

Introduction: Some Motivating Questions

“The shortest and best way between two truths of the real domain often passes through the imaginary one.”

- P. Painleve - J. Hadamard

1. Continuation of potentials

In the 1914 treatise [83] G. Herglotz studied the continuation of potentials inside the region occupied by masses. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded domain with a real analytic boundary, and $p(x)$ is a polynomial mass density, where $x = (x_1, x_2, x_3)$. (The case of uniform density $p(x) = 1$ is especially interesting.) Consider the Newtonian potential

$$U_{\Omega,p} = \frac{-1}{4\pi} \int_{\Omega} \frac{p(y)dy}{|x-y|},$$

which is a harmonic function for $x \notin \Omega$.

Question A (G. Herglotz, 1914): How far can $U_{\Omega,p}$ be analytically continued across $\partial\Omega$ as a harmonic function inside Ω (into the region occupied by mass, or charge)?

EXAMPLE 1.1 (a familiar example). $\Omega := \{x \in \mathbb{R}^n : |x| < 1\}$ is a ball, and density $p(x) = 1$ is constant. Then the Newtonian potential

$$(1.1) \quad U_{\Omega,p}(x) = \frac{-1}{4\pi} \int_{\Omega} \frac{dy}{|x-y|} = \frac{C}{|x|},$$

with C a constant, so $U_{\Omega,p}(x)$ extends into $\Omega \setminus \{0\}$. (This follows from the mean-value property for harmonic functions.) The right hand side of (1.1) is harmonic in $\mathbb{R}^3 \setminus \{0\}$. So $U_{\Omega,p}(x)$ extends to $\mathbb{R}^3 \setminus \{0\}$.

It is less well-known (and we shall prove it later in Chapter 14) that the same holds for any polynomial (or, even, entire) density p on the ball—the potential $U_{\Omega,p}(x)$ extends harmonically to $\mathbb{R}^3 \setminus \{0\}$.

As suggested in this familiar example, the problem can be stated in more physical terms:

Consider the exterior gravitational potential generated by a uniform massive object. Find a gravi-potentially equivalent object (of perhaps non-uniform density) occupying a smaller interior region.

EXAMPLE 1.2 (oblate spheroid—e.g., a planet with an “equatorial bulge”). Consider

$$\Omega := \left\{ x : \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} - 1 < 0 \right\},$$

with $a > b > 0$.

Then $U_{\Omega,p}(x)$ extends into $\Omega \setminus \{(x_1, x_2, 0) : x_1^2 + x_2^2 \leq a^2 - b^2\}$. Moreover, $U_{\Omega,p}(x)$ extends as a multi-valued algebraic function across the interior of the disk.

EXAMPLE 1.3 (prolate spheroid). Consider

$$\Omega := \left\{ x : \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{b^2} - 1 < 0 \right\},$$

with $a > b > 0$.

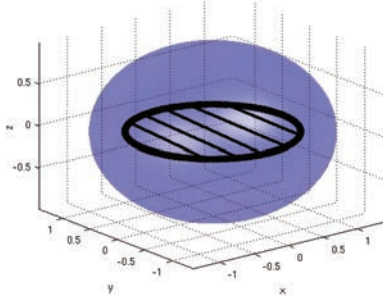


FIGURE 1. An oblate spheroid.

Then $U_{\Omega,p}(x)$ extends into $\Omega \setminus \{(x_1, 0, 0) : |x_1| \leq \sqrt{a^2 - b^2}\}$, yet singularities are worse, of logarithmic type.

One of the goals of this book is to explain these examples in a unified way.

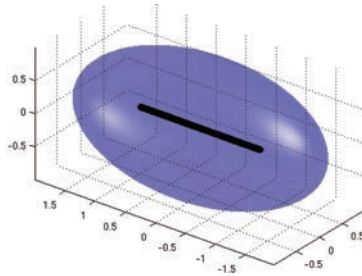


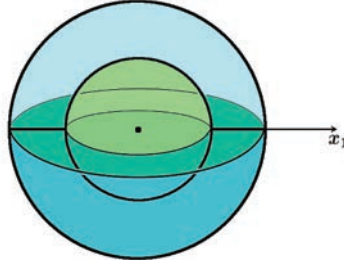
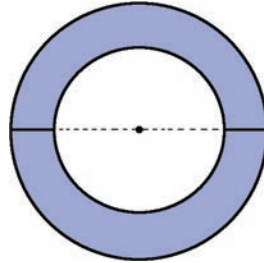
FIGURE 2. A prolate spheroid.

2. Uniqueness of potentials

Consider the spherical shell $\Omega := \{x \in \mathbb{R}^3 : r < |x| < R\}$, and let u be harmonic in Ω and vanish on the segment $(-R, -r)$ of, say, the x_1 -axis.

