
&= \int_G \int_G \overline{g(y)} f(x^{-1}y) d\mu(x) dy \\
&= \int_G (\mu * f)(y)\overline{g(y)} dy = \langle \lambda_G(\mu) f, g\rangle_2,
\end{aligned}$$

so that $T_{\varphi_\mu} = \lambda_G(\mu)$.

It remains to show that every $T \in VN(G)$ is of the form T_φ for some $\varphi \in A(G)^*$. Thus, let $T \in VN(G)$ and note first that given $f, g \in C_c(G)$, the function $T(u) = T(f * \widetilde{g}) = T(f) * \widetilde{g}$ is continuous. Therefore, for $u \in E_1$, we can put $\langle \varphi_T, u\rangle = T(\check{u})(e)$, and this definition does not depend on the representation of u. To show that φ_T is a bounded linear functional on E_1, recall that by Kaplansky's density theorem there exists a net $(h_\alpha)_\alpha$ in $C_c(G)$ such that $\|\lambda_G(h_\alpha)\| \leqslant \|T\|$ for all α and $\|Tg - \lambda_G(h_\alpha)g\| \to 0$ for every $g \in L^2(G)$. For u of the form $u = \sum_{j=1}^n (f_j * g_j^*)$, where $f_j, g_j \in C_c^2(G)$, it follows that

$$\langle \varphi_T, u\rangle = Tu(e) = \sum_{j=1}^n \langle Tf_j, g_j\rangle = \lim_\alpha \sum_{j=1}^n \langle \lambda_G(h_\alpha) f_j, g_j\rangle = \lim_\alpha \langle \varphi_{\lambda_G(h_\alpha)}, u\rangle$$

and hence

$$|\langle \varphi_T, u\rangle| = \lim_\alpha |\langle \varphi_{\lambda_G(h_\alpha)}, u\rangle| \leqslant \|u\| \cdot \sup_\alpha \|\lambda_G(h_\alpha)\| \leqslant \|u\| \cdot \|T\|.$$

Since E_1 is dense in $A(G)$, φ_T extends uniquely to a bounded linear functional on $A(G)$, also denoted φ_T, of norm $\leqslant \|T\|$. By definition of φ_T,

$$\begin{aligned}
\langle T_{\varphi_T}(f), g\rangle_2 &= \langle f\varphi_T, g\rangle_2 = \langle \varphi_T, \overline{g} * \check{f}\rangle \\
&= \langle \varphi_T, (f * \widetilde{g})\check{}\,\rangle = T(f * \widetilde{g})(e) \\
&= (T(f) * \widetilde{g})(e) = \langle T(f), g\rangle_2
\end{aligned}$$

and hence $T_{\varphi_T} = T$. □

Given $T \in VN(G)$, we let $\check{T}$ denote the operator in $VN(G)$ defined by $\langle \varphi_{\check{T}}, u\rangle = \langle \varphi_T, \check{u}\rangle$, $u \in A(G)$. Thus $T \to \check{T}$ is the transpose of the isometry $u \to \check{u}$ of $A(G)$. Then, for $\mu \in M(G)$,

$$\langle \varphi_{\widecheck{\lambda_G(\mu)}}, u\rangle = \int_G \check{u}(x) d\mu(x) = \langle \varphi_{\lambda_G(\check{\mu})}, u\rangle,$$

where $d\check{\mu}(x) = d\mu(x^{-1})$. Thus $\widecheck{\lambda_G(\mu)} = \lambda_G(\check{\mu})$, and passing to ultraweak limits, we deduce that the map $T \to \check{T}$ is an isometric and ultraweakly continuous involution on $VN(G)$.

Definition 2.3.10. For $T \in VN(G)$ and $u \in A(G)$, let Tu denote the unique element of $A(G)$ such that

$$\langle S, Tu\rangle = \langle \check{T}S, u\rangle$$

for all $S \in VN(G)$.

It is easily verified that the assignment $(T, u) \to Tu$ turns $A(G)$ into a left $VN(G)$-module.

Lemma 2.3.11. *Let G be a locally compact group and $T \in VN(G)$.*

(i) *The map $u \to Tu$ is a bounded linear operator on $A(G)$ with norm $\|T\|$.*
(ii) *For $u \in A(G)$ and $x \in G$, $T\check{u}(x) = \langle T, L_x u\rangle$.*
(iii) *If $u \in A(G) \cap L^2(G)$, then $Tu \in A(G) \cap L^2(G)$ and $Tu = T(u)$, the action of T on $L^2(G)$.*

Proof. (i) For every $u \in A(G)$, we have

$$\|Tu\| = \sup_{\|S\| \leqslant 1} |\langle S, Tu\rangle| = \sup_{\|S\| \leqslant 1} |\langle \check{T}S, u\rangle| \leqslant \|\check{T}\| \cdot \|u\| = \|T\| \cdot \|u\|.$$

So $u \to Tu$ is a bounded linear operator on $A(G)$. By definition, its transpose is the map $S \to \check{T}S$ which has norm $\|\check{T}\| = \|T\|$. Since the two maps have the same norm, (i) follows.

(ii) Consider first elements u of $A(G)$ of the form $u = (f * \tilde{g})\hat{}$, where $f, g \in C_c(G)$. Then

$$T(\check{u}) = T(f * \tilde{g}) = T(f) * \tilde{g}$$

is a continuous function and, for each $x \in G$,

$$\begin{aligned} T(\check{u})(x) &= (T(f) * \tilde{g})(x) = \langle T(f), L_x g\rangle \\ &= \varphi_T(\overline{L_x g} * \check{f}) = \varphi_T(L_x[(f * \tilde{g})\check{}\,]) \\ &= \langle T, L_x u\rangle. \end{aligned}$$

Using that $S(\check{u})(e) = \langle S, u\rangle$, it follows that

$$\begin{aligned} \langle T, L_x u\rangle &= T(\check{u})(x) = (\rho(x^{-1})T)(\check{u})(e) \\ &= \langle \rho(x^{-1})T, u\rangle = \langle \check{T}\rho(x), \check{u}\rangle = \langle \rho(x), T\check{u}\rangle \\ &= T\check{u}(x). \end{aligned}$$

By linearity, this shows that $T(\check{u}) = T\check{u}$ for all $u \in E_1$, where E_1 is the space defined in Proposition 2.3.3. Since E_1 is dense in $A(G)$ and the maps $u \to \check{u}$ and $u \to L_x u$ are continuous, we conclude that $T\check{u}(x) = \langle T, L_x u\rangle$ for all $u \in A(G)$.

(iii) Let $\mathcal{A}$ be the $*$-subalgebra of $VN(G)$ consisting of all operators T of the form $T = \lambda_G(\mu)$, where μ is a finite linear combination of Dirac measures. Note that, for $u \in A(G) \cap L^2(G)$ and $x \in G$, by (ii)

$$\lambda_G(x)u(y) = \langle \lambda_G(x), L_y\check{u}\rangle = L_y\check{u}(x) = u(x^{-1}y) = L_x u(y)$$

for all $y \in G$. Let $\mu = \sum_{j=1}^n c_j\delta_{x_j}$, $T = \lambda_G(\mu)$ and $f \in C_c(G)$. Then

$$\begin{aligned} \left|\int_G Tu(x)\overline{f(x)}dx\right| &= \left|\int_G \left(\sum_{j=1}^n c_j u(x_j^{-1}x)\overline{f(x)}\right) dx\right| \\ &= \left|\int_G u(x)\left(\sum_{j=1}^n c_j \overline{f(x_j x)}\right) dx\right| \\ &= \left|\int_G (\mu^* * f)(x)\overline{u(x)}dx\right| \\ &= \left|\int_G T^*(f)(x)\overline{u(x)}dx\right| \\ &\leqslant \|T\| \cdot \|f\|_2\|u\|_2 \end{aligned}$$

by Schwarz' inequality.

Now let $T \in VN(G)$ be arbitrary. By Kaplansky's density theorem, there exists a net $(T_\alpha)_\alpha$ in $\mathcal{A}$ such that $\|T_\alpha\| \leqslant \|T\|$ for all α and $T_\alpha \to T$ in the ultra weak topology. Then

$$\int_G T_\alpha u(x)\overline{f(x)}dx = \langle \lambda_G(f), T_\alpha u\rangle = \langle \check{T_\alpha}\lambda_G(f), u\rangle,$$

and passing to the limit,

$$\int_G Tu(x)\overline{f(x)}dx = \langle \lambda_G(f), Tu\rangle = \langle \check{T}\lambda_G(f), u\rangle.$$

The above estimate then yields

$$\left|\int_G Tu(x)\overline{f(x)}dx\right| \leqslant \|T\| \cdot \|u\|_2 \|f\|_2$$

for all $f \in C_c(G)$ and $u \in A(G) \cap L^2(G)$. This shows that $Tu \in L^2(G)$ and at the same time that the mapping $u \to Tu$ from $A(G) \cap L^2(G)$ into $L^2(G)$ is continuous with respect to the L^2-norms. On the other hand, we know that $Tu = T(u)$ for all $u \in E_1$. Since E_1 is dense in $A(G)$, continuity implies that $Tu = T(u)$ for all $u \in A(G) \cap L^2(G)$. This finishes the proof of (iii). □

2.4. Functorial properties and a description of $A(G)$

Let H be an open subgroup of G and let Haar measure on H be the one induced by Haar measure of G. As before, we denote by $\overset{\circ}{f}$ the trivial extension of a function f of H to all of G. To $T \in VN(G)$ and $f \in L^2(H)$ we associate the function $T|_H(f) = T(\overset{\circ}{f})|_H$ on H. Since

$$\int_H |T|_H(f)(x)|^2 dx \leqslant \int_G |T(\overset{\circ}{f})(x)|^2 dx \leqslant \|T\|^2 \|\overset{\circ}{f}\|_2^2 = \|T\|^2 \|f\|_2^2,$$

the map $T|_H : f \to T|_H(f)$ is an operator on $L^2(H)$ with $\|T|_H\| \leqslant \|T\|$. Moreover, $T|_H \in VN(H)$ since, for any $g \in C_c(H)$,

$$\begin{aligned} T|_H(f * g) &= T(f * g)^\circ|_H = T(\overset{\circ}{f} * \overset{\circ}{g})|_H = (T(\overset{\circ}{f}) * \overset{\circ}{g})|_H \\ &= T(\overset{\circ}{f})|_H * g = T_H(f) * g. \end{aligned}$$

Let H be a closed subgroup of G. Let $VN_H(G)$ denote the w^*-closure of the linear span of the set $\{\lambda_G(h) : h \in H\}$. Then $VN_H(G)$ is a von Neumann algebra.

Proposition 2.4.1. *Let H be an open subgroup of the locally compact group G.*

(i) *The map $\phi : u \to \overset{\circ}{u}$ is an isometric isomorphism of $A(H)$ into $A(G)$ and $\phi(P(H) \cap A(H)) \subseteq P(G) \cap A(G)$. The map $T \to T|_H$ is the adjoint ϕ^* of ϕ and*

$$\phi^*(VN(G)) = VN(H) \quad \text{and} \quad \phi^*(C^*_\lambda(G)) = C^*_\lambda(H).$$

(ii) *The restriction map $r : u \to u|_H$ maps $A(G)$ onto $A(H)$. Its adjoint r^* is an isomorphism of $VN(H)$ onto the von Neumann subalgebra of $VN(G)$ generated by the operators $\lambda_G(x)$, $x \in H$. Moreover, $r^*(C^*_\lambda(H)) \subseteq C^*_\lambda(G)$.*

Proof. (i) We know from Theorem 2.2.1(i) that $r : u \to u|_H$ is a norm decreasing map from $B(G)$ into $B(H)$. Since $r(B(G) \cap C_c(G)) \subseteq B(H) \cap C_c(H)$, continuity of r implies that $r(A(G)) \subseteq A(H)$. Let $\phi : T \to \phi(T)$ denote the adjoint of the map $r|_{A(G)} : A(G) \to A(H)$. Then ϕ is a norm decreasing linear map of $VN(H)$ into $VN(G)$ and it is also continuous for the ultraweak topologies. For $f \in C_c(H)$ and $u \in A(G)$ we have

$$\begin{aligned} \langle \phi(\lambda_H(f)), u \rangle &= \langle \lambda_H(f), u|_H \rangle = \int_H f(x) u(x) dx \\ &= \langle \lambda_G(\overset{\circ}{f}), u \rangle, \end{aligned}$$

and hence $\phi(\lambda_H(f)) = \lambda_G(\mathring{f})$. Since $\lambda_H(C_c(H))$ is uniformly dense in $C^*_\lambda(H)$, ϕ maps $C^*_\lambda(H)$ into $C^*_\lambda(G)$. Moreover, since $C^*_\lambda(H)$ is ultraweakly dense in $VN(H)$, it follows that $\phi(T^*) = \phi(T)^*$ and $\phi(ST) = \phi(S)\phi(T)$ for all $S, T \in VN(H)$. So ϕ is a von Neumann algebra homomorphism from $VN(H)$ into $VN(G)$. More precisely, the range of ϕ is contained in $VN_H(G)$ since $\phi(\lambda_H(h)) = \lambda_G(h)$ for all $h \in H$.

Let $f, g \in C_c(H)$ and put

$$u = f * \widetilde{g} \in E_1(H) \subseteq A(H) \subseteq VN(H)^*.$$

Then, for $T \in VN(G)$,

$$\begin{aligned} \langle \varphi_T, \check{\mathring{u}} \rangle &= \langle \varphi_T, \overline{g} * \check{f} \rangle = \langle \varphi_T, \overline{\mathring{g}} * \check{\mathring{f}} \rangle \\ &= \langle T(\mathring{f}), \mathring{g} \rangle_2 = \langle T(\mathring{f})|_H, g \rangle_2 \\ &= \langle T|_H(f), g \rangle_2 = \langle \varphi_{T|_H}, \overline{g} * \check{f} \rangle \\ &= \langle \varphi_{T|_H}, \check{u} \rangle. \end{aligned}$$

This shows that $R^*(u) = u_G \in E_1(G)$ for each $u \in E_1(H)$. Since R^* is continuous and $E_1(H)$ is dense in $A(H)$, it follows that $R^*(A(H)) \subseteq A(G)$. Actually, $R^*(u) = u_G$ for every $u \in A(H)$. In fact, this equation holds for all $u \in E_1(H)$, $E_1(H)$ is dense in $A(H)$ and the map $u \to u_G$ is continuous for the topologies of pointwise convergence. In particular, this implies that $u_G \in A(G)$ for every $u \in A(H)$. On the other hand,

$$\|u_G\| = \|R^*(u)\| \leqslant \|u\| = \|u_G|_H\| \leqslant \|u_G\|,$$

so that $\|u_G\| = \|u\|$. Observe next that if $u \in P(H) \cap A(H)$, then $u_G \in P(G) \cap A(G)$ since u_G is hermitian and

$$\|u_G\| = \|u\| = u(e) = u_G(e).$$

This proves (i).

(ii) Because the restriction map $r : v \to v|_H$ is surjective, it follows from duality theory that r^* is a topological isomorphism for the ultraweak topologies between $VN(H)$ and its range in $VN(G)$. Thus $r^*(VN(H))$ is ultraweakly closed in $VN(G)$ and hence coincides with $VN_H(G)$ since it contains all the operators $\lambda_G(h)$, $h \in H$. This completes the proof of (ii). □

Let K be a compact normal subgroup of G and let $q_K : G \to G/K$ denote the quotient homomorphism and μ_K the normalized Haar measure of K. Then the map $j_K : f \to f \circ q_K$ is a Hilbert space isomorphism between $L^2(G/K)$ and $L^2_K(G)$, the subspace of $L^2(G)$ consisting of all functions which are constant almost everywhere on xK for almost every coset $xK \in G/K$. In other words, $g \in L^2_K(G)$ if and only if $g \in L^2(G)$ and $g = g * \mu_K$. It follows that $T(L^2_K(G)) \subseteq L^2_K(G)$ for every $T \in VN(G)$. Therefore, to each $T \in VN(G)$ we can associate an operator T_K on $L^2(G/K)$ defined by

$$T_K(f) = (j_K^* \circ T \circ j_K)(f), \quad f \in L^2(G/K).$$

It is easy to see that $T_K \in VN(G/K)$. In addition,

$$(ST)_K = S_K T_K \quad \text{and} \quad (T^*)_K = (T_K)^*$$

for any $S, T \in VN(G)$.

Proposition 2.4.2. *Let K be a compact normal subgroup of the locally compact group G.*

(i) *The map* $u \to u \circ q_K$ *is an isometric isomorphism from* $A(G/K)$ *onto the subalgebra* $A_K(G)$ *of* $A(G)$ *consisting of all* $v \in A(G)$ *such that* $v(xk) = v(x)$ *for all* $x \in G$ *and* $k \in K$.

(ii) *The map* $T \to T_K$ *is the adjoint of the map* $u \to u \circ q_K$ *and it is an ultraweakly continuous homomorphism of* $VN(G)$ *onto* $VN(G/K)$.

Proof. (i) By Corollary 2.2.4, $\phi : u \to u \circ q_K$ maps $B(G/K)$ isometrically onto $B_K(G)$, the algebra of all $u \in B(G)$ which are constant on cosets of K. Clearly,

$$\phi(B(G/K) \cap C_c(G/K)) = B_K(G) \cap C_c(G).$$

Since $A(G/K)$ is the closure of $B(G/K) \cap C_c(G/K)$, it suffices to show that $B_K(G) \cap C_c(G)$ is dense in $A_K(G)$. Thus let $v \in A_K(G)$ and let $(v_n)_n$ be a sequence in $B(G) \cap C_c(G)$ converging to v. Then $v_n * \mu_K \in B_K(G) \cap C_c(G)$ and

$$\|v_n * \mu_K - v\|_{B(G)} = \|(v_n - v) * \mu_K\|_{B(G)} \leqslant \|\mu_K\| \cdot \|v_n - v\|_{B(G)} \to 0,$$

whence $v \in \overline{B_K(G) \cap C_c(G)}$.

