Preface

-Why do solutions of linear analytic PDE suddenly break down, while for ODE the solutions are as good as the coefficients? What is the source of these mysterious singularities, and how do they propagate?

-Is there a mean value property for harmonic functions in ellipsoids similar to the well-known mean value property for balls? What about other domains with algebraic boundaries?

-What makes the integrals of all summable harmonic functions over the non-convex complement of a parabola vanish but this "null mean value property" fails for hyperbolas?

-Is there a reflection principle for harmonic functions in higher dimensions similar to the celebrated Schwarz reflection in the plane?

-Given a series of zonal harmonics, is it possible to pinpoint where the singularities occur on the boundary of its ball of convergence?

-How can we understand a moving interface between two fluids, one of them viscous and one inviscid, and what does this nonlinear dynamical problem have to do with the static inverse problem of linear potential theory?

-Why does the logarithmic potential of a uniformly charged disk experience a logarithmic singularity at the center, while that of an ellipse has square-root-type algebraic singularities at the foci?

-How far outside of their natural domain can solutions of the classical Dirichlet problem be extended? Where do the continued solutions eventually break down and why?

This book is intended as an invitation for graduate students and young analysts to these and many other intriguing questions. In trying to make the book as accessible as possible to a wide audience, we do not assume that the readers are experts in the theory of holomorphic partial differential equations. Instead we have tried to develop all necessary tools to give a good first taste of a subject rich with deep and beautiful results that illustrate a nice interplay between various parts of modern analysis and attractive themes in classical "physical" mathematics of the nineteenth century. We hope that most of the book is accessible to anyone familiar with multivariate calculus and some basics in complex analysis and differential equations.

At no point at all have we tried to produce an encyclopedic treatise, so whenever a choice between clarity and simplicity vs. generality appeared we have most decidedly chosen the former but supply enough references to satisfy an engaged and more demanding reader. (Classic PDE books by J. Hadamard and R. Courant and D. Hilbert, and later by P. Garabedian, F. John, or a more recent treatise by L. Hormander can serve as excellent supplements for this volume.) Along the same line, wherever possible, we have tried to maintain an informal way of communicating with the reader following the "Socratic method" of a dialogue rather than a formal style of a textbook.

Although the book is not intended as a regular PDE textbook, one might adopt it as a text for a graduate topics course in holomorphic PDE. It is then intended that Chapters 1–8 be covered in order; after that, there is a lot of flexibility for the instructor to build his/her own subset of the remaining chapters. We have added some exercises to most chapters in order to simplify the task of building such a topics course. Concluding each chapter, we have included Notes that supply additional references. We also point out (a great many) tantalizing open problems that could tempt a young researcher.

The second part of the book deals with deeper, more recent results and, accordingly with our plan to keep the book accessible to students, we often resort to only treating the simplest possible cases of the results discussed there, hopefully providing enough guidance and references for the readers to do deeper research on their own.

The book has grown out of efforts in giving short and long courses on the subject over the years at conferences and as topics courses. The first author started research on these topics jointly with Professor Harold S. Shapiro from the Royal Institute of Technology in Stockholm almost three decades ago. Harold's influence on the choice of material and style is far beyond what one may see from the references to his works. Of course, he bears no responsibility for any possible errors.

The first author would also like to use this opportunity and express his gratitude to his father, the late S. Ya. Khavinson, Professor J. Wermer (Brown University), and Professor P. L. Duren (University of Michigan) for many years of guidance, friendship, and support.

The second author would like to thank A. Lerario (SISSA) for a close friendship and many dynamic collaborations.

Over the years we have benefited greatly from numerous stimulating discussions on a number of related topics with Professors B. Gustafsson, G. Johnsson, and H. Shahgholian of the Royal Institute of Technology, Professor Ebenfelt from UCSD at La Jolla, Professor L. Karp at Carmiel College in Galilee, Professors S. Gardiner and H. Render from University College Dublin, Professor R. Teodorescu at the University of South Florida, Professor M. Putinar at UCSB, and Professors S. R. Bell and A. Eremenko from Purdue University. It is our great pleasure to thank them here. We hope that the book will serve as a small token of our gratitude and appreciation to them all for sharing their time and ideas with us.

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