Question B: Does u also vanish on the remaining part of the x_1 -axis, the segment (r, R) ?

EXERCISE 1.1. Answer the same question in \mathbb{R}^2 (Hint: Use the Schwarz reflection principle). Does it matter that the line passes through the center. What happens if it doesn't?

FIGURE 3. Ω is a spherical shell.FIGURE 4. Exercise: Show that for $n = 2$ the answer is “Yes”.

3. The Schwarz reflection principle

Recall [176, Sec. 1.2] that if γ is a real analytic curve in \mathbb{R}^2 , then for any point A sufficiently close to γ , there exists B on the other side of γ such that $u(A) + u(B) = 0$ for all functions harmonic near γ and vanishing on γ . (The simplest cases being (i) when γ equals the real axis with $B = \bar{A}$ and (ii) when γ is the unit circle with $B = 1/\bar{A}$.)

Question C: What happens in \mathbb{R}^n for $n \geq 3$?

If $\gamma = \{x_n = 0\}$ is a hyperplane, the reflection principle holds with $A = (x_1, \dots, x_n)$, $B = (x_1, \dots, -x_n)$ the reflected point.

For spheres, e.g., $\gamma = \{|x| = 1\}$, the principle holds in the more complicated form

$$u(x) + |x|^{2-n} u\left(\frac{x}{|x|^2}\right).$$

So, $B = \frac{A}{|A|^2}$, but there is a coefficient $|A|^{2-n}$. This is known as Kelvin's reflection. What happens in \mathbb{R}^n for other surfaces? It turns out that the answer depends on the parity of n [40], and our goal is to explain why.

4. Szegő's theorem

Let $u = \sum_{n=0}^{\infty} a_n r^n P_n(\cos \theta)$ be an axially symmetric harmonic function in the open unit ball, $a_n \in \mathbb{R}$, $\overline{\lim} |a_n|^{1/n} = 1$, P_n are Legendre polynomials [99].

Since the expansion obviously diverges for $r > 1$ (it is a good exercise in classical analysis to check it.—cf. [186]) u must have singularities on the unit sphere. The question is, “where?” The remarkable theorem of Szegő pinpoints them precisely.

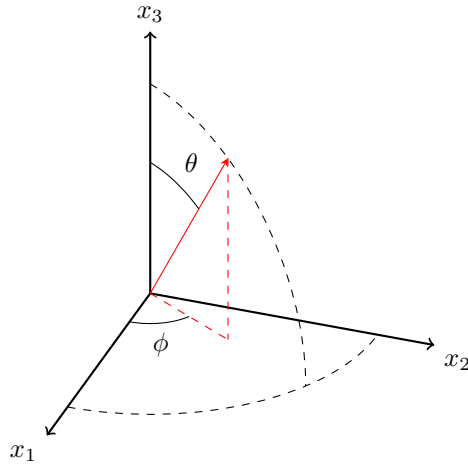


FIGURE 5. Spherical coordinates.

THEOREM 1.1 ([186]). *The function $u = \sum_{n=0}^{\infty} a_n r^n P_n(\cos \theta)$ extends harmonically across $p_0 = (1, \theta_0, \phi)$ if and only if the Taylor series $f(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n$ extends holomorphically across $\zeta_0 = e^{i\theta_0}$.*

It turns out that the two expansions appearing in the statement of Szegő's theorem can be viewed as solutions to a Cauchy problem with the same data but for different PDE (in two variables). This leads us to ask the following.

Question D: Can Szegő's theorem be understood from the high ground point of view of analytic continuation of solutions to holomorphic PDE?

5. PDE vs. ODE

Recall that for linear ODE

$$w^{(n)} + a_{n-1}(z)w^{(n-1)} + \dots + a_0 w(z) = f(z),$$

where a_j, f are assumed to be analytic in a domain $\Omega \subset \mathbb{C}$, all solutions of the Cauchy problem (assume $0 \in \Omega$)

$$w(0) = w_0, w'(0) = w_1, \dots, w^{(n-1)}(0) = w_{n-1},$$

for any data w_0, \dots, w_{n-1} extend analytically to all of Ω (solutions are as good as the coefficients [90]).

For PDE this fails miserably.

Example: Consider in \mathbb{C}^2 the Cauchy problem

$$\frac{\partial w}{\partial z_2} = z_1^2 \frac{\partial w}{\partial z_1}, \quad w(z_1, 0) = z_1.$$

The coefficients are analytic in all of \mathbb{C}^2 , yet the solution

$$w(z_1, z_2) = \frac{z_1}{1 - z_1 z_2}$$

blows up arbitrarily close to the initial curve $\{(z_1, 0)\}$.