(ii) The map $T \to T_K$ is the adjoint of the map $u \to u \circ q_K$. In fact, for $T \in VN(G)$ and $u \in A(G/K) \cap L^2(G/K)$ we have

$$\begin{aligned} \langle T, u \circ q_K \rangle &= T((u \circ q_K)\check{}\,)(e) = T(\check{u} \circ q_K)(e) \\ &= T(j_K(\check{u}))(e) = j_K(T_K(\check{u}))(e) \\ &= T_K(\check{u})(e) = \langle T_K, u \rangle. \end{aligned}$$

The second statement in (ii) then follows from duality theory of Banach spaces. □

Theorem 2.4.3. *Let* G *be a locally compact group. Then* $A(G)$ *is precisely the set of all functions* $f * \widetilde{g}$, *where* $f, g \in L^2(G)$.

Proof. Let us first assume that G is second countable. Then the space $L^2(G)$ is separable and hence the von Neumann algebra $VN(G)$ is countably generated. It follows from [**60**, Proposition 14.5.1] that every normal positive linear functional on $VN(G)$ is of the form $T \to \langle Tf, f \rangle$, where $f \in L^2(G)$. This in turn implies that every ultraweakly continuous linear functional on $VN(G)$ is of the form

$$T \to \langle Tf, g \rangle = \varphi_T(\overline{g} * \check{f}), \ \ f, g \in L^2(G).$$

On the other hand, as we have seen in Theorem 2.3.9, the ultraweakly continuous linear functionals on $VN(G)$ are exactly given by $T \to \varphi_T(u)$, where $u \in A(G)$. This establishes the theorem for second countable G.

Now suppose that G is σ-compact, and let $u \in A(G)$. Since u is continuous and vanishes at infinity, u is uniformly continuous. Then, by a theorem of Kakutani and Kodaira (Theorem 1.2.16), there exists a compact normal subgroup K of G such that G/K is second countable and u is constant on cosets of K. Then u is of the form $u = v \circ q$, where $v \in A(G/K)$ and $q : G \to G/K$ denotes the quotient homomorphism. The first part of the proof shows that there exist $f, g \in L^2(G/K)$ such that $v = f * \widetilde{g}$. However, this implies

$$u = v \circ q = (f \circ q) * (g \circ q)^{\sim},$$

where $f \circ q, g \circ q \in L^2(G)$.

Finally, let G be an arbitrary locally compact group. Since $u \in C_0(G)$, we find a sequence $(C_n)_n$ of compact subsets of G such that, for all n, $C_n \subseteq C_{n+1}$, C_n contains a neighbourhood of e in G and also the set of all $x \in G$ such that $|u(x)| \geqslant 1/n$. Let H_n denote the open subgroup generated by C_n and let $H = \bigcup_{n=1}^{\infty} H_n$. Then H is an open subgroup of G since $H_n \subseteq H_{n+1}$, and H is σ-compact because each H_n is σ-compact. Moreover, $u = 0$ on $G\backslash H$. Since $u|_H \in A(H)$, by the second part of the proof there exist $f, g \in L^2(H)$ such that $u|_H = f * \widetilde{g}$. Denoting by f_1 and g_1 the trivial extensions of f and g to all of G, it follows that $u = f_1 * \widetilde{g_1}$. This finishes the proof. □

Since $(f * \widetilde{g})(x) = \int_G f(xy)\overline{g(y)}dy = \langle \lambda_G(x^{-1})f, g\rangle$, the preceding theorem tells us that $A(G)$ coincides with the collection of coefficient functions of the left regular representation.

Corollary 2.4.4. *On $VN(G) \subseteq \mathcal{B}(L^2(G))$ the weak and the ultraweak operator topologies coincide.*

Proof. Let $(f_n)_n$ and $(g_n)_n$ be sequences in $L^2(G)$ such that $\sum_{n=1}^{\infty} \|f_n\|_2 \|g_n\|_2 < \infty$. Then, by Theorem 2.4.3, there exist $f, g \in L^2(G)$ such that

$$\sum_{n=1}^{\infty} \langle \lambda_G(x) f_n, g_n\rangle = \langle \lambda_G(x) f, g\rangle$$

for all $x \in G$, and hence

$$\sum_{n=1}^{\infty} \langle T f_n, g_n\rangle = \langle T f, g\rangle$$

for all $T \in VN(G)$. This implies the statement. □

Remark 2.4.5. Let G be a locally compact abelian group. Then

$$A(\widehat{G}) = \widehat{L^1(G)} = \{\widehat{f} : f \in L^1(G)\}$$

and $\|\widehat{f}\|_{A(G)} = \|f\|_1$. Since $\|\widehat{\mu}\|_{B(G)} = \|\mu\|$ for all $\mu \in M(G)$ (Remark 2.1.15), we only have to verify that $A(\widehat{G}) = \widehat{L^1(G)}$. Clearly, $\widehat{L^1(G)} \subseteq A(\widehat{G})$ since the set of all $f \in L^1(G)$ such that $\widehat{f} \in C_c(\widehat{G})$ is dense in $L^1(G)$ and $B(\widehat{G}) \cap C_c(\widehat{G}) \subseteq A(\widehat{G})$. Conversely, $u \in A(\widehat{G})$ is of the form $u = \xi * \widetilde{\eta}$, where $\xi, \eta \in L^2(\widehat{G})$. By the Plancherel theorem, $\xi = \widehat{f}$ and $\eta = \widehat{g}$ for certain $f, g \in L^2(G)$. Then $f\breve{g} \in L^1(G)$ and $\widehat{f\breve{g}} = \widehat{f} * \widehat{\breve{g}} = \xi * \widetilde{\eta} = u$.

2.5. The support of operators in $VN(G)$

The main theme of this section is to associate to each $T \in VN(G)$ a closed subset of G, the so-called support of T. This notion, which turns out to be a major tool in the sequel, will be studied thoroughly. We start by defining an action of $B(G)$ on $VN(G)$. Recall that, for $u \in B(G)$ and $T \in VN(G)$, the assignment $v \to \langle T, uv\rangle$ defines a bounded linear functional on $A(G)$.

Definition 2.5.1. Let $u{\cdot}T$ denote the operator in $VN(G)$ defined by $\langle u{\cdot}T, v\rangle = \langle T, uv\rangle$ for $v \in A(G)$.

It is clear that $\|u \cdot T\| \leqslant \|u\| \cdot \|T\|$ and that with this action $VN(G)$ becomes a left $B(G)$-module. Note that if $u \in B(G)$ and $v \in A(G)$, then by Lemma 2.3.11,

$(u \cdot T)v \in A(G)$ and, for every $x \in G$,

$$\begin{aligned}[][(u \cdot T)v](x) &= \varphi_{u \cdot T}(L_x \check{v}) \\ &= \varphi_{u \cdot T}((R_x v)\check{}) = \varphi_T((R_x v)\check{}\, u) \\ &= \varphi_T([(R_x v)\check{u}]\check{}) \\ &= T[vR_{x^{-1}}(\check{u})](x).\end{aligned}$$

REMARK 2.5.2. (1) If $u \in A(G)$ and $T \in VN(G)$, the reader should be careful to not mix up the operator $u \cdot T$ in $VN(G)$ with the function $Tu \in A(G)$.

(2) Let $T = \lambda_G(\mu)$ for some $\mu \in M(G)$. Then $u \cdot \lambda_G(\mu) = \lambda_G(u\mu)$ for $u \in B(G)$, where $u\mu$ denotes the product of the function u and the measure μ in the usual sense. Indeed, for any $x \in G$ and $v \in A(G)$, by the above formula

$$\begin{aligned}[][u \cdot \lambda_G(\mu)]v(x) &= \lambda_G(\mu)[vR_{x^{-1}}(\check{u})](x) \\ &= \mu * [vR_{x^{-1}}(\check{u})](x) = \int_G \check{u}(y^{-1}x)v(y^{-1}x)d\mu(y) \\ &= \int_G v(y^{-1}x)u(y)d\mu(y) = [u\mu * v](x) \\ &= [\lambda_G(u\mu)v](x).\end{aligned}$$

Let $T \in C^*_\lambda(G) \subseteq VN(G)$ and $T = T_\varphi$, $\varphi \in A(G)^*$. Then the proof of Theorem 2.3.9 shows that $\langle u, T\rangle = \langle \varphi, u\rangle$ for all $u \in A(G) \subseteq B_\lambda(G)$, where $\langle u, T\rangle$ refers to the duality $C^*_\lambda(G)^* = B_\lambda(G)$.

PROPOSITION 2.5.3. *Let $T \in VN(G)$ and $a \in G$. Then the following three conditions are equivalent.*

(i) *The operator $\lambda_G(a)$ is the w^*-limit in $VN(G)$ of operators of the form $v \cdot T$, where $v \in A(G)$.*
(ii) *For every neighbourhood V of a in G, there exists $v \in A(G)$ such that* $\operatorname{supp} v \subseteq V$ *and $\langle T, v\rangle \neq 0$.*
(iii) *If $u \in A(G)$ is such that $u \cdot T = 0$, then $u(a) = 0$.*

PROOF. (i) ⇒ (ii) Let $\lambda_G(a)$ be the w^*-limit of a net $(v_\alpha \cdot T)_\alpha$, $v_\alpha \in A(G)$, and let V be a neighbourhood of a in G. Since $A(G)$ is regular (Lemma 2.3.7), there exists $w \in A(G)$ such that $\operatorname{supp} w \subseteq V$ and $w(a) \neq 0$. Then

$$\langle v_\alpha \cdot T, w\rangle = \langle T, v_\alpha w\rangle \to \langle \lambda_G(a), w\rangle = w(a) \neq 0,$$

and hence $\langle T, v_\alpha w\rangle \neq 0$ eventually.

(ii) ⇒ (iii) Towards a contradiction, assume that there exists $u \in A(G)$ with $u \cdot T = 0$, but $u(a) \neq 0$. Then we can find $\delta > 0$ and a compact neighbourhood V of a such that $|u(x)| \geqslant \delta$ for all $x \in V$. Since $A(G)$ is regular, by Theorem 1.1.19 there exists $w \in A(G)$ such that $w(x) = \frac{1}{u(x)}$ for all $x \in V$. Now, by (ii), there exists $v \in A(G)$ with $\operatorname{supp} v \subseteq V$ and $\langle T, v\rangle \neq 0$. Then $v = vwu$ since $\operatorname{supp} v \subseteq V$. It follows that

$$\langle T, v\rangle = \langle u \cdot T, vw\rangle = 0.$$

This contradiction proves (iii).

(iii) ⇒ (i) Let $I = \{u \in A(G) : u \cdot T = 0\}$. Then I is a closed ideal in $A(G)$ since $\langle (uv) \cdot T, w\rangle = \langle u \cdot T, wv\rangle$ for all $v, w \in A(G)$ and $u \in I$. On the other hand, since $\langle u \cdot T, v\rangle = \langle v \cdot T, u\rangle$ for $v \in A(G)$, I is the annihilator in $A(G)$ of the subspace $A(G) \cdot T$ of $VN(G) = A(G)^*$. Consequently, $T^\perp$ is the w^*-closure of $A(G) \cdot T$

in $VN(G)$. Now, if $a \in G$ satisfies (iii) then $\lambda_G(a) \in I^\perp$ and hence $\lambda_G(a)$ is the w^*-limit of operators of the form $v \cdot T$, $v \in A(G)$. □

DEFINITION 2.5.4. Let $T \in VN(G)$. Then the *support of* T, supp T, is the set of all elements $a \in G$ satisfying one and hence all three conditions in Proposition 2.5.3.

It is clear that supp $0 = \varnothing$. On the other hand, we have

LEMMA 2.5.5. *If* $T \in VN(G)$, $T \neq 0$, *then* $\operatorname{supp} T$ *is a nonempty closed subset of* G.

PROOF. Let $a \in \overline{\operatorname{supp} T}$ and let V be an open neighbourhood of a. Then $V \cap \operatorname{supp} T \neq \varnothing$ and hence by Proposition 2.5.3(ii) there exists $v \in A(G)$ with $\operatorname{supp} v \subseteq V$ and $\langle T, v\rangle \neq 0$. Thus $a \in \operatorname{supp} T$.

Now, assume that $T \neq 0$. Since $A(G) \cap C_c(G)$ is dense in $A(G)$, there exists $v \in A(G) \cap C_c(G)$ with $\langle T, v\rangle \neq 0$. Towards a contradiction, suppose that $\operatorname{supp} T = \varnothing$. Then for each $a \in G$ there exists $u_a \in A(G)$ such that $u_a \cdot T = 0$, but $u_a(a) \neq 0$. Since $\operatorname{supp} v$ is compact, we find $a_1, \ldots, a_n \in \operatorname{supp} v$ such that $u_{a_i} \cdot T = 0$ and $u(x) = \sum_{i=1}^n u_{a_i}(x)^2 > 0$ for all $x \in$ supp v. There exists $w \in A(G)$ such that $w(x) = \frac{1}{u(x)}$ for all $x \in \operatorname{supp} v$ (Theorem 1.1.19). Then $v = vwu$ and hence

$$\langle T, v\rangle = \sum_{i=1}^n \langle u_{a_i} \cdot T, u_{a_i} wv\rangle = 0.$$

This contradiction finishes the proof. □

We now collect a number of useful facts about the supports of operators in $VN(G)$.

PROPOSITION 2.5.6. *Let* G *be a locally compact group and* $T \in VN(G)$.

(i) *For* $v \in B(G)$, *we have*

$$\operatorname{supp}(v \cdot T) \subseteq \operatorname{supp} T \cap \operatorname{supp} v.$$

In particular, $v \cdot T = 0$ *whenever* v *vanishes in some neighbourhood of* $\operatorname{supp} T$.

(ii) $\operatorname{supp} T$ *is the smallest closed subset* C *of* G *with the following property: If* $v \in A(G) \cap C_c(G)$ *vanishes in a neighbourhood of* C, *then* $\langle T, v\rangle = 0$.

(iii) $\operatorname{supp} T$ *is the smallest closed subset* C *of* G *with the following property: Given any closed neighbourhood* V *of* C *such that* $G\backslash V$ *is relatively compact, the operator* T *is a* w^**-limit in* $VN(G)$ *of finite linear combinations of operators* $\lambda_G(x)$, *where* $x \in V$.

(iv) *Let* F *be a closed subset of* G. *Suppose that* $(T_\alpha)_\alpha$ *is a net in* $VN(G)$ *converging to* T *in the* w^**-topology and satisfying* $\operatorname{supp} T_\alpha \subseteq F$ *for all* α. *Then* $\operatorname{supp} T \subseteq F$.

(v) *Let* $T, T_1, T_2 \in VN(G)$ *and* $\lambda \in \mathbb{C}$, $\lambda \neq 0$. *Then*

(1) $\operatorname{supp}(\lambda T) = \operatorname{supp} T$;

(2) $\operatorname{supp} T^* = (\operatorname{supp} T)^{-1}$;

(3) $\operatorname{supp}(T_1 + T_2) \subseteq \operatorname{supp} T_1 \cup \operatorname{supp} T_2$, *and equality holds whenever* $\operatorname{supp} T_1 \cap \operatorname{supp} T_2 = \varnothing$.

(4) $\operatorname{supp}(T_1 T_2) \subseteq (\operatorname{supp} T_1)(\operatorname{supp} T_2)$ *provided that one of* $\operatorname{supp} T_1$ *and* $\operatorname{supp} T_2$ *is compact.*

PROOF. (i) The inclusion $\operatorname{supp}(v \cdot T) \subseteq \operatorname{supp} T$ follows immediately from the description of the support in Proposition 2.5.3(i). In order to show that $\operatorname{supp}(v \cdot T) \subseteq \operatorname{supp} v$, let $a \in G \setminus \operatorname{supp} v$ and choose a neighbourhood V of a such that $v|_V = 0$. Then $uv = 0$ for every $u \in A(G)$ with $\operatorname{supp} u \subseteq V$ and hence $\langle v \cdot T, u\rangle = \langle T, vu\rangle = 0$ for every such u. Proposition 2.5.3(ii) implies that $a \notin \operatorname{supp} v \cdot T$.

If v vanishes in a neighbourhood of $\operatorname{supp} T$, then $\operatorname{supp} v \cap \operatorname{supp} T = \varnothing$ by (i) and therefore $\operatorname{supp}(v \cdot T) = \varnothing$. By Lemma 2.5.5, $v \cdot T = 0$.

(ii) Let $v \in A_c(G)$ and suppose that v vanishes in a neighbourhood of $\operatorname{supp} T$. There exists $w \in A(G)$ such that $w|_{\operatorname{supp} v} = 1$ and $w = 0$ in a neighbourhood of $\operatorname{supp} T$. Then $w \cdot T = 0$ by (i). On the other hand, $wv = v$ and hence

$$\langle T, v\rangle = \langle T, wv\rangle = \langle w \cdot T, v\rangle = 0.$$

This shows that $\operatorname{supp} T$ has the indicated property.

Now, let C be any closed subset of G satisfying the condition in (ii) and let $a \in G \setminus C$. Then there exist neighbourhoods V of a and U of C such that V is compact and $V \cap U = \varnothing$. If $v \in A(G)$ is such that $\operatorname{supp} v \subseteq V$, then $v \in A_c(G)$ and v vanishes in a neighbourhood of C. It follows that $\langle T, v\rangle = 0$ and consequently $a \notin \operatorname{supp} T$ by Proposition 2.5.3(ii).

(iii) Let C be a closed subset of G and V a closed neighbourhood of C such that $G \setminus V$ is relatively compact, and suppose that T is a w^*-limit of finite linear combinations of operators $\lambda_G(x), x \in V$. If v belongs to the ideal $I(V) = \{u \in A(G) : u|_V = 0\}$, then $\langle \lambda_G(x), v\rangle = v(x) = 0$ for all $x \in V$ and by w^*-continuity this implies $\langle T, v\rangle = 0$. It follows now from (ii) that $\operatorname{supp} T \subseteq C$, as required.

(iv) Let $u \in A(G) \cap C_c(G)$ be such that u vanishes in a neighbourhood of F. Then, for all α, $\langle T_\alpha, u\rangle = 0$ since $\operatorname{supp} T_\alpha \subseteq F$. Since $T_\alpha \to T$ in the w^*-topology, $\langle T, u\rangle = 0$, and this implies $\operatorname{supp} T \subseteq F$ by (ii).

(v) (1) is evident and (2) follows from (iii) taking into account that the map $T \to T^*$ is weakly continuous and that $\lambda_G(x)^* = \lambda_G(x^{-1})$.

To show (3) we apply (ii). Thus let $v \in A_c(G)$ vanish in a neighbourhood of $\operatorname{supp} T_1 \cup \operatorname{supp} T_2$. Then, by (ii),

$$\langle T_1 + T_2, v\rangle = \langle T_1, v\rangle + \langle T_2, v\rangle = 0$$

and hence, by (ii) again,

$$\operatorname{supp}(T_1 + T_2) \subseteq \operatorname{supp} T_1 \cup \operatorname{supp} T_2.$$

Suppose that in addition $\operatorname{supp} T_1 \cap \operatorname{supp} T_2 = \varnothing$. Let $a \in \operatorname{supp} T_1$ and let $v \in A(G)$ such that $v \cdot (T_1 + T_2) = 0$. Then $v \cdot T_1 = -v \cdot T_2 = S$, say. By (i),

$$\operatorname{supp} S \subseteq \operatorname{supp} T_1 \cap \operatorname{supp} T_2 = \varnothing.$$

This implies $S = 0$ by Lemma 2.5.5. Thus $v \cdot T_1 = 0$ and therefore $v(a) = 0$. So

$$\operatorname{supp} T_1 \subseteq \operatorname{supp}(T_1 + T_2)$$

by Proposition 2.5.3(iii). Similarly, $\operatorname{supp} T_2 \subseteq \operatorname{supp}(T_1 + T_2)$.

For (4), suppose that $\operatorname{supp} T_2$ is compact. It is a straightforward consequence of (iii) that

$$\operatorname{supp}(\lambda_G(x) T_2) = (\operatorname{supp}(\lambda_G(x))(\operatorname{supp} T_2)$$

for any $x \in G$. Hence, by (1) and (3), we get

$$\operatorname{supp}(T_1 T_2) \subseteq (\operatorname{supp} T_1)(\operatorname{supp} T_2)$$

whenever T_1 is a finite linear combination of operators $\lambda_G(x)$. If T_1 is arbitrary, then given any closed neighbourhood V of $\operatorname{supp} T_1$ with relatively compact complement, by (iii) there exists a net $(S_\alpha)_\alpha$ in $VN(G)$ such that $T_1 = w^*\text{-}\lim_\alpha S_\alpha$ and each S_α is a finite linear combination of operators $\lambda_G(x)$, $x \in V$. Then

$$\operatorname{supp}(S_\alpha T_2) \subseteq V \cdot \operatorname{supp} T_2$$

for all α, and since $V \cdot \operatorname{supp} T_2$ is closed and $T_1T_2 = w^*\text{-}\lim_\alpha(S_\alpha T_2)$, it follows from (iv) that

$$\operatorname{supp}(T_1T_2) \subseteq V \cdot \operatorname{supp} T_2.$$

Now let $\mathcal{V}$ denote the collection of all closed neighbourhoods of $\operatorname{supp} T_1$ with relatively compact complement. Then $\bigcap\{V : V \in \mathcal{V}\} = \operatorname{supp} T_1$, and since $\operatorname{supp} T_2$ is compact, it is easily verified that

$$\textstyle\bigcap_{V\in\mathcal{V}}(V \cdot \operatorname{supp} T_2) = (\bigcap_{V\in\mathcal{V}} V) \cdot \operatorname{supp} T_2.$$

This implies $\operatorname{supp}(T_1T_2) \subseteq (\operatorname{supp} T_1)(\operatorname{supp} T_2)$. □

LEMMA 2.5.7. *Let $T \in VN(G)$ and $u \in A(G) \cap C_c(G)$. Then*

$$\operatorname{supp}(Tu) \subseteq (\operatorname{supp} T)(\operatorname{supp} u).$$

PROOF. Let $a \notin (\operatorname{supp} T)(\operatorname{supp} u)$ and hence $a(\operatorname{supp} u)^{-1} \cap \operatorname{supp} T = \varnothing$. Since $\operatorname{supp} u$ is compact, there exist closed neighbourhoods V of a and U of $\operatorname{supp} T$ such that $V(\operatorname{supp} u)^{-1} \cap U = \varnothing$ and $G \backslash U$ is relatively compact. By Proposition 2.5.6(iii) there exists a net $(S_\alpha)_\alpha$ in $VN(G)$ such that $T = w^*\text{-}\lim_\alpha S_\alpha$ and each S_α is a finite linear combination of operators $\lambda_G(x)$, $x \in U$.

Fix α and let $S_\alpha = \sum_{j=1}^n c_j\lambda_G(x_j)$, $x_j \in U$. Then, for any $x \in G$,

$$\langle \varphi_{S_\alpha}, (R_xu)\check{}\,\rangle = \sum_{j=1}^n c_j\langle \lambda_G(x_j), (R_xu)\check{}\,\rangle = \sum_{j=1}^n c_j u(x_j^{-1}x).$$

Now, if $x \in V$ then $x_j^{-1}x \in U^{-1}V$ and hence $x_j^{-1}x \notin \operatorname{supp}\, u$. Thus $\langle \varphi_{S_\alpha}, (R_xu)\check{}\,\rangle = 0$ for all $x \in V$ and all α. Passing to the w^*-limit, it follows that

$$Tu(x) = \langle \varphi_T, (R_xu)\check{}\,\rangle = 0$$

for all $x \in V$. This shows that $a \notin \operatorname{supp}(Tu)$ and hence proves the statement of the lemma. □

The formula $T\check{u}(x) = \langle T, L_xu\rangle$ for $u \in A(G)$ and $x \in G$ in Lemma 2.3.11(ii) implies for any $\mu \in M(G)$

$$\begin{aligned}[\lambda(\mu)u](x) &= \langle \varphi_{\lambda(\mu)}, (R_xu)\check{}\,\rangle = \int_G (r_xu)\check{}(y)d\mu(y) \\ &= \int_G u(y^{-1}x)d\mu(y) = (\mu * u)(x)\end{aligned}$$

for all $x \in G$, so that $\lambda(\mu)u = \mu * u$.

LEMMA 2.5.8. *Let $T \in VN(G)$ and suppose that* $\operatorname{supp}(Tu) \subseteq \operatorname{supp} u$ *for all $u \in A(G) \cap C_c(G)$. Then $T = \lambda I$ for some $\lambda \in \mathbb{C}$.*

PROOF. The proof is divided into several steps.

We first show if U is a relatively compact open subset of G, then for any $u \in A(G) \cap C_c(G)$ such that u is constant on U, Tu is also constant on U. To verify this, fix any two points a and b in U and choose an open neighbourhood V

of the identity such that $Va \cup Vb \subseteq U$. If $x \in Vb$, then $x \in U$ and $xb^{-1}a \in U$. So the function $u - R_{b^{-1}a}u \in A(G) \cap C_c(G)$ vanishes on Vb and hence, by hypothesis, $T(u - R_{b^{-1}a}u)$ also vanishes on Vb. In particular,

$$Tu(b) = TR_{b^{-1}a}u(b) = R_{b^{-1}a}Tu(b) = Tu(a),$$

since T commutes with right translations.

Next we prove the existence of a constant λ, depending only on T, with the following property: For any relatively compact open subset U of G and any $u \in A(G) \cap C_c(G)$ which is identically 1 on U, the function Tu is identically λ on U. By what we have seen above, for every pair (U, u) there exists a constant $\lambda(U, u)$ such that $Tu(x) = \lambda(U, u)$ for all $x \in U$. Now, fix U and let $u_1, u_2 \in A(G) \cap C_c(G)$ such that $u_1|_U = u_2|_U = 1$. Then $u_1 - u_2$ vanishes on U and hence so does $Tu_1 - Tu_2$ by hypothesis. Thus $\lambda(U, u_1) = \lambda(U, u_2)$. Finally, let U_1 and U_2 be two relatively compact open subsets of G and let $u \in A(G) \cap C_c(G)$ be such that $u = 1$ on $U_1 \cup U_2$. Then

$$\lambda(U_j, u) = Tu(x) = \lambda(U_1 \cup U_2, u)$$

for $x \in U_j$, $j = 1, 2$, and hence $\lambda(U_1, u) = \lambda(U_2, u)$. This shows that $\lambda(U, u)$ does neither depend on U nor on u.

Let λ denote the constant associated to T by the preceding discussion. We proceed to show that $T(1_C) = \lambda 1_C$ for every compact subset of G with the property that the boundary of C has measure zero. Fix such a set C. Since Haar measure is regular, there exists an ascending sequence $(V_n)_n$ of open subsets of G such that $\overline{V_n} \subseteq C^\circ$ and

$$|C \backslash \overline{V_n}\,| = |C^\circ \backslash \overline{V_n}\,| \leqslant 1/n$$

for all n. For each n, we find $u_n \in A(G) \cap C_c(G)$ such that $0 \leqslant u_n \leqslant 1$, $u_n = 1$ on $\overline{V_n}$ and $u_n = 0$ on $G \backslash C$. Then the bounded continuous function Tu_n vanishes on $G \backslash C$ and takes the value λ on $\overline{V_n}$. Since $u_n \to 1_C$ in $L^2(G)$, it follows that

$$T(1_C) = \lim_{n \to \infty} Tu_n = \lambda 1_C.$$

Since the characteristic functions 1_C, where C is an arbitrary compact subset of G, form a total set in $L^2(G)$, to finish the proof it suffices to show that the conclusion of the preceding paragraph also holds if the hypothesis that the boundary of C be of measure zero is dropped.

Thus let C be an arbitrary compact subset of G. Then there exists a sequence $(U_n)_n$ of relatively compact open sets U_n in G such that, for each n, $C \subseteq U_n$, $|U_n| \leqslant |C| + \frac{1}{n}$ and the boundary of U_n is of measure zero. Then $1_C = \lim_{n\to\infty} 1_{U_n}$ in $L^2(G)$ and hence $T(1_C) = \lim_{n\to\infty} T(1_{U_n})$. Since we already know that $T(1_{U_n}) = \lambda 1_{U_n}$, we conclude that $T(1_C) = \lambda 1_C$, as was to be shown. □

COROLLARY 2.5.9. *Let G be a locally compact group and $a \in G$. If* $\operatorname{supp} T = \{a\}$*, then* $T = \alpha\lambda_G(a)$ *for some* $\alpha \in \mathbb{C}$.

PROOF. Since, by Proposition 2.5.6(v)(4),

$$\operatorname{supp}(\lambda_G(a^{-1})T) \subseteq \operatorname{supp} \lambda_G(a^{-1}) \cdot \operatorname{supp} T = \{e\},$$

we can assume that $a = e$. Then $\operatorname{supp}(Tu) \subseteq \operatorname{supp} u$ for all $u \in A(G)$ by Lemma 2.5.7 since $\operatorname{supp} T = \{e\}$. Lemma 2.5.8 now shows that $T = \alpha \cdot I = \alpha\lambda_G(e)$, as required. □

The preceding corollary allows a quick application to the ideal theory of Fourier algebras. Recall that an ideal I of a commutative Banach algebra A is called primary if the hull of I in $\sigma(A)$ is a singleton.

COROLLARY 2.5.10. *Every closed primary ideal of $A(G)$ is maximal.*

PROOF. Let I be a closed ideal of $A(G)$ such that $h(I) = \{x\}$ for some $x \in G = A(G)$. The annihilator $I^{\perp}$ of I in $VN(G) = A(G)^*$ is weakly closed in $VN(G)$ and invariant under the transformations $T \to u \cdot T$, $u \in A(G)$, because I is an ideal. For $T \in VN(G)$, let $J_T = \{u \in A(G) : \langle T, u\rangle = 0\}$. Then, for $T \in I^{\perp}$, $J_T \supseteq I$ and hence $h(J_T) \subseteq \{x\}$. Thus, if $T \in I^{\perp}$ and $T \neq 0$, then $h(J_T) = \{x\}$ and so $\operatorname{supp} T = \{x\}$. Corollary 2.5.9 implies that T is a multiple of $\lambda_G(x)$. This shows that $I^{\perp}$ is 1-dimensional and therefore I is maximal and $I = k(x)$. □

COROLLARY 2.5.11. *Let $x \in G$ and $u \in A(G)$ such that $u(x) = 0$. Then there exists a sequence $(u_n)_n$ in $A(G)$ such that $\|u_n - u\|_{A(G)} \to 0$ and u_n vanishes on some neighbourhood of x.*

PROOF. By Corollary 2.5.10, $k(x)$ is the only closed ideal in $A(G)$ with hull $\{x\}$. The statement follows now from Section 1.1 since $A(G)$ is regular and semisimple. □

COROLLARY 2.5.12. *Let $u \in A(G)$ be such that $u(e) = 0$ and let $\epsilon > 0$. Then there exists $w \in P^1(G) \cap C_c(G)$ such that $\|uw\| \leqslant \epsilon$.*

PROOF. By Corollary 2.5.11 there exists $v \in A(G)$ such that $\|v - u\| \leqslant \epsilon$ and $v = 0$ in a neighbourhood U of e in G. Choose a compact symmetric neighbourhood V of e such that $V^2 \subseteq U$ and set

$$w = \|1_V\|_2^{-2}(1_V * \widetilde{1}_V) \in P(G) \cap C_c(G).$$

Then $w(e) = 1$, $\operatorname{supp} w \subseteq V^2$ and

$$\|wu - wv\| \leqslant \|w\| \cdot \|u - v\| = \|u - v\| \leqslant \epsilon.$$

Since $\operatorname{supp} w \cap \operatorname{supp} v = \varnothing$, $wv = 0$ and hence $\|wu\| \leqslant \epsilon$. □

2.6. The restriction map from $A(G)$ onto $A(H)$

Let H be a closed subgroup of the locally compact group G. In the study of Fourier algebras, a natural question arising is whether functions in $A(H)$ extend to functions in $A(G)$. In this section we show that this question admits an affirmative answer. The corresponding problem for Fourier-Stieltjes algebras is much more involved and will be investigated in Chapter 7.

In the following, let H be a closed subgroup of G and let $G\backslash H$ denote the space of all right cosets of H in G, endowed with the quotient topology defined through the map $p : G \to G\backslash H, x \to Hx$.

LEMMA 2.6.1. *Retain the above notation and let C be a compact, second countable subset of G. Then there exists a Borel subset M of C such that $p(M) = p(C)$ and p is one-to-one on M.*

PROOF. From general topology, it is known that there exist a perfect subset T of $[0, 1]$ and a continuous map φ from T onto the metric space C. For each $n \in \mathbb{N}$, define T_n to be the set of all $t \in T$ with the following property: for every $s \in T$ with $s \leqslant t - 1/n$, we have $p(\varphi(s)) \neq p(\varphi(t))$. Using that T is closed in $[0, 1]$, it is easily verified that T_n is relatively open in T. Then $M = \bigcap_{n=1}^{\infty} \varphi(T_n)$ is a Borel subset of C. To show that $p(M) = p(C)$, fix $x \in C$. Since $\varphi(T) = C$ and T is closed in $[0,1]$, the real number

$$t = \inf\{s \in T : p(\varphi(s)) = p(x)\} \in T$$

satisfies $p(\varphi(t)) = p(x)$. It is also clear that $t \in T_n$ for all n and hence $\varphi(t) \in M$.

It remains to show that φ is one-to-one on M. To see this, assume that $y \in C$ is such that $y \neq x$ and $p(y) = p(x)$ and set $S = \{s \in T : \varphi(s) = y\}$. Then, since $t \in T_n$ for all n, $S \subseteq [t, 1]$, and since S is closed in T and $y \neq x$, it even follows that $S \subseteq [t + \delta, 1]$ for some $\delta > 0$. For every $n > 1/\delta$ we then have $S \subseteq T \backslash T_n$ and consequently

$$y \in \varphi(S) \subseteq \varphi(T) \backslash \varphi(T_n) = C \backslash \varphi(T_n) \subseteq C \backslash M.$$

This contradiction completes the proof. □

PROPOSITION 2.6.2. *Let G be a second countable locally compact group and let H be a closed subgroup of G. Then there exists a Borel set S in G with the following properties.*

(i) *S intersects each right coset of H in exactly one point.*
(ii) *For each compact subset C of G, $HC \cap S$ has a compact closure.*
(iii) *$H \cap S = \{e\}$.*

Moreover, there is a closed neighbourhood V of e in G such that $HV = V$ and $V \cap S$ is relatively compact.

PROOF. We first show the existence of a set S satisfying properties (i), (ii) and (iii). Choose a compact symmetric neighbourhood V of e in G. Then $L = \bigcup_{n=1}^{\infty} V^n$ is an open subgroup of G and L has at most countably many right cosets in G. Since every compact subset of G is contained in the union of finitely many cosets of L, we can find a sequence $C_1 \subseteq C_2 \subseteq \cdots$ of compact subsets of G such that every compact subset of G is contained in some C_j. By Lemma 2.6.1, for each j there exists a Borel subset $S_j \subseteq C_j$ such that $p(S_j) = p(C_j)$ and p is one-to-one on S_j. Observe next that the sets S_j can be chosen so that $S_j \subseteq S_{j+1}$ for every $j \in \mathbb{N}$. Indeed, suppose that we already have arranged for $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_j$. Then choose any Borel set $M_{j+1} \subseteq C_{j+1}$ such that p is one-to-one on M_{j+1} and $p(M_{j+1}) = p(C_{j+1})$ and set

$$S_{j+1} = (M_{j+1} \backslash p^{-1}(p(S_j))) \bigcup S_j.$$

Then S_{j+1} is a Borel set. To see this, since S_j and M_{j+1} are Borel sets, it suffices to verify that $p^{-1}(p(S_j))$ is a Borel set. Since S_j is a Borel set in the complete metric space C_j and p is continuous and one-to-one on S_j, $p(S_j)$ is a Borel set [**170**], and hence $p^{-1}(p(S_j))$ is a Borel set as well.