Question E: Where do these singularities come from, and what positive statements can we make in general about the size of the domain of analyticity for solutions to PDE with holomorphic coefficients?

6. Laplacian growth and the inverse potential problem

A one-parameter family $\Omega_t \subset \mathbb{R}^n$ of smoothly-varying domains each with smooth boundary is referred to as a *Hele-Shaw flow with source (or sink) at x_0* if the normal velocity of the moving boundary $\partial\Omega_t$ coincides (at each boundary point at each time) with the normal derivative of the classical harmonic Green function of Ω_t with singularity at x_0 .

This nonlinear dynamical system arises as a simplified model for the growth of a region of viscous fluid in a porous medium with injection at x_0 . In the case when Ω_t is unbounded with a sink located at infinity, the process is referred to as *Laplacian growth* and arises (at least as a first-approximation) in many physical processes involving a moving-boundary, such as viscous fingering, electrodeposition, formation of ice in the presence of undercooling, growth of mineral dendrites, growth of certain types of cancerous tumors, and generally diffusion-limited growth.

S. Richardson [160] discovered infinitely many conservation laws for this system and established that the problem of finding the shape of Ω_T at some final time $t = T$ (given the initial shape Ω_0) is equivalent to the inverse potential problem, that is, it reduces to answering the following question.

Question F: What is the shape of a uniformly massive object given the Newtonian gravitational potential it creates outside of itself?

7. Some basic notation

Here we list some basic notation. Further notations will be introduced later in the text as needed.

Let $z \in \mathbb{C}^n$, $z = (z_1, z_2, \dots, z_n) = x + iy$, where $x, y \in \mathbb{R}^n$. Then,

$$\langle z, \xi \rangle := \sum_{j=1}^n z_j \bar{\xi}_j,$$

$$z \cdot \xi := \sum_{j=1}^n z_j \xi_j,$$

$$\|z\| := \langle z, z \rangle^{1/2}.$$

A multi-index α is a vector $(\alpha_1, \alpha_2, \dots, \alpha_n)$, with $\alpha_j \in \mathbb{Z}_+$ for $j = 1, 2, \dots, n$. We set

$$|\alpha| := \alpha_1 + \alpha_2 + \dots + \alpha_n,$$

$$\alpha! := \alpha_1! \alpha_2! \dots \alpha_n!,$$

$$z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n},$$

$$\partial_j := \frac{\partial}{\partial z_j},$$

$$\partial^\alpha := \left(\frac{\partial}{\partial z_1} \right)^{\alpha_1} \left(\frac{\partial}{\partial z_2} \right)^{\alpha_2} \dots \left(\frac{\partial}{\partial z_n} \right)^{\alpha_n}.$$

We will sometimes use D^α in place of ∂^α .

We say that a function $f(z)$ is holomorphic near z^0 if

$$f(z) = \sum_{|\alpha|=0}^{\infty} \frac{\partial^\alpha f(z^0)}{\alpha!} (z - z^0)^\alpha,$$

and the series converges absolutely and uniformly in some neighborhood of z^0 .

The polydisk $D(z^0, R)$ of radius $R > 0$ centered at z^0 is the set

$$D(z^0, R) := \{z : |z_j - z_j^0| < R, j = 1, 2, \dots, n\}.$$

Finally, we sometimes use $z' = (z_1, z_2, \dots, z_{n-1})$ to abbreviate the projection of z onto the hyperplane $z_n = 0$.

Notes

The classical monographs of O. Kellogg [99] and G. Evans [50] or later ones by N. Landkof [120] and T. Ransford [153] could serve as excellent references for potential theory in depth. The rudiments of elementary several complex variables can be found in B.V. Shabat [171] or L. Hormander [87].

The problem discussed in Section 1 is stated in the Herglotz memoir [83], although there are earlier works of C. Neumann, E. Schmidt, and others. A fairly detailed survey of the literature can be found in [110] and even more so in [176]. Herglotz has completely solved the problem in two dimensions; in higher dimensions it remains unsolved even today (cf. [110, 176]). Examples (1.2) and (1.3) are studied in great detail in G. Johnsson's thesis [94], cf. Chapters 19 and 20. The question in Section 2 was posed and answered in the affirmative by N. Nadirashvili [144] by making use of symmetrization techniques. At the end of Chapter 9 we shall outline a different approach to it, which will allow us to consider more general, not necessarily symmetric, configurations. We return to the discussion of Question E in Section 5 in Chapters 18 - 20. [176] contains a detailed discussion of various topics associated with the Schwarz Reflection Principle and gives detailed references. A complete solution of the problem posed in Section 3 has been obtained in [40]. In Chapter 21, we will return to the fluid dynamic problem stated in Section 6 while giving special attention to quadrature domains, an important class of exact solutions.