Now, set $S = \bigcup_{j=1}^{\infty} S_j$. Then S is a Borel set satisfying (i) and (ii). Indeed,

$$p(S) = \bigcup_{j=1}^{\infty} p(S_j) = \bigcup_{j=1}^{\infty} p(C_j) = G \backslash H,$$

and since p is one-to-one on each S_j and $S_j \subseteq S_{j+1}$ for each j, p is one-to-one on S. To verify (ii), let $(s_n)_n$ be a sequence in $HC \cap S$ and choose $j \in \mathbb{N}$ such that $C \subseteq C_j$. Then, since $p(S_j) = p(C_j)$, for each n there exist $x_n \in H$ and $t_n \in S_j$ such that $s_n = x_n t_n$. This implies that $s_n = t_n$ and, since $S_j \subseteq C_j$, the sequence $(t_n)_n$ has a convergent subsequence. This shows that $HC \cap S$ is relatively compact.

Finally, translating S if necessary, we can arrange for $S \cap H = \{e\}$.

For the remaining statement, choose an open neighbourhood U of e in G such that $\overline{U}$ is compact and set $V = H\overline{U}$. Then V is a closed neighbourhood of e in G such that $HV = H(H\overline{U}) = H\overline{U} = V$. We also have $\overline{V \cap S} = \overline{(H\overline{U}) \cap S}$, which is compact by (ii). $\square$

The following proposition, the proof of which is fairly long and technical, is a major step towards Theorem 2.6.4 below.

Proposition 2.6.3. *Let G, H, S and V be as in Proposition 2.6.2 and for $x \in G$, let $\beta(x)$ be the unique element in H such that $x = \beta(x)s$ for some $s \in S$. For any complex-valued function f on H define f_V on G by*

$$f_V(x) = f(\beta(x))1_V(x), \ x \in G.$$

Then the following hold.

(i) *If f is a measurable function on H, then f_V is measurable on G.*
(ii) *If f has compact support, then so does f_V.*
(iii) *There exists a constant $c > 0$ such that $f \to c f_V$ is a linear isometry of $L^2(H)$ into $L^2(G)$.*
(iv) *The mapping $f \to f_V$ is a linear isometry of $L^\infty(H)$ into $L^\infty(G)$.*
(v) *If f and g are in $L^2(H)$, then for all $h \in H$, we have*

$$c^2(f_V * \widecheck{g_V})(h) = (f * \check{g})(h),$$

the convolution on the left and on the right being over G and H, respectively.

Proof. In the sequel, m_G and m_H will denote left Haar measures on G and H, respectively.

(i) Assume first that f is real-valued. Then f_V is real-valued, and for every $r \in \mathbb{R}$, we have

$$\begin{aligned} \{x \in G : f(\beta(x)) > r\} &= \{x \in G : \beta(x) \in \{h \in H : f(h) > r\}\} \\ &= \{h \in H : f(h) > r\} \cdot S. \end{aligned}$$

Since f is measurable, $T = \{h \in H : f(h) > r\}$ is measurable in H and hence TS is a measurable subset of G. Thus $f \circ \beta$ is measurable, and so is 1_V since V is closed in G. So f_V is measurable. If f is an arbitrary complex-valued measurable function, then standard arguments on $\mathrm{Re} f$ and $\mathrm{Im} f$ show that f_V is measurable.

(ii) If $f_V(x) \neq 0$, then $x \in V$ and $f(\beta(x)) \neq 0$. This implies

$$\operatorname{supp} f_V \subseteq \overline{V \cap \beta^{-1}(\operatorname{supp} f)} = \overline{V \cap (\operatorname{supp} f)S}.$$

If $y \in V \cap (\operatorname{supp} f)S$, then $\beta(y) \in \operatorname{supp} f$ and $\gamma(y) = \beta(y)^{-1}y \in HV = V$, and therefore $y \in (\operatorname{supp} f)(S \cap V)$. This shows that

$$\operatorname{supp} f_V \subseteq \overline{V \cap (\operatorname{supp} f)S} \subseteq (\operatorname{supp} f) \cdot \overline{S \cap V},$$

which is compact since both $\operatorname{supp} f$ and $\overline{S \cap V}$ are compact.

(iii) It will be convenient to prove the following fact first. Let f be any real-valued measurable function on H, let $\delta > 0$ and suppose that

$$m_H(\{h \in H : f(h) > \delta\}) > 0.$$

Then $m_G(\{x \in G : f_V(x) > \delta\}) > 0$. To see this, let C be a compact neighbourhood of e in G such that $C \subseteq V$ and set $D = \{h \in H : f(h) > \delta\}$. Since $m_H(D) > 0$, we have $m_G(D(CH \cap S)) > 0$. Because $C \subseteq V$ and $D \subseteq V$,

$$D(CH \cap S) \subseteq D(V \cap S) \subseteq DS \cap V,$$

so that $m_G(V \cap DS) > 0$ (clearly, $DS \cap V$ is a measurable set). Now note that if $y \in DS \cap V$, then $\beta(y) \in D$ and $x \in V$. Therefore,

$$DS \cap V \subseteq \{x \in G : f_V(x) > \delta\},$$

and hence the latter set has positive measure.

We next define a positive and additive functional on $C_c^+(H)$ as follows. If $f \in C_c^+(H)$, then f_V is bounded and measurable by (i) and f_V has compact support by (ii). Thus we may define

$$I(f) = \int_G f_V(x)dx.$$

If $f \neq 0$, then for some $\delta > 0$ the set $\{h \in H : f(h) > \delta\}$ is nonempty and open in H. The previous paragraph then shows that $m_G(\{y \in G : f_V(y) > \delta\}) > 0$, whence $I(f) > 0$. Furthermore, since $HV = V$, for $h \in H$ we have

$$\begin{aligned} I(L_h f) &= \int_G 1_V(x)(L_h f)(\beta(x))dx = \int_G 1_V(x) f(h^{-1}\beta(x))dx \\ &= \int_G 1_V(h^{-1}x) f(\beta(h^{-1}x))dx = \int_G 1_V(x) f(\beta(x))dx \\ &= I(f). \end{aligned}$$

Hence I is left invariant on $C_c^+(H)$. By the uniqueness theorem for the left Haar integral there exists a constant $c > 0$ such that, for all $f \in C_c^+(H)$,

$$\int_H f(h)dh = c\int_G f_V(x)dx.$$

From this equation, (iii) follows by routine arguments from integration theory.

(iv) Since G and H are σ-compact, in both groups there is no distinction between locally null sets and null sets. Let $f \in L^\infty(H)$. Then, for each $\delta < \|f\|_\infty$, the set $\{h \in H : |f(h)| > \delta\}$ is not a null set. Thus by what we have shown in the first paragraph of the proof of (iii), $\{x \in G : |f_V(x)| > \delta\}$ is not a null set. Since this holds for all $\delta < \|f\|_\infty$, it follows that $\|f\|_\infty \leqslant \|f_V\|_\infty$.

For the reverse inequality, let $\delta < \|f_V\|_\infty$. There is a measurable subset M of G such that $m_G(M) > 0$ and $|f_V(x)| > \delta$ for all $x \in M$. Then $M \subseteq V$ and $|f(x)| > \delta$ for all $x \in \beta(M)$. Since $M \subseteq \beta(M) \subseteq S$, we have $m_G(\beta(M)S) \geqslant m_G(M) > 0$, which in turn implies $m_H(\beta(M)) > 0$. Therefore, the set $\{h \in H : f(h) > \delta\}$ contains $\beta(M)$ which has positive measure, and this implies $\|f\|_\infty > \delta$. As before, since this holds for all $\delta < \|f_V\|_\infty$, we conclude that $\|f\|_\infty \geqslant \|f_V\|_\infty$.

(v) Let $f, g \in L^2(H)$. Then $f_V, g_V \in L^2(G)$ by (iii). Therefore, the convolution products $(f * \check{g})(h)$ and $(f_V * \widetilde{g_V})(x)$ exist for all $h \in H$ and $x \in G$, respectively.

For $h \in H$, it follows from (iii) that

$$\begin{aligned}(f_V * \widecheck{g_V})(h) &= \int_G f_V(hx)g_V(x)dx \\ &= \int_G 1_V(x)f(h\beta(x))g(\beta(x))dx \\ &= \int_G 1_V(x)(L_{h^{-1}}f)(\beta(x))g(\beta(x))dx \\ &= \int_G 1_V(x)(L_{h^{-1}}f)\cdot g)(\beta(x))dx \\ &= \int_G ((L_{h^{-1}}f)g)_V(y)\, dy \\ &= \frac{1}{c^2}\int_H (L_{h^{-1}}f)\cdot g)(y)dy \\ &= \frac{1}{c^2}(f * \check{g})(h).\end{aligned}$$

This completes the proof of the proposition. □

With Proposition 2.6.3 at hand, we are now ready for the main result of this section.

THEOREM 2.6.4. *Let G be a locally compact group and H a closed subgroup of G. For every $u \in A(H)$ there exists $v \in A(G)$ such that $v|_H = u$ and $\|v\|_{A(G)} = \|u\|_{A(H)}$. If u is positive definite, then v can be chosen to be positive definite.*

PROOF. (a) To begin with, we consider the case when G is second countable. Let $u \in A(H)$ and let $f, g \in L^2(H)$ be such that $u = f * \check{g}$ and $\|u\|_{A(H)} = \|f\|_2\|g\|_2$. Define $v : G \to \mathbb{C}$ by $v(x) = c^2(f_V * \widecheck{g_V})(x)$, $x \in G$. Then $v \in A(G)$ and, by Proposition 2.6.3(v), $v(h) = (f * \check{g})(h) = u(h)$ for all $h \in H$. Moreover, by assertion (iii) of Proposition 2.6.3,

$$\|v\|_{A(G)} \leqslant c^2\|f_V\|_2\|g_V\|_2 = \|f\|_2\|g\|_2 = \|u\|_{A(H)}.$$

Since $\|u\|_{A(H)} = \|v|_H\|_{A(H)} \leqslant \|v\|_{A(G)}$, it follows that $\|v\|_{A(G)} = \|u\|_{A(H)}$.

(b) Next, suppose that G is σ-compact. Since G is a normal topological space, u extends to some uniformly continuous function h on G. The theorem of Kakutani and Kodaira 1.2.16 then assures that there exists a compact normal subgroup K of G such that G/K is second countable and h is constant on cosets of G/K. Then u is constant on cosets of $H \cap K$ in H. Indeed, if $x, y \in H$ are such that $y^{-1}x \in K$, then

$$u(x) = h(x) = h(y(y^{-1}x)) = h(y) = u(y).$$

Since K is compact, HK is a closed subgroup of G. Moreover, since H is σ-compact, the map $x(H \cap K) \to xK$ is a topological isomorphism from $H/H \cap K$ onto HK/K. Therefore we can define a function w on HK/K by $w(xK) = u(x)$, $x \in H$. Then w belongs to $A(HK/K)$. Since G/K is second countable, (a) applies to the closed subgroup HK/K and w and yields the existence of some $w' \in A(G/K)$ with $w'|_{HK/K} = w$ and $\|w'\|_{A(G/K)} = \|w\|_{A(HK/K)}$. Now define $v \in A(G)$ by $v(x) = w'(xK)$. For $h \in H$, we then have

$$v(h) = w'(hK) = w(hK) = u(h).$$

It is also clear that

$$\|v\|_{A(G)} = \|w'\|_{A(G/K)} = \|w\|_{A(HK/K)} = \|u\|_{A(H)}.$$

(c) Now let G be an arbitrary locally compact group and let $u = f * \check{g} \in A(H)$. We prove the existence of a σ-compact open subgroup K of G such that f and g both vanish almost everywhere on $H \backslash K$. Since $C_c(H)$ is dense in $L^2(H)$ and $C_c(H) = C_c(G)|_H$, there exist sequences $(f_n)_n$ and $(g_n)_n$ in $C_c(G)$ such that

$$\|f_n|_H - f\|_2 \to 0 \quad \text{and} \quad \|g_n|_H - g\|_2 \to 0.$$

Now simply take for K the open subgroup of G generated by the σ-compact set

$$\bigcup_{n=1}^{\infty}(\text{supp } f_n \cup \text{supp } g_n).$$

Then K has the required properties. It follows that $u = 0$ on $H \backslash K$ and

$$u|_{H \cap K} = f|_{H \cap K} * \check{g}|_{H \cap K}.$$

Now, since K is σ-compact, by (b) $u|_{H \cap K}$ admits an extension $w \in A(K)$ with $\|w\|_{A(K)} = \|u|_{H \cap K}\|_{A(H \cap K)}$.

Finally, define v to be the trivial extension of w to all of G. Then $v|_H = u$ since both functions agree on $H \cap K$ and vanish on $H \backslash K$. Furthermore,

$$\|v\|_{A(G)} = \|w\|_{A(K)} = \|u|_{H \cap K}\|_{A(H \cap K)} = \|u\|_{A(H)},$$

and we are done.

Following the construction performed in (a), (b) and (c), it is easily seen that v is positive definite whenever u is positive definite, that is, if $u = f * \check{f}$ for some $f \in L^2(H)$ with $\|u\|_{A(H)} = \|f\|_2 \|\check{f}\|_2$. □

Corollary 2.6.5. *Let G be a locally compact group and H a closed subgroup of G. Let $I(H) = \{u \in A(G) : u|_H = 0\}$. Then the restriction map $u \to u|_H$ from $A(G)$ onto $A(H)$ induces an isometric isomorphism $u + I(H) \to u|_H$ from the quotient algebra $A(G)/I(H)$ onto $A(H)$.*

Proof. The map $u + I(H) \to u|_H$ is an algebra isomorphism of $A(G)/I(H)$ into $A(H)$. Proposition 2.4.1(ii) shows that this map is onto and an isometry since

$$\|u|_H\|_{A(H)} = \inf\{\|v\|_{A(G)} : v \in A(G), v - u \in I(H)\} = \|u + I(H)\|$$

for all $u \in A(G)$. □

Let H be a closed subgroup of G and let

$$r : A(G) \to A(H), \; u \to r(u) = u|_H$$

be the restriction map. We conclude this section with briefly studying the adjoint map

$$r^* : A(H)^* = VN(H) \to A(G)^* = VN(G)$$

given by $\langle r^*(S), u \rangle = \langle S, r(u) \rangle$ for $u \in A(G)$ and $S \in VN(H)$. Since r is surjective by Proposition 2.4.1(ii), r^* is injective. Recall that λ_G and λ_H denote the regular representation of G and H, respectively.

Proposition 2.6.6. *Let H be a closed subgroup of the locally compact group G. The map r^* is a w^*-w^*-continuous isomorphism from $VN(H)$ onto $VN_H(G)$ satisfying $r^*(\lambda_H(x)) = \lambda_G(x)$ for all $x \in H$.*

PROOF. It is clear that r^* is w^*-w^*-continuous. To see that r^* is a homomorphism, we first observe that if $x \in H$, then $r^*(\lambda_H(x)) = \lambda_G(x)$. So r^* preserves products on the set $D = \{\lambda_H(x) : x \in H\}$. Since multiplication in a von Neumann algebra is separately continuous in the w^*-topology, it follows that $r^*(ST) = r^*(S)r^*(T)$ for all $S, T \in VN(H)$.

Observe next that r^* also preserves involution. Indeed, if $T \in D$ then clearly $r^*(T^*) = (r^*(T))^*$. If $T \in VN(H)$ is arbitrary, let $(T_\alpha)_\alpha$ be a net in the linear span of D such that $T_\alpha \to T$ in the w^*-topology. Then $T_\alpha^* \to T^*$ in the w^*-topology and hence

$$r^*(T^*) = \lim_\alpha r^*(T_\alpha^*) = \lim_\alpha (r^*(T_\alpha))^* = (r^*(T))^*.$$

To see that r^* is surjective, it suffices to show that $X = r^*(VN(H))$ is w^*-closed in $VN(G)$. Since r^* is a $*$-homomorphism of the C^*-algebra $VN(H)$ into the C^*-algebra $VN(G)$, X must be norm-closed. By the open mapping theorem, the unit ball X_1 of X is contained in $r^*(VN(H)_\delta)$ for some $\delta > 0$, where

$$VN(H)_\delta = \{S \in VN(H) : \|S\| \leqslant \delta\}.$$

We claim that X_1 is w^*-closed. For that, let $(T_\alpha)_\alpha$ be a net in X_1 converging to some $T \in VN(G)$ in the w^*-topology. Then, for each α, there exists $S_\alpha \in VN(H)_\delta$ such that $r^*(S_\alpha) = T_\alpha$. After passing to a subnet if necessary, we can assume that $S_\alpha \to S$ in the w^*-topology for some $S \in VN(H)_\delta$. Since r^* is w^*-w^*-continuous, it follows that $T_\alpha = r^*(S_\alpha) \to r^*(S)$ and hence that $T = r^*(S) \in X$. As $VN(G)_1$ is w^*-closed, $T \in X_1$. Thus X_1 is w^*-closed, and consequently X must be w^*-closed by the Krein-Šmulian theorem. This finishes the proof. $\square$

2.7. Existence of bounded approximate identities

Given any nonunital Banach algebra A, it is important to know whether A at least admits a bounded approximate identity. The theme of this section is to solve this problem for the Fourier algebra of a locally compact group G. It turns out that the existence of a bounded approximate identity in $A(G)$ is equivalent to the amenability of G (Theorem 2.7.2).

Of course, the reader will be aware of that there are several different properties that are equivalent to amenability of a locally compact group (Section 1.8). We are going to present a very much focused approach to Theorem 2.7.2 mainly using that amenability is equivalent to that the trivial representation is weakly contained in the left regular representation.

The following proposition provides the main step towards showing that amenability of G implies the existence of a bounded approximate identity.

PROPOSITION 2.7.1. *Let G be a locally compact group and let $(u_\alpha)_\alpha$ be a net in $P(G)$ that converges to some $u \in P(G)$ uniformly on compact subsets of G. Then*

$$\lim_\alpha \|(u_\alpha - u)v\|_{A(G)} = 0$$

for every $v \in A(G)$.

PROOF. Note first that since $u_\alpha(e) \to u(e)$, we can assume that the net $(u_\alpha(e))_\alpha$ is bounded. Because $P(G) \cap C_c(G)$ spans a dense subspace of $A(G)$ and

$$\sup_\alpha \|(u_\alpha - u)w\|_{A(G)} \leqslant (u(e) + \sup_\alpha u_\alpha(e))\|w\|_{A(G)}$$

for every $w \in A(G)$, it is sufficient to show that

$$\lim_{\alpha} \|(u_\alpha - u)v\|_{A(G)} = 0$$

for each $v \in P(G) \cap C_c(G)$. Fix such v and put $w = uv$ and $w_\alpha = u_\alpha v$. Then $w_\alpha, w \in P(G)$ and the supports of w_α and w are all contained in the compact set $K = \operatorname{supp} v$. Since $u_\alpha \to u$ uniformly on K, we have $\|w_\alpha - w\|_\infty \to 0$. Since $w \in P(G) \cap C_c(G)$, $\lambda_G(w)$ is a positive bounded operator on $L^2(G)$ and there exists $g \in L^2(G)$ such that

$$w = g * \widetilde{g} \quad \text{and} \quad g * f = \lambda_G(w)^{1/2} f$$

for all $f \in C_c(G)$. Similarly, there exist functions $g_\alpha \in L^2(G)$ such that

$$w_\alpha = g_\alpha * \widetilde{g_\alpha} \quad \text{and} \quad g_\alpha * f = \lambda_G(w_\alpha)^{1/2} f$$

for all $f \in C_c(G)$.

We are going to show that $\|g_\alpha - g\|_2 \to 0$. For $f \in C_c(G)$, we have $\|R_y f\|_2 = \Delta(y)^{1/2}\|f\|_2$ and therefore, using vector-valued integration,

$$\begin{aligned}
\|(w_\alpha - w) * f\|_2 &= \left\| x \to \int_G [w_\alpha(xy) - w(xy)] f(y^{-1})\, dy \right\|_2 \\
&= \left\| x \to \int_G [w_\alpha(y) - w(y)] \Delta(y^{-1}) L_{y^{-1}} f(x)\, dy \right\|_2 \\
&\leqslant \int_G |w_\alpha(y) - w(y)| \cdot \|f\|_2\, dy \\
&= \int_K |w_\alpha(y) - w(y)| \cdot \|f\|_2\, dy \\
&\leqslant \|w_\alpha - w\|_\infty \|f\|_2 |K|.
\end{aligned}$$

This implies $\|\lambda_G(w_\alpha) - \lambda_G(w)\| \to 0$ and hence also that $c = \sup_\alpha \|\lambda_G(w_\alpha)\| < \infty$. Now, employing the continuous functional calculus for C^*-algebras and approximating the function $\sqrt{t}$ through polynomials uniformly on the interval $[0, c]$, we conclude that also

$$\|\lambda_G(w_\alpha)^{1/2} - \lambda_G(w)^{1/2}\| \to 0.$$

This in turn yields

$$\lim_{\alpha} \|(w_\alpha - w) * g\|_2 = \lim_{\alpha} \|(\lambda_G(w_\alpha)^{1/2} - \lambda_G(w)^{1/2}) g\|_2 = 0$$

for all $g \in C_c(G)$, and hence also

$$\lim_{\alpha} \langle g_\alpha - g, f * h \rangle = \lim_{\alpha} \langle (g_\alpha - g) * h^*, f \rangle = 0$$

for all $f, h \in C_c(G)$. Since $C_c(G) * C_c(G)$ is dense in $L^2(G)$ and $\sup_\alpha \|g_\alpha\|_2^2 = \sup_\alpha w_\alpha(e) < \infty$, it follows that

$$\lim_{\alpha} \langle g_\alpha, f \rangle = \langle g, f \rangle$$

for all $f \in L^2(G)$. Since also $\|g_\alpha\|_2^2 = w_\alpha(e) \to w(e) = \|g\|_2^2$, we get

$$\lim_{\alpha} \|g_\alpha - g\|_2^2 = 2\,\|g\|_2^2 - 2 \lim_{\alpha} \langle g_\alpha, g \rangle = 0.$$

This finally implies that

$$\begin{aligned}
\|u_\alpha v - uv\|_{A(G)} &= \|w_\alpha - w\|_{A(G)} \\
&= \|g_\alpha * \widetilde{g}_\alpha - g * \widetilde{g}\|_{A(G)} \\
&= \|g_\alpha * (\widetilde{g}_\alpha - \widetilde{g}) + (g_\alpha - g) * \widetilde{g}\|_{A(G)} \\
&\leqslant \|g_\alpha - g\|_2(\|g_\alpha\|_2 + \|g\|_2),
\end{aligned}$$

which converges to 0, as was to be shown. □

We are now ready for the main result of this section.

THEOREM 2.7.2. *For a locally compact group G, the following three conditions are equivalent.*

(i) *G is amenable.*
(ii) *$A(G)$ has an approximate identity bounded by 1 and consisting of compactly supported positive definite functions.*
(iii) *$A(G)$ has a bounded approximate identity.*

PROOF. (i) ⇒ (ii) Since G is amenable, by Theorem 1.8.18 the trivial representation of G is weakly contained in the left regular representation. Thus, given any $K \in \mathcal{K}(G)$, the collection of all compact subsets of G, and $\epsilon > 0$, there exists $u_{K,\epsilon} \in P(G)$ associated with λ_G such that $|u_{K,\epsilon}(x) - 1| \leqslant \epsilon$ for all $x \in K$. Clearly, since $C_c(G)$ is dense in $L^2(G)$, we can in addition assume that $u_{K,\epsilon}$ has compact support. Now, order the set of all pairs $\alpha = (K, \epsilon)$, $K \in \mathcal{K}, \epsilon > 0$, by $\alpha \geqslant \alpha' = (K', \epsilon')$ if $K \supseteq K'$ and $\epsilon \leqslant \epsilon'$. Then the net $(u_\alpha)_\alpha$ satisfies the hypotheses of Proposition 2.7.1 with $u = 1_G$ and therefore forms an approximate identity for $A(G)$. Finally, since $u_\alpha(e) \to 1$, replacing u_α by $u_\alpha(e)^{-1}u_\alpha$, we can assume that $1 = u_\alpha(e) = \|u_\alpha\|_{A(G)}$. This shows that (ii) holds.

It remains to prove (iii) ⇒ (i). Suppose that $(u_\alpha)_\alpha$ is an approximate identity for $A(G)$ with $\|u_\alpha\|_{A(G)} \leqslant c < \infty$ for all α. We are going to show that $\|\lambda_G(f)\| = \|f\|_1$ for every $f \in C_c^+(G)$. By Proposition 1.8.20 and Theorem 1.8.18 , this property implies that G is amenable.

Let K be any compact subset of G, choose a compact neighbourhood V of e in G, and put

$$u = |V|^{-1}(1_V * 1_{V^{-1}K}) \in A(G).$$

Then, for $x \in K$,

$$u(x) = |V|^{-1}\int_G 1_V(y)1_{V^{-1}K}(y^{-1}x)dy = 1.$$

Since $\|u_\alpha u - u\|_{A(G)} \to 0$, it follows that $u_\alpha u \to u$ uniformly on K. Thus, given $\epsilon > 0$, there exists an index α such that $\mathrm{Re}(u_\alpha(x)) \geqslant 1 - \epsilon$ for all $x \in K$. We now apply the preceding with $K = \operatorname{supp} f$, where $f \in C_c^+(G)$. Then

$$\mathrm{Re}\langle u_\alpha, f\rangle = \int_G f(x)\mathrm{Re}(u_\alpha(x))dx \geqslant (1-\epsilon)\|f\|_1.$$

On the other hand, we have

$$|\langle u_\alpha, f\rangle| = |\langle \lambda_G(f), u_\alpha\rangle| \leqslant c\,\|\lambda_G(f)\|.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\|f\|_1 \leqslant c\,\|\lambda_G(f)\|$ for every $f \in C_c^+(G)$. Replacing f with the n-fold convolution product f^n in the preceding calculations, it follows that

$$\|f\|_1^n = \|f^n\|_1 \leqslant c\,\|\lambda_G(f^n)\| \leqslant c\,\|\lambda_G(f)\|^n$$

and therefore

$$\|f\|_1 \leqslant \|\lambda_G(f)\| \cdot \lim_{n\to\infty} c^{1/n} = \|\lambda_G(f)\| \leqslant \|f\|_1.$$

This finishes the proof of (iii) ⇒ (i). □

Theorem 2.7.2 raises the question of whether the Fourier algebra $A(G)$ of a nonamenable locally compact group G possesses an approximate identity which is bounded in some norm weaker than the $A(G)$-norm or at least an unbounded approximate identity. This appears to be a very difficult problem, which we are going to touch in Chapter 5. However, it should be mentioned that even the weakest possible variant, namely whether $u \in \overline{uA(G)}$ holds for every $u \in A(G)$, appears to be a widely open problem.

We continue with several interesting applications of Theorem 2.7.2.

Corollary 2.7.3. *Let G be an amenable locally compact group and C a compact subset of G. Then, given $\epsilon > 0$, there exists $u \in A(G) \cap C_c(G)$ such that $u = 1$ on C and $\|u\| \leqslant 1 + \epsilon$.*

Proof. Choose $0 < \delta < 1$ such that $(1 + 2\delta)(1 - \delta)^{-1} \leqslant 1 + \epsilon$. Since $A(G)$ is regular, there exists $w \in A(G)$ such that $w = 1$ on C. Because $A(G)$ has an approximate identity bounded by 1, by Cohen's factorization theorem [**126**] we can decompose w as $w = vw_1$, where $\|v\| \leqslant 1$ and $\|w_1 - w\| \leqslant \delta$. Let $u_1 = v - v(w - w_1) \in A(G)$. Then $u_1 = 1$ on C and $\|u_1\| \leqslant 1 + \delta$. Finally, choose $u_2 \in A(G) \cap C_c(G)$ such that $\|u_2 - u_1\| \leqslant \delta$ and define $u \in A(G)$ by the norm-convergent sum

$$u = u_2 \cdot \sum_{n=0}^{\infty} (u_1 - u_2)^n.$$

Then u has compact support, $u = 1$ on C and

$$\|u\| \leqslant \frac{\|u_2\|}{1 - \|u_1 - u_2\|} \leqslant \frac{1 + 2\delta}{1 - \delta} \leqslant 1 + \epsilon,$$

as required. □

Corollary 2.7.4. *Let H be a proper closed subgroup of a locally compact G such that the ideal $I(H)$ has a bounded approximate identity. Then H is amenable.*

Proof. By Theorem 2.7.2, it suffices to show that $A(H)$ has a bounded approximate identity. Choose any $x \in G\backslash H$. The ideal $I(xH)$ also has a bounded approximate identity, say $(u_\alpha)_\alpha$. For each α, let $v_\alpha = u_\alpha|_H \in A(H)$. Then $\|v_\alpha\|_{A(H)} \leqslant \|u_\alpha\|_{A(G)}$, and since $A(H) \cap C_c(H)$ is dense in $A(H)$, it is enough to verify that $\|v_\alpha v - v\|_{A(H)} \to 0$ for every $v \in A(H) \cap C_c(H)$. Fix such a v and let $C = \operatorname{supp} v$ and choose a neighbourhood V of C such that $V \cap xH = \varnothing$. Since $A(G)$ is regular, there exists $u_0 \in A(G)$ such that $u_0|_C = 1$ and $\operatorname{supp} u_0 \subseteq V$. Since the restriction map $A(G) \to A(H)$ is surjective, there exists $u \in A(G)$ with $u|_H = v$. Then $w = u_0 u$ belongs to $I(xH)$ and $w|_H = v$. Finally,

$$\lim_\alpha \|v_\alpha v - v\|_{A(H)} \leqslant \lim_\alpha \|w_\alpha w - w\|_{A(G)} = 0,$$

which completes the proof. □

THEOREM 2.7.5. *For a locally compact group G, $G \neq \{e\}$, the following are equivalent.*

(i) *G is amenable.*
(ii) *The ideal $I = I(\{e\})$ has a bounded approximate identity.*
(iii) *There exists a proper closed subgroup H of G such that $I(H)$ has a bounded approximate identity.*
(iv) *There exists an amenable proper closed subgroup H of G such that $I(H)$ has a bounded approximate identity.*

PROOF. (i) $\Rightarrow$ (ii) Let $(u_\alpha)_\alpha$ be an approximate identity for $A(G)$ with $\|u_\alpha\| \leqslant 1$ for all α and let $(W_\beta)_\beta$ be a neighbourhood basis of e. For each β, there exists $w_\beta \in P(G)$ such that supp $w_\beta \subseteq W_\beta$ and $w_\beta(e) = 1$. For each pair (α, β), let

$$v_{\alpha,\beta} = u_\alpha - u_\alpha(e) w_\beta \in A(G).$$

Then $v_{\alpha,\beta} \in I(\{e\})$ and

$$\|v_{\alpha,\beta}\| \leqslant \|u_\alpha\| + |u_\alpha(e)| \cdot \|w_\beta\| \leqslant 2.$$

Now let $u \in I(\{e\})$ and $\epsilon > 0$ be given. Since the singleton $\{e\}$ is set of synthesis (see Chapter 6), there exists $v \in j(\{e\})$ such that $\|v - u\| \leqslant \epsilon$. For β large enough, $W_\beta \cap \operatorname{supp} v = \varnothing$ and hence $w_\beta v = 0$. It follows that

$$\begin{aligned} \|v_{\alpha,\beta} u - u\| &\leqslant \|v_{\alpha,\beta}(u - v)\| + \|v_{\alpha,\beta}\, v - v\| + \|v - u\| \\ &\leqslant 3\epsilon + \|v_{\alpha,\beta} v - v\| = 3\epsilon + \|u_\alpha v - v\| \end{aligned}$$

since $w_\beta v = 0$. Since $(u_\alpha)_\alpha$ is an approximate identity for $A(G)$ and $\epsilon > 0$ is arbitrary, it follows that the net $(v_{\alpha,\beta})_{\alpha,\beta}$ is an approximate identity for $I(\{e\})$.

(ii) $\Rightarrow$ (iii) being trivial, assume that (iii) holds. Then the subgroup H must be amenable by Corollary 2.7.4, so (iv) holds.

(iv) $\Rightarrow$ (i) Since H is amenable, $A(H)$ has a bounded approximate identity. Then $A(G)/I(H)$, being isometrically isomorphic to $A(H)$ (Corollary 2.7.4), also has a bounded approximate identity. Since both $I(H)$ and $A(G)/I(H)$ have bounded approximate identities, the same is true for $A(G)$ (Proposition 1.1.5). Therefore, G is amenable (Theorem 2.7.2). □

Corollary 2.7.4 and Theorem 2.7.5 suggest the question of which impact on the group G the existence of just some closed ideal of $A(G)$ with bounded approximate identity might have. The remaining part of this section is devoted to clarify this question.

LEMMA 2.7.6. *Let I be a closed ideal of $A(G)$ with bounded approximate identity. Then the interior $h(I)^\circ$ of the hull $h(I)$ is closed in G and the boundary $\partial(h(I))$ has measure zero. Moreover, $1_{G \setminus h(I)^\circ} \in B_\lambda(G) = C^*_\lambda(G)^*$.*

PROOF. Let $E = h(I)$ and let $(u_\alpha)_\alpha$ be a bounded approximate identity for I. After passing to a subnet if necessary, we can assume that $w^* - \lim_\alpha u_\alpha = u$ for some $u \in C^*_\lambda(G)^* = B_\lambda(G)$. We claim that $u = 1_{G \setminus E^\circ}$. To see this, let $v \in I$, $f \in L^1(G)$ and $\epsilon > 0$. Then

$$\langle uv, f \rangle = \langle u, vf \rangle = \lim_\alpha \langle u_\alpha, vf \rangle = \lim_\alpha \langle u_\alpha v, f \rangle.$$

Now choose α such that $|\langle uv, f \rangle - \langle u_\alpha v, f \rangle| \leqslant \epsilon$ and $\|u_\alpha v - v\| \leqslant \epsilon$. Then

$$\begin{aligned} |\langle uv, f \rangle - \langle v, f \rangle| &\leqslant |\langle uv, f \rangle - \langle u_\alpha v, f \rangle| + |\langle u_\alpha v, f \rangle - \langle v, f \rangle| \\ &\leqslant \epsilon + \|f\|_1 \|u_\alpha v - v\| \leqslant \epsilon(1 + \|f\|_1). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $\langle uv, f\rangle = \langle v, f\rangle$ for every $f \in L^1(G)$, and hence $uv = v$ for every $v \in I$. This implies that $u(x) = 0$ on E° and $u(x) = 1$ on $G\backslash E$, and therefore $u = 1$ on $G\backslash E^\circ$. Thus $u = 1_{G\backslash E^\circ}$. In particular, E° is closed in G.

It remains to observe that $|\partial(E)| = 0$. For a contradiction, assume that $|\partial(E)| > 0$. Choose a Borel subset V of $\partial(E)$ such that $0 < |V| < \infty$, and let $f = 1_V$. Then, since $u = 1$ on $\partial(E)$ and $(u_\alpha)_\alpha \subseteq I$,

$$0 < |V| = \langle u, f\rangle = \lim_\alpha \langle u_\alpha, f\rangle = 0.$$

This contradiction finishes the proof. □

Lemma 2.7.7. *Let I be a non-zero closed ideal in $A(G)$ with bounded approximate identity. Then either G is amenable or $h(I)$ has positive measure.*

Proof. Assume that $|h(I)| = 0$. To conclude that G is amenable, it suffices to show that $\|f\|_1 = \|\lambda_G(f)\|$ for every $f \in C_c(G)$ with $f \geqslant 0$, where λ_G denotes the left regular representation of G (Section 1.8). Let M be a norm bound for an approximate identity of I.

Let $f \in C_c^+(G)$ and let C be any compact subset of G such that $\operatorname{supp} f \subseteq C$, and let $\epsilon > 0$. Since $|h(I)| = 0$, there exists an open neighbourhood V of $h(I)$ such that $|V| \cdot \|f\|_\infty \cdot M \leqslant \epsilon$. Since the bounded approximate identity of I converges to 1 uniformly on compact subsets, which are disjoint from $h(I)$, there exists $u \in I$ such that

$$\inf\{\mathrm{Re}(u(x)) : x \in C\backslash V\} \geqslant 1 - \epsilon.$$

Then $|\langle u, f\rangle| \leqslant \|u\| \cdot \|\lambda_G(f)\| \leqslant M \cdot \|\lambda_G(f)\|$. On the other hand,

$$\begin{aligned}\mathrm{Re}(\langle u, f\rangle) &= \int_C \mathrm{Re}(u(x))f(x)dx \\ &\geqslant (1-\epsilon)\|f\|_1 + \int_V \mathrm{Re}(u(x))f(x)dx \\ &\geqslant (1-\epsilon)\|f\|_1 - |V| \cdot \|f\|_\infty \cdot M \geqslant (1-\epsilon)\|f\|_1 - \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that

$$M \cdot \|\lambda_G(f)\| \geqslant |\langle u, f\rangle| \geqslant \|f\|_1$$

for every $f \in C_c^+(G)$. Thus, for $g \in C_c^+(G)$ and any $n \in \mathbb{N}$,

$$\|g\|_1^n = \|g^n\|_1 \leqslant M\|\lambda_G(g^n)\| \leqslant M \cdot \|\lambda_G(g)\|^n,$$

and hence $\|g\|_1 \leqslant \|\lambda_G(g)\|$, as required. □

Corollary 2.7.8. *Suppose that G is connected and I is a nonzero closed ideal of $A(G)$ with bounded approximate identity. Then G is amenable.*

Proof. Assuming that G is not amenable, $|h(I)| > 0$ by Lemma 2.7.7. On the other hand, by Lemma 2.7.6, $|\partial(h(I))| = 0$ and $h(I)^\circ$ is closed in G. Since G is connected, either $h(I)^\circ = \varnothing$ or $h(I) = G$. However, since $I \neq \{0\}$, $h(I) \neq G$. Thus $0 < |h(I)| = |h(I)^\circ \cup \partial(h(I))| = |\partial(h(I))| = 0$, a contradiction. □

Theorem 2.7.9. *Suppose that $A(G)$ has a nonzero closed ideal which possesses a bounded approximate identity. Then G has an amenable open subgroup.*

PROOF. We first observe that G can be assumed to be almost connected. In fact, since $I \neq \{0\}$, there exists $x \in G \backslash h(I)$ and, by translating if necessary, we can assume that $x \in G_0$, the connected component of the identity. Fix an open subgroup H of G such that H/G_0 is compact. Then $1_H \cdot I$ can be viewed as a closed ideal of $A(H)$, which is nonzero and has a bounded approximate identity. If G_0 has been shown to be amenable, then H is amenable as well because H/G_0 is compact. Thus we can assume that G is almost connected.

Recall that the restriction map $r : A(G) \to A(G_0), u \to u|_{G_0}$ is surjective and norm decreasing. Therefore, $J = \overline{r(I)}$ is a closed ideal of $A(G_0)$ and J has a bounded approximate identity. Finally, $J \neq \{0\}$ since $x \in G_0 \backslash h(I)$. Corollary 2.7.8 now implies that G_0 is amenable. □

In Section 6.5, employing operator space theory, we shall explicitly describe all the closed ideals of $A(G)$ when G is amenable in terms of the closed coset ring of G.

2.8. The subspaces $A_\pi(G)$ of $B(G)$

In this section we associate to any unitary representation π of the locally compact group G a closed linear subspace $A_\pi(G)$ of $B(G)$ and present several results about these spaces, which will be used to determine $B(G)$ for some specific groups G (Section 2.9) and also in Chapters 3 and 4.

DEFINITION 2.8.1. Let π be a unitary representation of G. Let

$$A_\pi(G) = \overline{\operatorname{span}\{\varphi_{\xi,\eta} : \xi, \eta \in \mathcal{H}(\pi)\}}^{\|\cdot\|} \subseteq B(G),$$

where $\varphi_{\xi,\eta}(x) = \langle \pi(x)\xi, \eta \rangle$, $x \in G$. The space $A_\pi(G)$ is often called the *Fourier space* associated with the representation π. Moreover, let

$$VN_\pi(G) = \{\pi(x) : x \in G\}'' = \overline{\operatorname{span}\{\pi(x) : x \in G\}}^{w^*} \subseteq \mathcal{B}(\mathcal{H}(\pi)).$$

Note that when $\pi = \lambda_G$, then the set $A(G)$ of all coordinate functions of λ_G is already a closed linear subspace of $B(G)$ and hence $A(G) = A_{\lambda_G}(G)$.

If σ is a subrepresentation of π, then $A_\sigma(G)$ is a subspace of $A_\pi(G)$ by the very definition. For a more precise statement, see Lemma 2.8.3 below. We first identify the dual space of $A_\pi(G)$.

LEMMA 2.8.2. *For any representation π of G, the dual space $A_\pi(G)^*$ of $A_\pi(G)$ is isometrically isomorphic to $VN_\pi(G)$.*

PROOF. Let $\mathcal{B}(\mathcal{H}(\pi))_*$ denote the norm closure in $\mathcal{B}(\mathcal{H}(\pi))^*$ of the linear span of all linear functionals on $\mathcal{B}(\mathcal{H}(\pi))$ of the form

$$\varphi_{\xi,\eta}(T) = \langle T\xi, \eta \rangle, \quad T \in \mathcal{B}(\mathcal{H}(\pi)),\ \xi, \eta \in \mathcal{H}(\pi).$$

Then, as is well known, $\mathcal{B}(\mathcal{H}(\pi))_*$ is the unique predual of $\mathcal{B}(\mathcal{H}(\pi))$. Moreover, let $\mathcal{E}_\pi = \{\varphi|_{VN_\pi(G)} : \varphi \in \mathcal{B}(\mathcal{H}(\pi))_*\}$; then $\mathcal{E}_\pi^* = VN_\pi(G)$. For each $\varphi \in \mathcal{E}_\pi$, $\varphi \circ \pi \in A_\pi(G)$ and

$$\|\varphi|_{VN_\pi(G)}\| = \sup\left\{\left|\sum_{j=1}^n \lambda_j \varphi(\pi(x_j))\right| : \left\|\sum_{j=1}^n \lambda_j \pi(x_j)\right\| \leqslant 1\right\} = \|\varphi \circ \pi\|$$

by Lemma 2.1.8. □

Let V be a closed subspace of $B(G) = C^*(G)^*$, it follows readily from [**270**, Theorem 2.7, p. 123] that V is invariant, that is, $f \cdot V \cup V \cdot f \subseteq V$ for every $f \in C^*(G)$, if and only if V is left and right translation invariant. In this case there exists a unique central projection p in $C^*(G)^{**}$ such that $V = B(G) \cdot p$ [**270**, Theorem 2.7(iii), p. 123]. Since $A_\pi(G)$ is two-sided translation invariant, it is an invariant subspace of $B(G)$. We shall see next that conversely every closed invariant subspace of $B(G)$ is of this form.

LEMMA 2.8.3. *Let π be a unitary representation of G.*

(i) *If σ is a subrepresentation of π, then there exists a unique central projection P in*

$$\{T \in VN_\pi(G) : T \cdot A_\pi(G) \cup A_\pi(G) \cdot T \subseteq A_\pi(G)\} \subseteq \mathcal{B}(\mathcal{H}(\pi))$$

such that $A_\sigma(G) = P \cdot A_\pi(G)$.

(ii) *If V is a closed translation invariant subspace of $A_\pi(G)$, then $V = A_\sigma(G)$ for some subrepresentation σ of π.*

PROOF. (i) follows from [**270**, Theorem 2.7(iii), p. 123].

(ii) Given V, there exists a central projection P in $VN_\pi(G)$ such that $V = P \cdot A_\pi(G)$. Let $\mathcal{L} = P(\mathcal{H}(\pi))$ and define σ by $\sigma(x) = \pi(x)|_{\mathcal{L}}$, $x \in G$. Then σ is a subrepresentation of π on L and $A_\sigma(G) = V$. □

The following theorem gives an explicit description of the functions in $A_\pi(G)$.

THEOREM 2.8.4. *Let π be a representation of the locally compact group G.*

(i) *Let $(\xi_n)_n$ and $(\eta_n)_n$ be sequences in $\mathcal{H}(\pi)$ such that $\sum_{n=1}^\infty \|\xi_n\| \cdot \|\eta_n\|_n < \infty$. Then*

$$u(x) = \sum_{n=1}^\infty \langle \pi(x)\xi_n, \eta_n\rangle, \ x \in G,$$

defines an element of $A_\pi(G)$ and $\|u\| \leqslant \sum_{n=1}^\infty \|\xi_n\| \cdot \|\eta_n\|_n$.

(ii) *For each $u \in A_\pi(G)$ there exist sequences $(\xi_n)_n$ and $(\eta_n)_n$ in $\mathcal{H}(\pi)$ such that*

$$u(\cdot) = \sum_{n=1}^\infty \langle \pi(\cdot)\xi_n, \eta_n\rangle \quad \text{and} \quad \|u\| = \sum_{n=1}^\infty \|\xi_n\| \cdot \|\eta_n\|_n.$$

PROOF. (i) is clear since the series defining $u(x)$ is absolutely convergent and $\|\langle \pi(\cdot)\xi, \eta\rangle\| \leqslant \|\xi\| \cdot \|\eta\|$ for each $\xi, \eta \in \mathcal{H}(\pi)$. (ii) As in Section 2.1, we now use the polar decomposition of elements in the predual of a von Neumann algebra. For every $u \in A_\pi(G) = VN_\pi(G)_*$, there exist a partial isometry $V \in VN_\pi(G)$ and an element $|u| \in A_\pi(G)$ such that $u = V \cdot |u|$, $\|u\| = \||u|\|$ and $|u|$ defines a positive normal linear functional on $VN_\pi(G)$. Since the linear functional $|u|$ is positive, there exists a sequence $(\eta_n)_n$ in $\mathcal{H}(\pi)$ such that $|u|(x) = \sum_{n=1}^\infty \langle \pi(x)\eta_n, \eta_n\rangle$ for all $x \in G$ [**59**, Theorem 1, p.54]. It follows that $\||u|\| = |u|(e) = \sum_{n=1}^\infty \|\eta_n\|^2$. Now,

$$u(x) = (V \cdot |u|)(x) = \sum_{n=1}^\infty \langle \pi(x)V\eta_n, \eta_n\rangle,$$

and hence

$$\||u|\| = \|u\| \leqslant \sum_{n=1}^\infty \|V\eta_n\| \cdot \|\eta_n\| \leqslant \sum_{n=1}^\infty \|\eta_n\| \cdot \|\eta_n\| = \||u|\|.$$

Thus the stated decomposition follows by setting $\xi_n = V\eta_n$. □

LEMMA 2.8.5. *Let H and G be locally compact groups, $\phi : H \to G$ a continuous homomorphism and $j(f) = f \circ \phi$ for every function f on G. Let π be any representation of G.*

(i) $A_{\pi\circ\phi}(H) = j(A_\pi(G))$.
(ii) *For every $v \in A_{\pi\circ\phi}(H)$ there exists $u \in A_\pi(G)$ such that $v = j(u)$ and $\|v\|_{B(H)} = \|u\|_{B(G)}$.*

PROOF. (i) By Theorem 2.8.4, $A_\pi(G)$ consists precisely of those functions u in $B(G)$ for which there exist sequences $(\xi_n)_n$ and $(\eta_n)_n$ in $\mathcal{H}(\pi)$ such that

$$\sum_{n=1}^{\infty} \|\xi_n\| \cdot \|\eta_n\| < \infty \quad \text{and} \quad u(x) = \sum_{n=1}^{\infty} \langle \pi(x)\xi_n, \eta_n\rangle$$

for all $x \in G$. Consequently, $j(A_\pi(G))$ coincides with the set of all functions of the form $v(y) = \sum_{n=1}^{\infty}\langle \pi(\phi(y))\xi_n, \eta_n\rangle$, $y \in H$. This shows that $j(A_\pi(G)) = A_{\pi\circ\phi}(H)$.

(ii) Given $v \in A_{\pi\circ\phi}(H)$, by Theorem 2.8.4 there exist sequences $(\xi_n)_n$ and $(\eta_n)_n$ in $\mathcal{H}(\pi)$ such that $v(y) = \sum_{n=1}^{\infty}\langle \pi(\phi(y))\xi_n, \eta_n\rangle$ for all $y \in H$ and $\|v\| = \sum_{n=1}^{\infty} \|\xi_n\| \cdot \|\eta_n\|$. Thus $v = j(u)$ for some $u \in A_\pi(G)$ with $\|u\| \leqslant \sum_{n=1}^{\infty} \|\xi_n\| \cdot \|\eta_n\| = \|j(u)\|$. On the other hand, by Theorem 2.2.1(ii) we always have $\|j(u)\| \leqslant \|u\|$. □

REMARK 2.8.6. (1) Let H be the group G equipped with a stronger locally compact topology than the given one (e.g., the discrete topology), and let $\phi : H \to G$ be the identity map. Then the preceding lemma tells us that $A_\pi(G) = A_{\pi\circ\phi}(H)$.

(2) Let H be a closed subgroup of G and $\phi : H \to G$ the embedding. Then, by Lemma 2.8.5, $A_{\pi\circ\phi}(H) = A_\pi(G)|_H$. In particular, $A_\pi(G)|_H$ is closed in $B(H)$ for every representation π of G. Taking for π the universal representation of G, we conclude that $B(G)|_H$ is closed in $B(G)$.

We now turn to various properties of the assignment $\pi \to A_\pi(G)$.

LEMMA 2.8.7. *Let π and ρ be representations of G. Then $A_\pi(G) \cap A_\rho(G) = \{0\}$ if and only π and ρ are disjoint.*

PROOF. Suppose that π and ρ are not disjoint. Then there exists a subrepresentation σ of π with $\mathcal{H}(\sigma) \neq \{0\}$, which is equivalent to a subrepresentation of ρ, implemented by a unitary map $U : \mathcal{H}(\sigma) \to \phi(\mathcal{H}(\sigma)) \subseteq \mathcal{H}(\rho)$. Then

$$\langle \sigma(x)\xi, \eta\rangle = \langle\, U(\sigma(x)\xi), U\eta\rangle = \langle \rho(x)(U\xi), U\eta\rangle$$

for all $\xi, \eta \in \mathcal{H}(\sigma)$ and $x \in G$. This contradicts $A_\pi(G) \cap A_\rho(G) = \{0\}$.

For the converse, assume that π and ρ are disjoint and let $\sigma = \pi \oplus \rho$. Let P_π and P_ρ be the central projections in $VN_\sigma(G)$ provided by Lemma 2.8.3. Then P_π and P_ρ are the orthogonal projections from $\mathcal{H}(\sigma)$ onto $\mathcal{H}(\pi)$ and $\mathcal{H}(\rho)$, respectively. Since π and ρ are disjoint, it follows that $P_\pi P_\rho = 0$, and this implies that

$$\begin{aligned} A_\pi(G) \cap A_\rho(G) &= P_\pi A_\sigma(G) \cap P_\rho A_\sigma(G) = P_\pi A_\sigma(G) \cap (I - P_\pi)P_\rho A_\sigma(G) \\ &\subseteq P_\pi A_\sigma(G) \cap (I - P_\pi)A_\sigma(G) = \{0\}, \end{aligned}$$

as was to be shown. □

PROPOSITION 2.8.8. *Let $(\pi_\iota)_{\iota\in I}$ be a family of unitary representations of G and $\pi = \oplus_{\iota\in I}\pi_\iota$ their direct sum. Then $A_\pi(G)$ consists precisely of those functions in $B(G)$ which can be written as $u = \sum_{\iota\in I} u_\iota$, where $u_\iota \in A_{\pi_\iota}(G)$ and $\sum_{\iota\in I} \|u_\iota\| < \infty$.*

PROOF. Since each $A_{\pi_\iota}(G)$ is a subspace of $A_\pi(G)$, every family $(u_\iota)_{\iota\in I}$ satisfying $u_\iota \in A_{\pi_\iota}(G)$ and $\sum_{\iota\in I}\|u_\iota\| < \infty$, defines an element of $A_\pi(G)$ by setting $u(x) = \sum_{\iota\in I} u_\iota(x)$ for $x \in G$.

Conversely, let $u \in A_\pi(G)$ be given. Then, by Theorem 2.8.4, there exist sequences $(\xi_n)_n$ and $(\eta_n)_n$ in $\mathcal{H}(\pi)$ such that $\sum_{n=1}^\infty \|\xi_n\|\cdot\|\eta_n\| < \infty$ and

$$u(x) = \sum_{n=1}^\infty \langle \pi(x)\xi_n, \eta_n\rangle, \quad x \in G.$$

Since $\mathcal{H}(\pi) = \oplus_{\iota\in I}\mathcal{H}(\pi_\iota)$, ξ_n and η_n can be decomposed as $\xi_n = \sum_{\iota\in I}\xi_{n\iota}and\eta_n = \sum_{\iota\in I}\eta_{n\iota}$, where $\xi_{n\iota}, \eta_{n\iota} \in \mathcal{H}(\pi_\iota)$ and

$$\|\xi_n\|^2 = \sum_{\iota\in I}\|\xi_{n\iota}\|^2 \quad \text{and} \quad \|\eta_n\|^2 = \sum_{\iota\in I}\|\eta_{n\iota}\|^2.$$

This implies, for each $x \in G$ and $n \in \mathbb{N}$,

$$\langle \pi(x)\xi_n, \eta_n\rangle = \sum_{\iota\in I}\langle \pi_\iota(x)\xi_{n\iota}, \eta_{n\iota}\rangle.$$

Now the family of complex numbers $(\langle \pi_\iota(x)\xi_{n\iota}, \eta_{n\iota}\rangle)_{\iota\in I, n\in\mathbb{N}}$ is absolutely summable. Indeed, since $|\langle \pi_\iota(x)\xi_{n\iota}, \eta_{n\iota}\rangle)| \leqslant \|\xi_{n\iota}\|\cdot\|\eta_{n\iota}\|$ and

$$\sum_{\iota\in I}\|\xi_{n\iota}\|\cdot\|\eta_{n\iota}\| \leqslant \left(\sum_{\iota\in I}\|\xi_{n\iota}\|^2\right)^{1/2}\left(\sum_{\iota\in I}\|\eta_{n\iota}\|^2\right)^{1/2},$$

it follows that

$$\sum_{n=1}^\infty\left(\sum_{\iota\in I}\|\xi_{n\iota}\|\cdot\|\eta_{n\iota}\|\right) \leqslant \sum_{n=1}^\infty\|\xi_n\|\cdot\|\eta_n\| < \infty.$$

Thus we can write, for $x \in G$,

$$u(x) = \sum_{\iota\in I}\left(\sum_{n=1}^\infty \langle \pi_\iota(x)\xi_{n\iota}, \eta_{n\iota}\rangle\right).$$

Since $\sum_{n=1}^\infty \|\xi_{n\iota}\|\cdot\|\eta_{n\iota}\| < \infty$, we can now define $u_\iota \in A_{\pi_\iota}(G)$ by

$$u_\iota(x) = \sum_{n=1}^\infty \langle \pi_\iota(x)\xi_{n\iota}, \eta_{n\iota}\rangle.$$

Then $\|u_\iota\| \leqslant \sum_{n=1}^\infty \|\xi_{n\iota}\|\cdot\|\eta_{n\iota}\|$ and therefore

$$\begin{aligned}\sum_{\iota\in I}\|u_\iota\| &\leqslant \sum_{n=1}^\infty\left(\sum_{\iota\in I}\|\xi_{n\iota}\|\cdot\|\eta_{n\iota}\|\right)\\ &\leqslant \sum_{n=1}^\infty\left(\sum_{\iota\in I}\|\xi_{n\iota}\|^2\right)^{1/2}\left(\sum_{\iota\in I}\|\eta_{n\iota}\|^2\right)^{1/2}\\ &= \sum_{n=1}^\infty\|\xi_n\|\cdot\|\eta_n\|.\end{aligned}$$

This finishes the proof. □

PROPOSITION 2.8.9. *Let* π *and* ρ *be* disjoint representations *of* G. *Then*

(i) $A_{\pi\oplus\rho}(G) = A_\pi(G) \oplus_1 A_\rho(G)$.

(ii) $V_{\pi\oplus\rho}(G) = V_\pi(G) \oplus_\infty V_\rho(G)$.

PROOF. (i) We know from Lemma 2.8.7 that $A_\pi(G) \cap A_\rho(G) = \{0\}$. On the other hand, $A_{\pi\oplus\rho}(G) = A_\pi(G) + A_\rho(G)$ by Proposition 2.8.8. It thus remains to show that if $u \in A_{\pi\oplus\rho}(G)$, $u = u_1 + u_2$, where $u_1 \in A_\pi(G)$ and $u_2 \in A_\rho(G)$, then $\|u\| \geqslant \|u_1\| + \|u_2\|$.

To that end, let $\sigma = \pi \oplus \rho$ and $\xi, \eta \in \mathcal{H}(\sigma)$ such that $u(x) = \langle \sigma(x)\xi, \eta\rangle$ for all $x \in G$ and $\|u\| = \|\xi\| \cdot \|\eta\|$. Then, with P_π and P_ρ as in the proof of Lemma 2.8.7,

$$u_1(x) = \langle \sigma(x)P_\pi\xi, P_\pi\eta\rangle \quad \text{and} \quad u_2(x) = \langle \sigma(x)P_\rho\xi, P_\rho\eta\rangle$$

for all $x \in G$. Since $P_\pi + P_\rho = I_{\mathcal{H}(\sigma)}$,

$$\begin{aligned} \|u_1\| + \|u_2\| &\leqslant \|P_\pi\xi\| \cdot \|P_\pi\eta\| + \|P_\rho\xi\| \cdot \|P_\rho\eta\| \\ &\leqslant \left(\|P_\pi\xi\|^2 + \|P_\pi\eta\|^2\right)^{1/2} \left(\|P_\rho\xi\|^2 + \|P_\rho\eta\|^2\right)^{1/2} \\ &= \|\xi\| \cdot \|\eta\| = \|u\|. \end{aligned}$$

(ii) follows from (i) and Lemma 2.8.2. since

$$A_{\pi\oplus\rho}(G)^* = (A_\pi(G) \oplus_1 A_\rho(G))^* = V_\pi(G) \oplus_\infty V_\rho(G) = V_{\pi\oplus\rho}(G)$$

by disjointness of π and ρ. □

COROLLARY 2.8.10. *Let V be the closed linear span of all coefficient functions of those unitary representations of G which are disjoint from the left regular representation λ of G. Then $V = A_\sigma(G)$ for some representation σ of G and $B(G) = A_{\lambda_G}(G) \oplus_1 A_\sigma(G)$.*

PROOF. Clearly, V is translation invariant. Therefore, by Lemma 2.8.3, $V = A_\sigma(G)$ for some representation σ of G. The second statement follows from Proposition 2.8.9. □

PROPOSITION 2.8.11. *Let $(\pi_\iota)_{\iota\in I}$ be a family of pairwise disjoint representations of G and let $\sigma = \oplus_{\iota\in I}\pi_\iota$ and $u \in A_\sigma(G)$. Then there exists a unique decomposition $u = \sum_{\iota\in I} u_\iota$, where $u_\iota \in A_{\pi_\iota}(G)$. Moreover, $\|u\| = \sum_{\iota\in I} \|u_\iota\|$.*

PROOF. To show the uniqueness, fix $\lambda \in I$ and set $\rho = \sum_{\iota\neq\lambda} \pi_\iota$. Then π_λ and ρ are disjoint and hence $A_\sigma(G) = A_{\pi_\lambda}(G) \oplus A_\rho(G)$ by Proposition 2.8.9. Let $u = \sum_{\iota\in I} u_\iota$ be any decomposition of u as in Proposition 2.8.8, and let $v = \sum_{\iota\in I,\iota\neq\lambda} u_\iota$. Then $u = u_\lambda + v$, $u_\lambda \in A_{\pi_\lambda}(G)$, $v \in A_\rho(G)$, and this decomposition is unique. Thus u_λ is uniquely determined for every $\lambda \in I$.

Since $\|u\| \leqslant \sum_{\iota\in I} \|u_\iota\|$, it suffices to show that $\|u\| \geqslant \sum_{\iota\in J} \|u_\iota\|$ for every finite subset J of I. Let $\pi = \oplus_{\iota\in J}\pi_\iota$ and $\rho \oplus_{\iota\in I\setminus J} \pi_\iota$, and let $v = \sum_{\iota\in J} u_\iota \in A_\pi(G)$ and $w = \sum_{\iota\in I\setminus J} u_\iota \in A_\rho(G)$. Since π and ρ are disjoint, $\|u\| = \|v\| + \|w\|$ by Proposition 2.8.9. Moreover, as J is finite and the π, $\iota \in I$, are pairwise disjoint, repeated application of Proposition 2.8.9 yields $\|v\| = \sum_{\iota\in J} \|u_\iota\|$ and hence $\|u\| \geqslant \sum_{\iota\in J} \|u_\iota\|$, as required. □

We conclude this section by answering the natural question of when, for two representations π and ρ of G, the spaces $A_\pi(G)$ and $A_\rho(G)$ coincide. To that end, we have to introduce the following notion. Two representations π and ρ of G are called *quasi-equivalent* if there is no subrepresenation of π which is equivalent to a subrepresentation of ρ and no subrepresentation of ρ which is equivalent to a subrepresentation of π. Obviously, equivalence implies quasi-equivalence, and the two notions coincide for irreducible representations.

PROPOSITION 2.8.12. *Let π and ρ be representations of G. Then $A_\pi(G) = A_\rho(G)$ if and only if π and ρ are quasi-equivalent.*

PROOF. By [**60**, Proposition 5.3.1], π and ρ are quasi-equivalent if and only if there exists an isomorphism ϕ from $VN_\pi(G)$ onto $VN_\rho(G)$ such that $\rho(f) = \phi(\pi(f))$ for every $f \in L^1(G)$.

Suppose first that $A_\pi(G) = A_\rho(G)$. Then $A_\pi(G)^* = A_\rho(G)^*$, and hence there exists a Banach space isomorphism ϕ from $VN_\pi(G)$ onto $VN_\rho(G)$. We have to show that ϕ is an algebra isomorphism. For $T \in VN_\pi(G)$, the operator $\phi(T)$ satisfies $\langle \phi(T), u\rangle = \langle T, u\rangle$ for every $u \in A_\pi(G) = A_\rho(G)$. Thus

$$\langle \phi(\pi(f)), u\rangle = \langle \pi(f), u\rangle = \int_G f(x)u(x)dx = \langle \rho(f), u\rangle$$

for every $f \in L^1(G)$, and hence $\phi(\pi(f)) = \rho(f)$. For $S = \pi(f)$ and $T = \pi(g)$, $f, g \in L^1(G)$, it follows that

$$\phi(ST) = \phi(\pi(f)\pi(g)) = \phi(\pi(f * g)) = \rho(f * g) = \phi(S)\phi(T).$$

Now, $\pi(L^1(G))$ is ultraweakly dense in $VN_\pi(G)$ and the map $(S, T) \to ST$ is separately continuous for the ultraweak topologies. Moreover, the map ϕ remains to be continuous when $VN_\pi(G)$ and $VN_\rho(G)$ are endowed with their ultraweak topologies, because these topologies equal the weak topologies $\sigma(VN_\pi(G), A_\pi(G))$ and $\sigma(VN_\rho(G), A_\rho(G))$, respectively. From these facts we conclude that $\phi(ST) = \phi(S)\phi(T)$ for all $S, T \in VN_\pi(G)$. Thus ϕ is an algebra isomorphism, and consequently π and ρ are quasi-equivalent by the criterion mentioned above.

Conversely, assume that π and ρ are quasi-equivalent, and let ϕ be an algebra isomorphism from $VN_\pi(G)$ onto $VN_\rho(G)$ satisfying $\phi(\pi(f)) = \rho(f)$ for every $f \in L^1(G)$. Since the von Neumann algebras $VN_\pi(G)$ and $VN_\rho(G)$ are isomorphic, so are their preduals. Thus there exists an isometry $\phi_* : A_\pi(G) \to A_\rho(G)$ which is defined by the following property: $\langle u, T\rangle = \langle \phi_*(u), \phi(T)\rangle$ for $u \in A_\pi(G)$ and $T \in VN_\pi(G)$. For every $f \in L^1(G)$, it follows that

$$\begin{aligned}\int_G u(x)f(x)dx &= \langle u, \pi(f)\rangle = \langle \phi_*(u), \phi(\pi(f))\rangle = \langle \phi_*(u), \rho(f)\rangle \\ &= \int_G \phi_*(u)(x)f(x)dx.\end{aligned}$$

Since u and $\phi_*(u)$ are continuous, this equation implies $u = \phi_*(u)$. This shows that $A_\pi(G) = A_\rho(G)$. □

2.9. Some examples

When G is a noncompact locally compact abelian group then, according to common understanding, the spectrum of $B(G) = M(\widehat{G})$ is an intractable object. However, there are locally compact groups G, actually certain semidirect products of abelian groups with compact groups, for which $B(G)$ turns out to be an extension of $A(G)$ by another Fourier algebra and consequently $\sigma(B(G))$ can be determined. The situation is as follows.

Let G be a semidirect product $G = N \rtimes K$, where

(1) K is a compact group and N is a locally compact abelian group and both are second countable, and

(2) the dual space $\widehat{G}$ of G is countable and decomposes as

$$\widehat{K} \circ q \cup \{\pi_k : k \in \mathbb{N}\},$$

where $q : G \to K$ is the quotient map and each π_k is a subrepresentation of the left regular representation of G.

PROPOSITION 2.9.1. *Let $G = N \rtimes K$ as above. Then*

$$B(G) = A(K) \circ q + A(G).$$

PROOF. Let $u \in B(G)$, so that $u = \langle \pi(\cdot)\xi, \eta\rangle$ for some unitary representation π of G and $\xi, \eta \in \mathcal{H}(\pi)$. By condition (2) above, π is completely decomposable. Thus

$$\pi = (\oplus_{\sigma\in\widehat{K}} m_\sigma(\sigma \circ q)) \oplus (\oplus_{k=1}^{\infty} n_k \pi_k),$$

where m_σ and n_k denote the multiplicity of $\sigma \circ q$ and π_k as subrepresentations of π. For $\sigma \in \widehat{K}$ and $k \in \mathbb{N}$, let P_σ and P_k denote the orthogonal projections associated with $m_\sigma(\sigma \circ q)$ and $n_k\pi_k$, respectively. By Proposition 2.8.11, there is a unique decomposition

$$u(x) = \sum_{\sigma\in\widehat{K}} \langle m_\sigma(\sigma \circ q)(x)P_\sigma\xi, P_\sigma\eta\rangle + \sum_{k=1}^{\infty} \langle n_k\pi_k(x)P_k\xi, P_k\eta\rangle.$$

By Proposition 2.8.11, this is an ℓ^1-direct sum, that is,

$$\|u\|_{B(G)} = \sum_{\sigma\in\widehat{K}} \|v_\sigma\|_{B(G)} + \sum_{k=1}^{\infty} \|w_k\|_{B(G)},$$

where $v_\sigma = \langle m_\sigma(\sigma \circ q)(x)P_\sigma\xi, P_\sigma\eta\rangle$ and $w_k = \langle n_k\pi_k(x)P_k\xi, P_k\eta\rangle$. Now each v_σ lies in $B(K) \circ q$ and each w_k belongs to $A(G)$. So

$$u_1 = \sum_{\sigma\in\widehat{K}} v_\sigma \in A(K) \circ q \quad \text{and} \quad u_2 = \sum_{k=1}^{\infty} w_k \in A(G),$$

and $u = u_1 + u_2$. □

For locally compact groups G as above the spectrum of $B(G)$ and its topology can be described as follows.

PROPOSITION 2.9.2. *Let $G = N \rtimes K$ as above with $B(G) = A(K) \circ q + A(G)$ and N noncompact. Then G and K embed topologically into $\sigma(B(G))$ by $x \to \varphi_x$, where $\varphi_x(v) = v(x)$ for $v \in B(G)$, and $a \to \psi_a$, where $\psi_a(v \circ q + u) = v(a)$ for $v \in A(K)$ and $u \in A(G)$, respectively. Moreover, $\sigma(B(G)) = G \cup K$, G is open in $\sigma(B(G))$ and K is closed in $\sigma(B(G))$, and a net $(\varphi_{x_\alpha})_\alpha$, $x_\alpha = y_\alpha a_\alpha$, $y_\alpha \in N$, $a_\alpha \in K$, converges to ψ_a for some $a \in K$ if and only if $a_\alpha \to a$ in K and $y_\alpha \to \infty$ in N.*

PROOF. Since $A(G)$ is an ideal in $B(G)$ and $A(K) \circ q = B(G)/A(G)$, it is clear from Theorem 2.3.7 and from general Gelfand theory that $x \to \varphi_x$ and $a \to \psi_a$ are both topological embeddings of G and of K into $\sigma(B(G))$, respectively, that $\sigma(B(G)) = G \cup K$, G is open in $\sigma(B(G))$ and K is closed in $\sigma(B(G))$.

Now let $(x_\alpha)_\alpha$ be a net in G such that $\varphi_{x_\alpha} \to \psi_a$ for some $a \in K$, and let $x_\alpha = y_\alpha a_\alpha$, $y_\alpha \in N$, $a_\alpha \in K$. For each $u \in A(G)$ and $v \in A(K)$ we then have

$$\begin{aligned} v(a) &= \psi_a(v \circ q) = \psi_a(v \circ q + u) \\ &= \lim_\alpha \varphi_{x_\alpha}(v \circ q + u) \\ &= \lim_\alpha (v(a_\alpha) + u(y_\alpha a_\alpha)). \end{aligned}$$

Taking $u = 0$, this gives $v(a_\alpha) \to v(a)$ for every $v \in A(K)$ and hence $u(y_\alpha a_\alpha) \to 0$ for each $u \in A(G)$ and regularity of $A(K)$ implies that $a_\alpha \to a$.

Towards a contradiction, assume that there exist a compact subset C of N and a subnet $(y_{\alpha_\beta})_\beta$ of $(y_\alpha)_\alpha$ such that $y_{\alpha_\beta} \in C$ for all β. Then, passing to a further subnet if necessary, we can assume that $y_{\alpha_\beta} \to y$ for some $y \in C$. Then $y_{\alpha_\beta} a_{\alpha_\beta} \to ya$ and hence

$$u(ya) = \lim_\beta u(y_{\alpha_\beta} a_{\alpha_\beta}) = 0$$

for all $u \in A(G)$. This contradiction shows that $y_\alpha \to \infty$ in N.

Conversely, if $y_\alpha \to \infty$ in N and $a_\alpha \to a$ in K, then

$$\varphi_{x_\alpha}(v \circ q + u) = v(a_\alpha) + u(x_\alpha) \to v(a) = \psi_a(v \circ q + u)$$

for all $v \in A(K)$ and $u \in A(G)$, so that $\varphi_{x_\alpha} \to \psi_a$ in $\sigma(B(G))$. □

We close this section by presenting a class of examples of locally compact groups which satisfy the conditions (1) and (2) set out ahead of Proposition 2.9.1.

EXAMPLE 2.9.3. Let p be a prime number, Ω_p the locally compact field of p-adic numbers and Δ_p the compact open subring of p-adic integers. Let $SL(n, \Delta_p)$ be the multiplicative group of $n \times n$ matrices with entries in Δ_p and determinant of valuation 1. This compact group acts on the vector space Ω_P^n by matrix multiplication. Form the semidirect product $G = \Omega_p^n \rtimes SL(n, \Delta_p)$. When $n = 1$, this group was presented by Fell as a noncompact group which nevertheless has a countable dual space, and it is therefore usually referred to as the *Fell group*.

Since $SL(n, \Delta_p)$ is compact, the semidirect product G satisfies the hypotheses that allow to determine $\widehat{G}$ by applying the Mackey machine outlined in Section 1.4. Since Ω_p^n is self-dual, we only have to determine the orbits in Ω_p^n under the matrix action of $SL(n, \Delta_p)$.

For $x = (x_1, \ldots, x_n) \in \Omega_p^n$, let $S(x)$ denote the l^∞-sphere through x, that is,

$$S(x) = \{y = (y_1, \ldots, y_n) \in \Omega_p^n : \max_{1 \leqslant j \leqslant n} |y_j| = \max_{1 \leqslant j \leqslant n} |x_j|\}.$$

We claim that the nontrivial orbits in Ω_p^n are such spheres. Note first that $SL(n, \Delta_p)$ preserves spheres. Indeed, if $A \in SL(n, \Delta_p)$ and $A(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, then

$$\max_{1 \leqslant j \leqslant n} |y_j| = \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^n a_{ji} x_i \right| \leqslant \max_{1 \leqslant j \leqslant n} |x_j|$$

because $|a_{ji}| \leqslant 1$ for all j, i and the valuation is nonarchimedean, and then $\max_{1 \leqslant j \leqslant n} |x_j| = \max_{1 \leqslant j \leqslant n} |y_j|$ since $SL(n, \Delta_p)$ is a group. Furthermore, $SL(n, \Delta_p)$ acts transitively on the spheres. To see this, let $(x_1, \ldots, x_n)$ be a nonzero element of Ω_p^n and choose $w \in \Omega_p$ with $|w| = \max_{1 \leqslant j \leqslant n} |x_j|$. Then there exists $A \in SL(n, \Delta_p)$ such that

$$A(w, 0, \ldots, 0) = (x_1, \ldots, x_n).$$

In fact, set $a_{j1} = w^{-1}x_j$ for $1 \leqslant j \leqslant n$. Then $a_{j1} \in \Delta_p$ since $|w^{-1}x_j| \leqslant 1$, and since $\max_{1\leqslant j\leqslant n} |w^{-1}x_j| = 1$, it is possible to find a_{ji}, $1 \leqslant j \leqslant n$, $2 \leqslant i \leqslant n$, so that $A = (a_{ji})_{1\leqslant j\leqslant n, 1\leqslant i\leqslant n} \in SL(n, \Delta_p)$. The important fact now is that all these spheres are open in Ω_p^n and that they cover $\Omega_p^n \backslash \{(0, \ldots, 0)\}$.

For each $k \in \mathbb{Z}$, choose $w_k \in \Omega_p$ with $|w_k| = p^{-k}$ and let $\chi_k \in \widehat{\Omega_p^n}$ be the character corresponding to $(w_k, 0, \ldots, 0)$. Let S_k denote the stability group of χ_k in $SL(n, \Delta_p)$ and let $G_k = \Omega_p^n \rtimes S_k$. Then

$$\widehat{G} = \widehat{SL(n, \Delta_p)} \circ q \cup \left(\bigcup_{k\in\mathbb{Z}} \{\pi_{k,\tau} : \tau \in \widehat{S_k}\} \right),$$

where $q : G \to SL(n, \Delta_p)$ is the quotient map and

$$\pi_{k,\tau} = \mathrm{ind}_{G_k}^G (\chi_k \otimes \tau), \quad k \in \mathbb{Z}, \tau \in \widehat{S_k}.$$

Then each $\pi_{k,\tau}$ is a subrepresentation of the left regular representation of G since the restriction of $\pi_{k,\tau}$ to Ω_p^n is supported on the open orbit through χ_k. Thus the group G satisfies the hypotheses of Proposition 2.9.1.

2.10. Notes and references

The Fourier and Fourier-Stieltjes algebras of a locally compact group have been introduced by Eymard as generalizations of the L^1- and measure algebras, $L^1(G)$ and $M(G)$, respectively, of a locally compact abelian group G and they have been extensively studied in his seminal paper [**73**]. Actually, [**73**] has not only initiated, but also enormously influenced what has since become one of the most popular research areas in abstract harmonic analysis. All the material presented in Sections 2.1 to 2.5 is taken from [**73**], and our treatment follows very closely the excellent exposition in [**73**].

Given a commutative Banach algebra A, the first relevant problem is to determine the Gelfand spectrum of $\sigma(A)$ of A. It is a classical result that, for a locally compact abelian group G with dual group $\widehat{G}$, $\sigma(L^1(\widehat{G})) = G$. This is generalized by Proposition 2.3.2, which states that for an arbitrary locally compact group G, $\sigma(A(G))$ can be canonically identified with G and which may be considered as one of the most fundamental results of the subject area. For example, it forms the basis for Walter's [**280**] isomorphism theorems presented in Section 3.4 as well as for the study of ideal theory in $A(G)$ (Chapter 6). Equally important is the fact that $A(G)$ is a regular function algebra (Proposition 2.3.2).

In contrast, the Gelfand spectrum of the Fourier-Stieltjes algebra $B(G)$ is much less understood. It was extensively investigated by J.L. Taylor (see [**274**] and the references therein) for abelian groups and by Walter [**280**], [**281**] for general locally compact groups. It was shown, for instance, that $\sigma(B(G))$ is a semigroup with multiplication inherited from $W^*(G)$, the dual of $B(G)$, and that then G identifies with all unitary elements in $\sigma(B(G))$. Part of this is treated in detail in Section 3.2.

The identification of the Banach space dual of $A(G)$ with the von Neumann algebra $VN(G)$, generated by the left regular representation of G on the Hilbert space $L^2(G)$, plays a central role in the study of $A(G)$. In particular, the various results on the support of operators in $VN(G)$, established in this section, proved to be invaluable tools subsequently. The fact that $A(G)^* = VN(G)$ is also of great importance for the operator space structure of $A(G)$ as developed by Ruan [**245**]. It has also motivated the investigation of uniformly continuous and weakly almost periodic functionals on $A(G)$ [**102**], [**174**], which are analogues of weakly almost periodic functions on $\widehat{G}$ when G is abelian [**66**].

Let H be a closed subgroup of the locally compact group G. Then the assignment $r : u \to u|_H$ maps $A(G)$ into $A(H)$. It was proved by McMullen [**212**], and independently by Herz [**123**], that r is actually surjective. More precisely, any $v \in A(H)$ admits an extension $u \in A(G)$ with the same norm, $\|u\|_{A(G)} = \|v\|_{A(H)}$. The proof is fairly technical in that it involves the reduction to second countable groups and the existence of appropriate Borel cross-sections. Our presentation in Section 2.7 follows [**212**]. As a consequence one obtains that the adjoint map $r^* : VN(H) \to VN(G)$ is a w^*-w^*-continuous isomorphism from $VN(H)$ onto $VN_H(G)$, the w^*-closure in $VN(G)$ of the linear span of the set of all left regular representation operators $\lambda_G(x), x \in H$. This latter fact as well as surjectivity of r are frequently used later in the book, especially in Section 3.3 and Chapter 6.

There are various properties equivalent to amenability of a locally compact group G, which have been employed to give different proofs of Leptin's theorem (Theorem 2.7.2), such as

(1) $\|f\|_1 = \|\lambda_G(f)\|$ for every $f \in L^1(G)$, $f \geqslant 0$;

(2) given any compact subset K of G and $\delta > 1$, there exists a compact subset U of G such that $|KU| < \delta|U|$;

(3) the constant one function can be uniformly on compact subsets of G approximated by functions of the form $f * f^*$, $f \in C_c(G)$.

Leptin, who actually was the first to characterize a Banach algebraic property of $A(G)$ in terms of the group G, used (1) and (2). An alternative proof of Theorem 2.7.2 was given by Derighetti [**50**], based on (3) and on his own result that on the unit sphere of $B(G)$ the compact-open topology coincides with the weak topology $\sigma(B(G), L^1(G))$. The proof presented in Section 2.7 builds on (3) and on Proposition 2.7.1, which can be found in [**27**] and is attributed there to an unpublished thesis of Nielson. The important question for which locally compact groups G, the Fourier algebra $A(G)$ possesses an approximate identity which is bounded in some norm weaker than the $A(G)$-norm, has been studied by several authors (see Chapter 5). The remaining results of Section 2.7 (Theorem 2.7.5 and 2.7.9), dealing with the impact on G of the existence of some ideal in $A(G)$ with bounded approximate identity, are due to Forrest [**79, 80**].

The subspaces $A_\pi(G)$ of $B(G)$, which we treated in Section 2.8, were introduced and studied in [**5**]. Arsac proved many results beyond those we have, mainly following the exposition in [**5**], presented here. For instance, he intensively investigated the assignment $\pi \to A_\pi(G)$ under various aspects, such as forming tensor products and direct integrals of representations and inducing representations. He also clarified the natural question of when $A_\pi(G)$ is a subalgebra of $B(G)$.

Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p(G)$ and $g \in L^q(G)$, the convolution product

$$(f * \check{g})(x) = \int_G f(xy)g(y)dy, \quad x \in G,$$

defines a function in $C_0(G)$ such that $\|f * \check{g}\|_\infty \leqslant \|f\|_p \|g\|_q$. Since $(f * \check{g})(x) = \int_G g(x^{-1}y)f(y)dy$, u can be viewed as a coefficient function of the left regular representation of G on $L^q(G)$. Define $A_p(G)$ to be the set of all functions $u \in C_0(G)$

for which there exist sequences $(f_n)_n$ in $L^p(G)$ and $(g_n)_n$ in $L^q(G)$ such that

$$\sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q < \infty \quad \text{and} \quad u(x) = \sum_{n=1}^{\infty} (f_n * \widecheck{g_n})(x)$$

for all $x \in G$. For $u \in A_p(G)$, let

$$\|u\|_{A_p(G)} = \inf \left\{ \sum_{n=1}^{\infty} \|f_n\|_p \|g_n\|_q \right\},$$

where $(f_n)_n$ and $(g_n)_n$ are sequences with the above two properties. Then, of course, $\|u\|_\infty \leqslant \|u\|_{A_p(G)}$.

THEOREM 2.10.1. $(A_p(G), \|\cdot\|_{A_p(G)})$ *is a Banach algebra with respect to pointwise operations,* $A_p(G) \cap C_c(G)$ *is dense in* $A_p(G)$ *and* $A_p(G)$ *is uniformly dense in* $C_0(G)$.

THEOREM 2.10.2. *The spectrum of* $A_p(G)$ *can be canonically identified with* G. *More precisely, the map* $x \to \gamma_x$, *where* $\gamma_x(u) = u(x)$ *for* $u \in A_p(G)$, *is a homeomorphism from* G *onto* $\sigma(A_p(G))$. *Moreover,* $A_p(G)$ *is regular and Tauberian.*

The algebras $A_p(G)$ are usually referred to as the *Figà-Talamanca-Herz algebras*. Note that $A_2(G) = A(G)$. The preceding two theorems are due to Herz and have been shown in [**122**] and [**123**], respectively.

Several other results presented in this chapter for $A(G)$, essentially extend to the algebras $A_p(G)$, $1 < p < \infty$. We mention two significant ones. As shown by Herz [**123**], surjectivity of the restriction map $u \to u|_H$ remains true for $A_p(G)$. However, when $p \neq 2$, given $\epsilon > 0$, the existence of an extension $u \in A_p(G)$ of $v \in A_p(H)$ could only be shown to satisfy the norm inequality $\|u\|_{A_p(G)} \leqslant \|v\|_{A_p(H)} + \epsilon$. In addition, as also shown in [**123**], if H is normal and $v \in A_p(H)$ has compact support, then given an open subset U of G such that $\operatorname{supp} v \subseteq U \cap H$, there exists such an extension u of v satisfying $\operatorname{supp} u \subseteq \overline{U}$. If G is amenable, then $A_p(G)$ has an approximate identity of norm bound 1 for every p. Conversely, if $A_p(G)$ has a bounded approximate identity for some p, then G is amenable.

The notions of the Fourier and the Fourier-Stieltjes algebra of a locally compact group have been generalized in various different directions, which we now briefly indicate, confining ourselves to Fourier algebras.

Firstly, let K be a compact subgroup of the locally compact group G. Forrest [**83**] has introduced the Fourier algebra $A(G/K)$ of the left coset space G/K. This algebra can simultaneously be viewed as an algebra of functions on G/K and as the subalgebra of $A(G)$ consisting of all those functions in $A(G)$ which are constant on left cosets of K. In many respects, $A(G/K)$ behaves as nicely as does $A(G)$. For instance, as shown in [**83**], $A(G/K)$ is regular and semisimple, $\sigma(A(G/K)) = G/K$ and $A(G/K)$ has a bounded approximate identity if and only if G is amenable. The algebras $A(G/K)$ are precisely the norm closed left translation invariant $*$-subalgebras of $A(G)$ [**271**].

Secondly, let H be a locally compact hypergroup with left Haar measure. The Fourier space $A(H)$ was defined in analogy to the description of functions in $A(G)$ in terms of L^2-functions (Proposition 2.3.3). In general, $A(H)$ need not be closed under pointwise multiplication. However, it is an algebra for many important classes of hypergroups, such as double coset hypergroups. See [**217**] and the references therein.

Let G be a topological group, let $P(G)$ denote the collection of all continuous positive definite functions on G, and let $B(G)$ denote the linear span of $P(G)$.

By a σ-continuous representation of G into a W*-algebra M, we shall mean a pair (ω, M) such that ω is a homomorphism of G into

$$Mu := \{x \in M : x^*x = xx^* = 1\},$$

the group of unitaries in M, where 1 is the identity of M, and σ is the weak*-topology $\sigma(M, M_*)$ defined by the unique predual of M.

Let $\Omega(G)$ denote the collection of all σ-continuous representations $\alpha = (\omega, M)$ of G, such that $\overline{\langle \omega(G) \rangle}^{\alpha} = M$. Then $B(G)$ is precisely the collection of all complex-valued functions ϕ on G such that $\phi = \hat{f}_\alpha$ for some $f \in M_*$ and some $\alpha = (\omega, M) = (\omega_\alpha, M_\alpha)$ in $\Omega(G)$, where $\hat{f}_\alpha(a) = \langle \omega(a), f \rangle$ for all $a \in G$. For each $\phi \in B(G)$, define

$$\|\phi\| := \|\phi\|_{B(G)} = \inf \left\{ \|f_\alpha\| : f_\alpha \in M_*,\ \phi = \hat{f}_\alpha, \text{ and } \alpha = (\omega, M) \in \Omega(G) \right\}.$$

Also let $M_\Omega := \sum \bigoplus M_{\omega_\alpha}$, the direct summand of the W*- algebras $M_\alpha := M_{\omega_\alpha}, \alpha \in \Omega(G)$. Define a σ-continuous homomorphism of G into M_Ω by $\omega_\Omega(a)(\alpha) := \omega_\alpha(a)$ for each $\alpha = (\omega_\alpha, M_\alpha)$ in $\Omega(G)$. Write

$$W^*(G) = \overline{\langle \omega_\Omega(G) \rangle}^{\sigma}.$$

THEOREM 2.10.3. (a) *$B(G)$ is a subalgebra of $WAP(G)$, the space of continuous weakly almost periodic functions on G, containing the constant functions. Furthermore, $\|\cdot\|$ is a norm on $B(G)$ and $(B(G), \|\cdot\|)$ is a commutative Banach algebra. More specifically, the map $\rho : W^*(G)_* \to B(G)$ defined by $\rho(f) := \hat{f}$, $f \in W^*(G)_*$, is a linear isometry from $W^*(G)_*$ onto $B(G)$. Furthermore, $\rho(f)$ is positive definite if and only if f is positive.*

(b) *If (ω, M) is any σ-continuous representation of G, then there is a W*-homomorphism h_ω from $W^*(G)$ into M such that the diagram*

$$\begin{array}{ccc} G & \xrightarrow{\ \omega_\Omega\ } & W^*(G) \\ & \searrow^{\omega} & \downarrow{\scriptstyle h_\omega} \\ & & M \end{array}$$

is commutative. Also if $f \in M_*$, then $\hat{f}(x) = \langle h_\omega(x), f \rangle$ for all $x \in W^*(G)$.

(c) *If $\phi \in B(G)$ and $a \in G$, then the functions $l_a\phi$, $r_a\phi$, ϕ^*, $\bar{\phi}$ are all in $B(G)$ and*

$$\|l_a\phi\| = \|\phi\|, \quad \|r_a\phi\| = \|\phi\|, \quad \|\phi^*\| = \|\phi\|, \quad \|\bar{\phi}\| = \|\phi\|,$$

where $\phi^*(y) = \phi(y^*)$ for each $y \in M$.

Let A be a C^*-algebra and (π, H_π) be a non-degenerate representation of A. Let π^e be the unique extension of π to $M(A)$, the multiplier algebra of A is the locally convex topology on $M(A)$ defined by the semi-norms: $\{P_a : a \in A\}$, where

$$P_a(m) = \|a \cdot m\| + \|m \cdot a\|, \quad m \in M(A).$$

DEFINITION 2.10.4. Let G be a topological group. We call a *host algebra* of G a pair (A, η), where A is a C^*-algebra and $\eta : G \mapsto U(M(A))$ is a continuous homomorphism from G into the unitary group of $M(A)$, such that the mapping $\eta^* : \mathrm{Rep(A)} \to \mathrm{Rep(G)}, \eta^*(\pi) := \pi^{\mathrm{e}} \circ \eta$ is injective, where Rep(A) denotes the collection of $*$- representations of A, and Rep(G) the group of the continuous unitary representations of G.

Given any C*- algebra A, let $\widetilde{A}$ denote the universal enveloping von Neumann algebra A^{**} of A.

We say that (A, η) is a *full host algebra* of a topological group G, if it is a host algebra of G and if η^* is also surjective.

THEOREM 2.10.5. *Let G be a topological group with a full host algebra A. Then the von Neumann algebras $\widetilde{A}$ and $W^*(G)$ are isomorphic. In particular, $\widetilde{A}_*$ is isometrically isomorphic to $B(G)$. Furthermore, if A_1 is any C^*-algebra such that A_1^* is isometrically isomorphic to the Banach space $B(G)$, then $\widetilde{A_1}$ and $\widetilde{A}$ are either isomorphic or anti-isomorphic.*

REMARK 2.10.6. Theorem 2.10.3 was also proved in [**173**], and Theorem 2.10.5 in [**181**]. See also [**182**].